CSE 421 Winter 2025
Lecture 11: Quicksort and Medians

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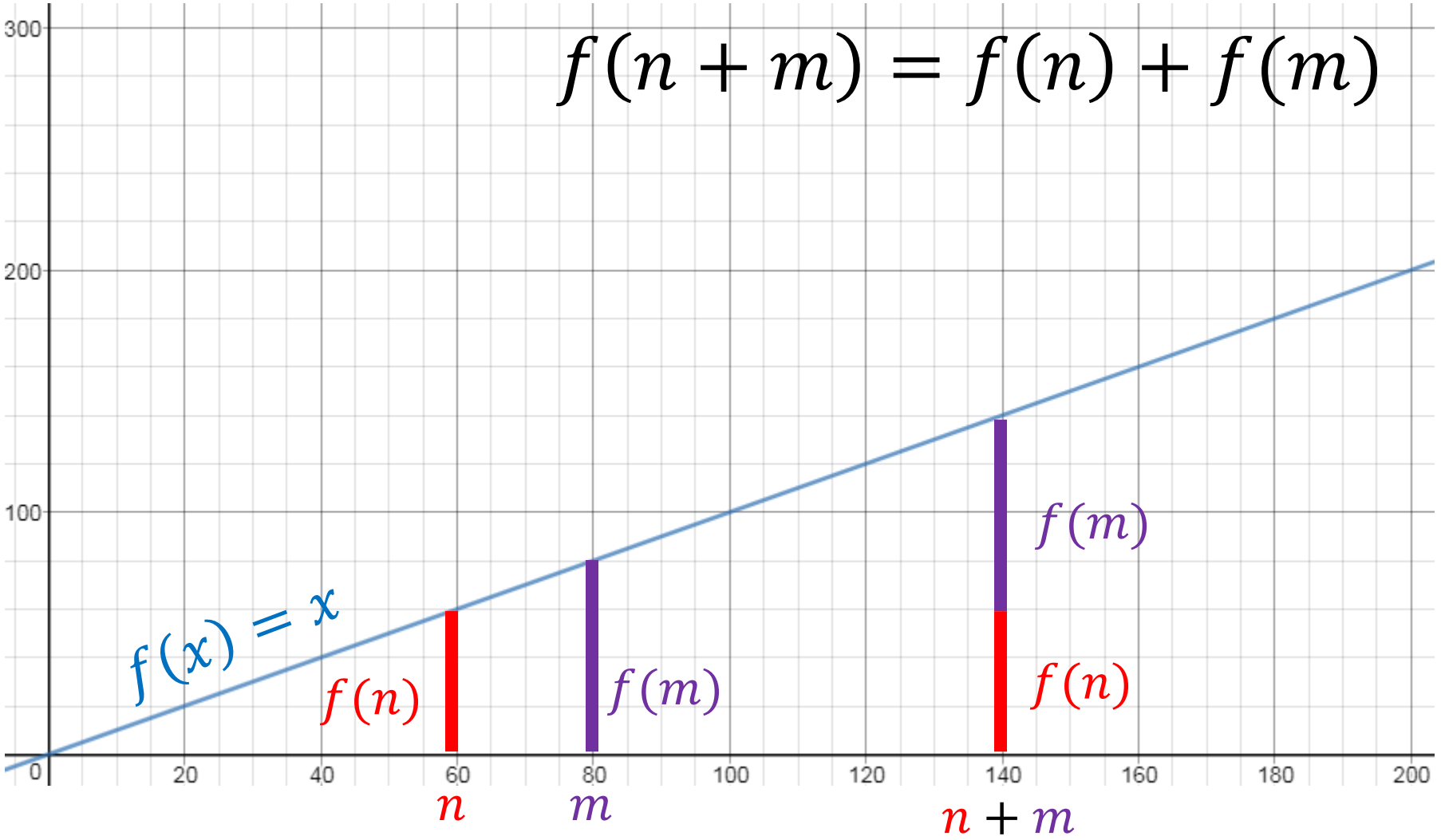
http://www.cs.uw.edu/421

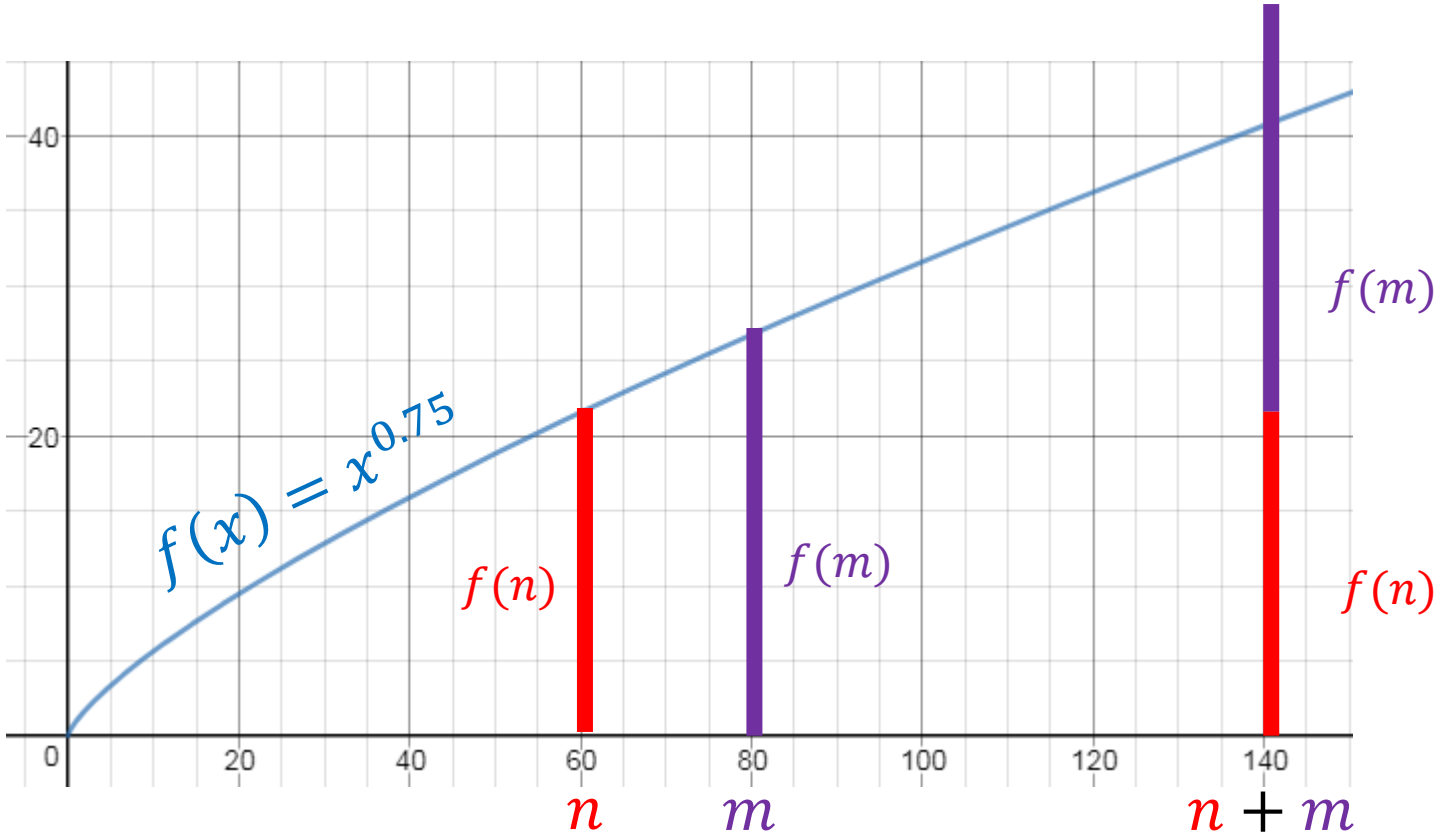
$f(n+m)$ vs. $f(n)+f(m)$

- When is each true?
  - $f(n+m)=f(n)+f(m)$
  - $f(n+m)<f(n)+f(m)$
  - $f(n+m)>f(n)+f(m)$

$f(n) = \Theta(n)$


$f(n+m) = f(n) + f(m)$
300
200
100
0
20
40
60
80
100
120
140
160
180
200
$f(x) = x$
$f(n)$
$f(m)$
$f(m)$
$f(n)$
n
m
$n+m$

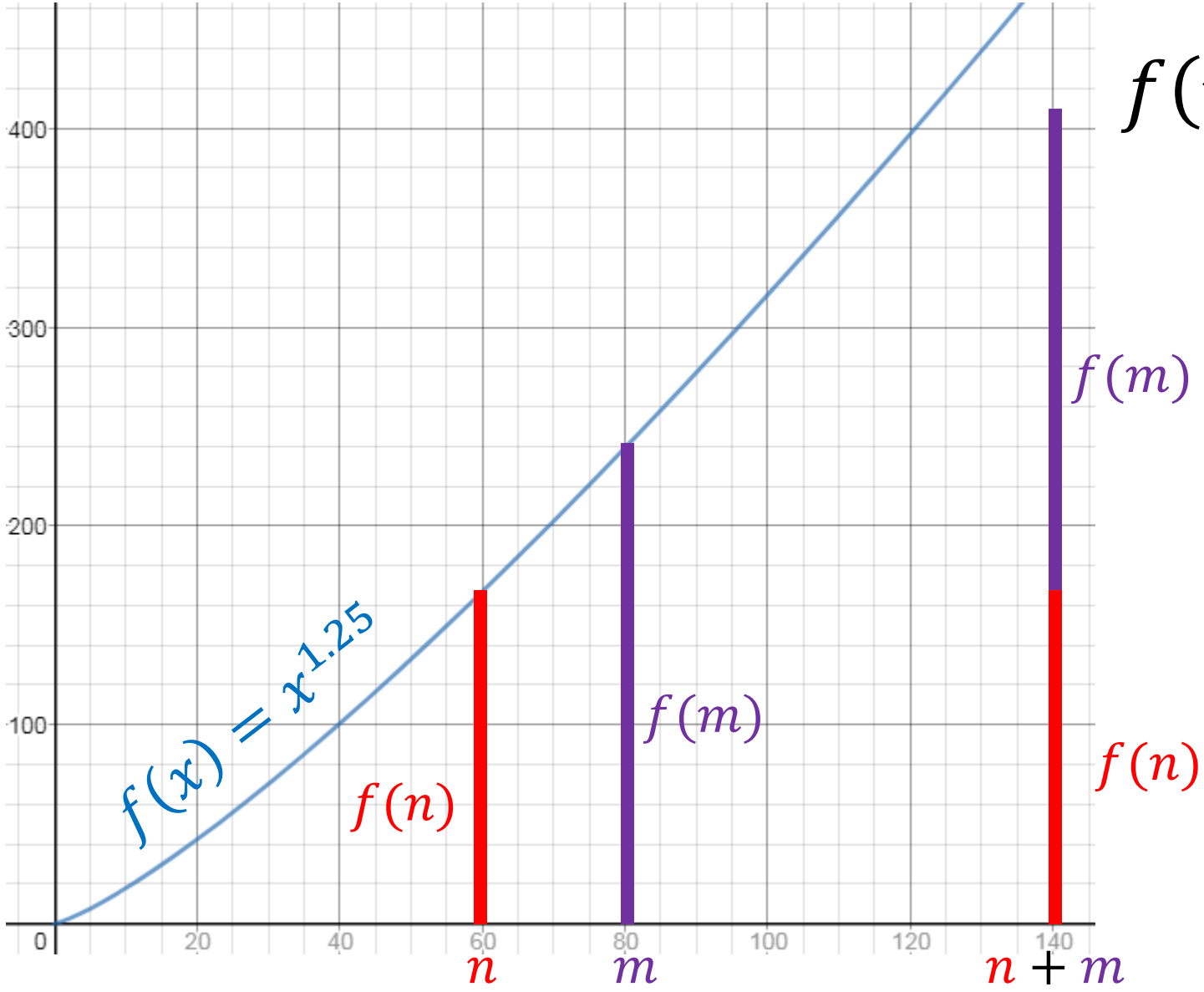
$$f(n) \in O(n)$$



$$f(n+m) \leq f(n) + f(m)$$

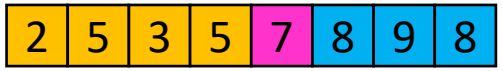
$f(n) \in \Omega(n)$

$$f(n+m) \geq f(n) + f(m)$$



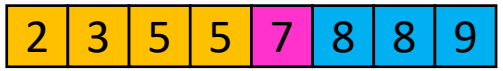
Divide and Conquer (Quick Sort)

- **Base Case**:
  - If the list is of length 1 or 0, it's already sorted, so just return
  - (Alternative: when length is ≤ 15, use insertion sort)



- **Divide**:
  - Select an element to use as a "pivot"
  - Partition: rearrange the list so that all elements less than the pivot are to the left of the pivot, all elements greater are to the right

- **Conquer**:
  - Sort the sublists to the left and right of the pivot recursively.



- **Combine**:
  - Nothing!

Partition

1. Put p at beginning of list
2. Put a pointer (Begin) just after p, and a pointer (End) at the end of the list
3. While Begin < End:
    1. If Begin value $< p$, move Begin right
    2. Else swap Begin value with End value, move End Left
4. If pointers meet at element $< p$: Swap p with pointer position
5. Else If pointers meet at element $> p$: Swap p with value to the left

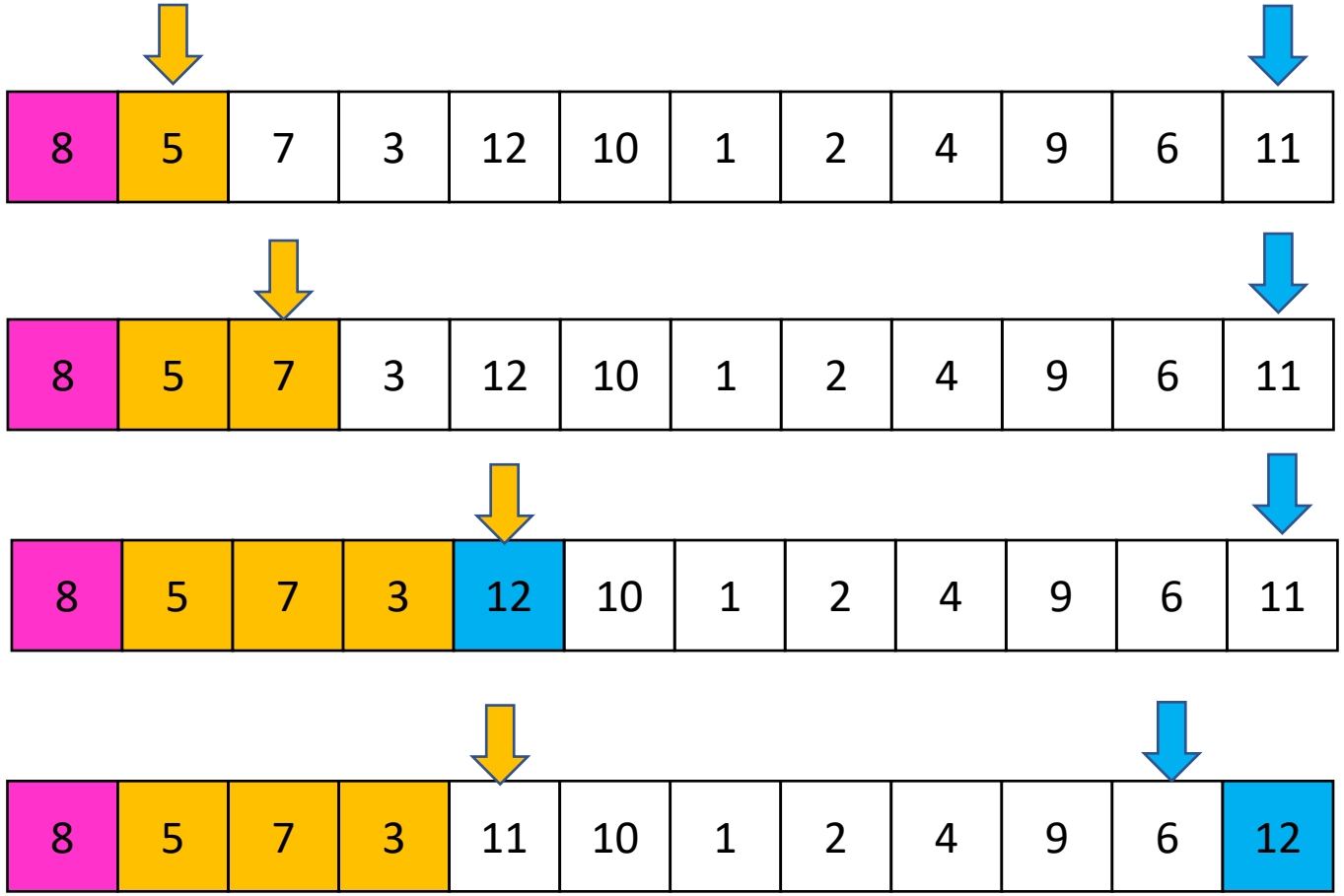
Run time? $O(n)$

Partition, Procedure

If Begin value $<$ p, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End

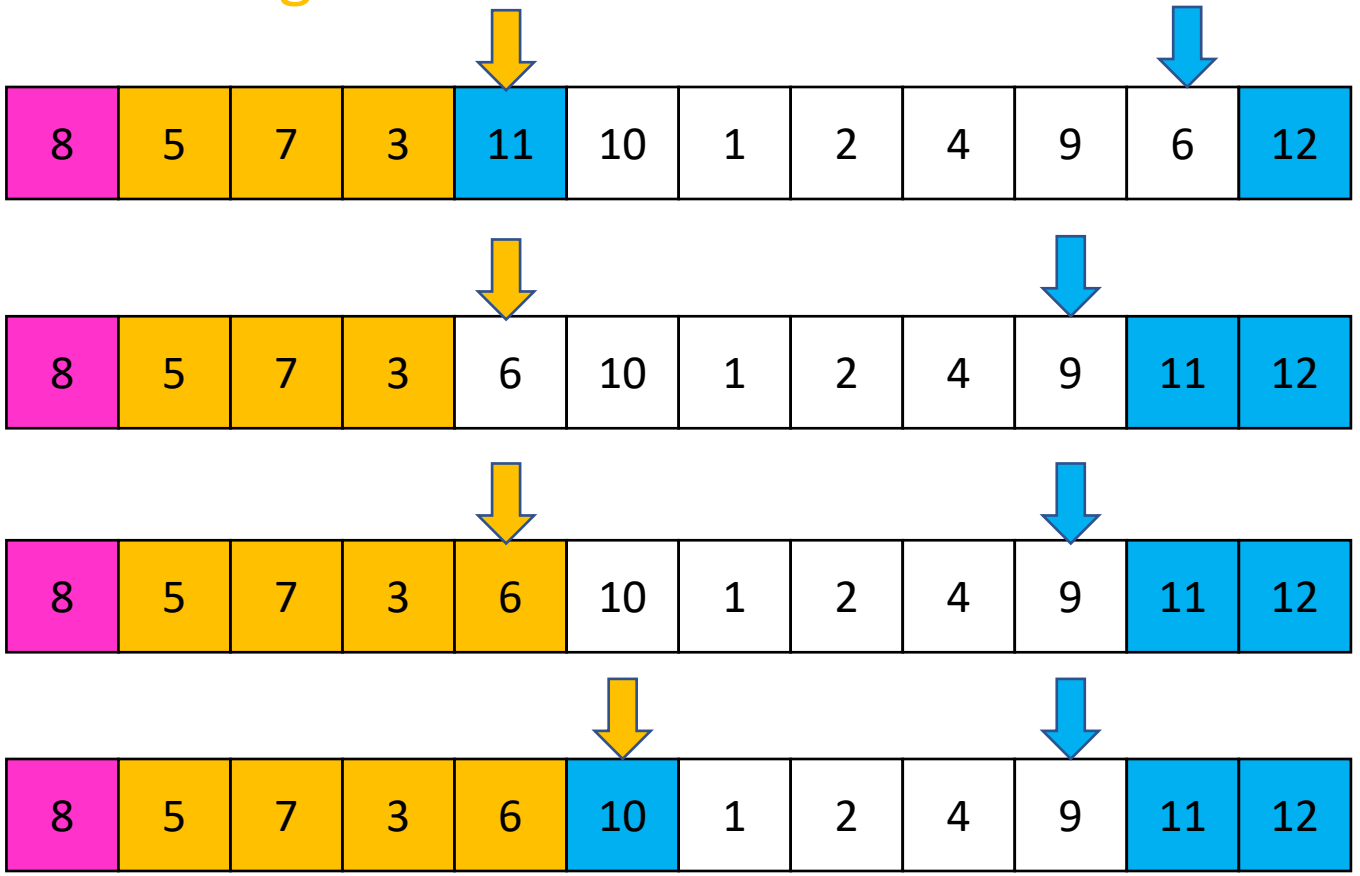


Partition, Procedure

If Begin value $<$ p, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End



Partition, Procedure

If Begin value $<$ p, move Begin right

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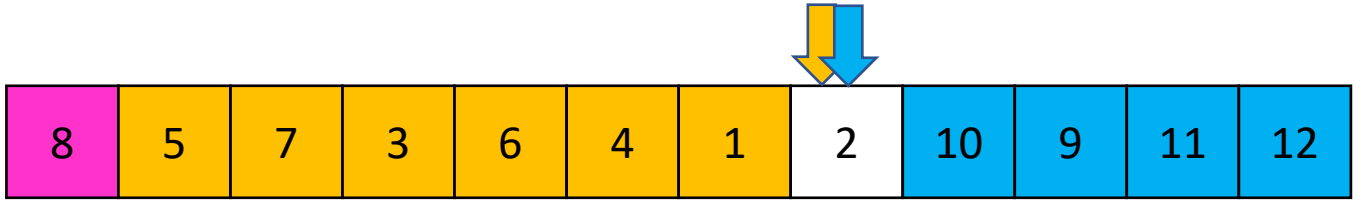
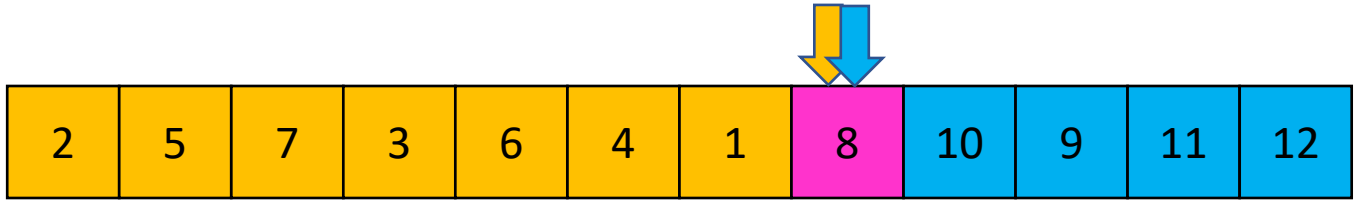


Case 1: meet at element $< p$

Swap p with pointer position (2 in this case)

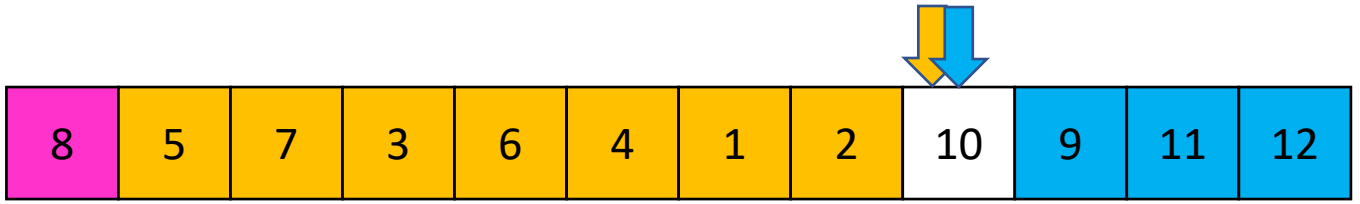


Partition, Procedure

If Begin value $<$ p, move Begin right

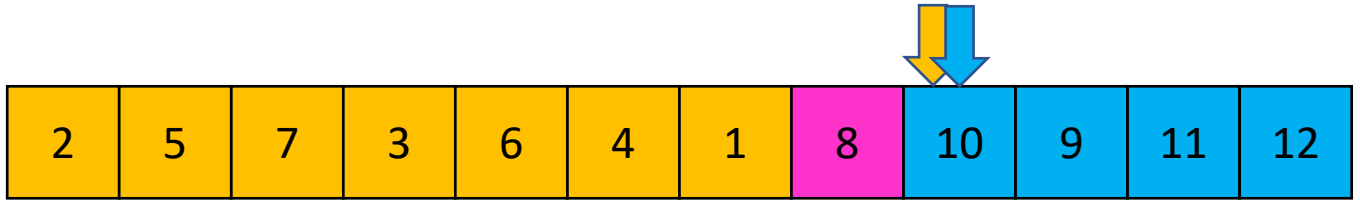
Else swap Begin value with End value, move End Left

Done when Begin = End



Case 2: meet at element $>$ p

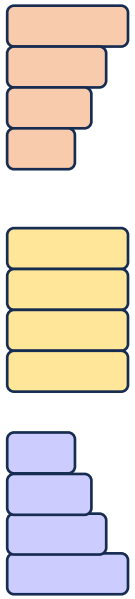
Swap p with value to the left (2 in this case)



Quick Sort Running Time

Master Theorem: Suppose that $T(n) = a \cdot T(n/b) + O(n^k)$ for $n > b$.

- If $a < b^k$ then $T(n)$ is $O(n^k)$
  - Cost is dominated by work at top level of recursion
- If $a = b^k$ then $T(n)$ is $O(n^k \log n)$
  - Total cost is the same for all $\log_b n$ levels of recursion
- If $a > b^k$ then $T(n)$ is $O(n^{\log_b a})$
  - Note that $\log_b a > k$ in this case
  - Cost is dominated by total work at lowest level of recursion



Ideally: $T(n) = 2T\left(\frac{n}{2}\right) + n$

$a = 2, b = 2, k = 1$ so $a = b^k$: Solution: $O(n \log n)$

Worst Case: $T(n) = T(n-1) + n$

The master theorem does not apply, but the solution is $O(n^2)$

Expected: $O(n \log n)$ … Our first task today is to show this!

Expected Runtime for QuickSort: “Global analysis”

Runtime is the # of comparisons

Recurrence & Master Theorem kind of analysis won’t work ...

Instead, use a “global” analysis:

- Number elements $a_1, a_2, \ldots, a_n$ based on **final** sorted order
- Let $p_{i,j}$ = Probability that QuickSort compares a_i and a_j

Expected number of comparisons:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_{i,j}$$

Observation – We only compare to pivot

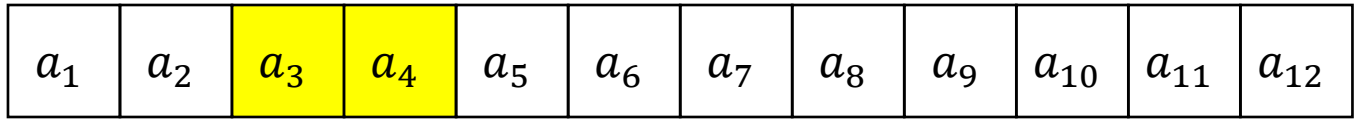
1. Put p at beginning of list
2. Put a pointer (Begin) just after p, and a pointer (End) at the end of the list
3. While Begin < End:
    1. If Begin value $< p$, move Begin right
    2. Else swap Begin value with End value, move End Left
4. If pointers meet at element $< p$: Swap p with pointer position
5. Else If pointers meet at element $> p$: Swap p with value to the left

Conclusion: we only compare a_i with a_j when both of these are true:

- a_i and a_j are in the same subproblem (no previous pivot fell between them)
- One of a_i and a_j is the pivot

Finding $p_{i,j}$ - Adjacent Values

We always compare consecutive elements



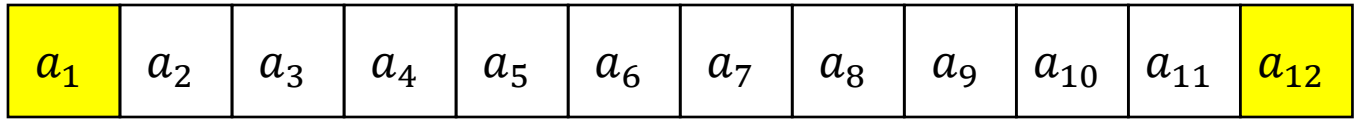
$p_{i,i+1} = 1$

Why?

- Intuitively, we MUST compare adjacent items to guarantee we get them in the right order
- Precisely, a_i and a_{i+1} can't be on opposite sides of any third value, so they'll only end up in separate subproblems when one was the pivot

Finding $p_{i,j}$ - Extreme Values

We rarely compare the min and the max



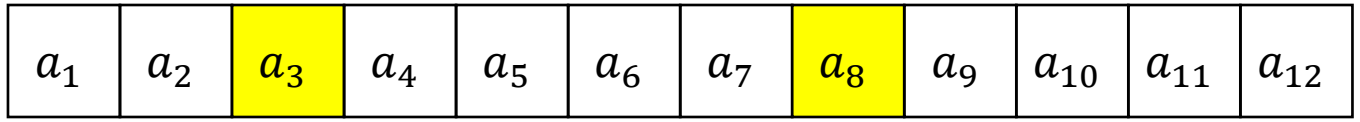
$$p_{1,n} = \frac{2}{n}$$

Why?

– The only way we will compare the smallest and largest items is if one or the other was the very first pivot chosen ($\frac{1}{n}$ chance of each)

Finding $p_{i,j}$ - In General

For any pair of elements, the probability we compare them in proportional to their distance



$$p_{i,j} = \frac{2}{j-i+1} \quad \text{if } i < j$$

Why?

- a_i and a_j will only be compared if one of them was the very first pivot chosen from among the range $a_i, a_{i+1}, \ldots, a_j$
- There are $j - i + 1$ items in this range, two of which result in this comparison

Expected Runtime for QuickSort: "Global analysis"

For $i < j$ we have $p_{i,j} = \frac{2}{j-i+1}$.

Harmonic series sum:

$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

Fact: $H_n = \ln n + O(1)$

Expected number of comparisons:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_{i,j} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i+1} \frac{2}{k+1} \qquad \text{for } k = j - i$$

$$< 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k}$$

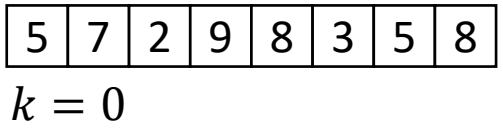
$$< 2\, n\, H_n$$

$$= 2\, n \ln n + O(n) \leq 1.387\, n \log_2 n$$

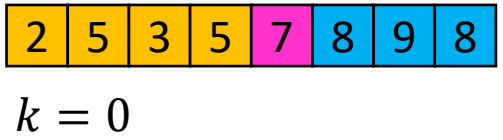
QuickSelect

- Finds k^{th} order statistic
  - k^{th} smallest element in the list
  - 1^{st} order statistic: minimum
  - n^{th} order statistic: maximum
  - $\left\lfloor \frac{n}{2} \right\rfloor^{\text{th}}$ order statistic: median (for odd n)

QuickSelect(S, k)



- **Base Case**:
  - If $k = 0$ then return $S[0]$



- **Divide**:
  - Select an element to use as a "pivot"
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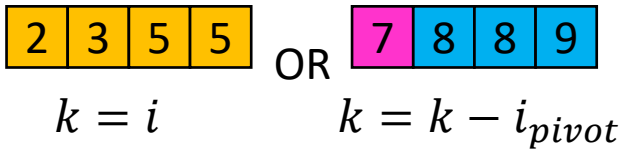

- **Conquer**:
  - Let i_{pivot} be the index of the pivot after partitioning
  - If $k < i_{pivot}$ then call QuickSelect(S_{left}, k)
  - Otherwise call Quickselect(S_{right},$k - i_{pivot}$)
- **Combine**:
  - Nothing!

QuickSelect Running Time

Master Theorem: Suppose that $\boldsymbol{T(n) = a \cdot T(n/b) + O(n^k)}$ for $\boldsymbol{n > b}$.

- If $\boldsymbol{a < b^k}$ then $\boldsymbol{T(n)}$ is $\boldsymbol{O(n^k)}$
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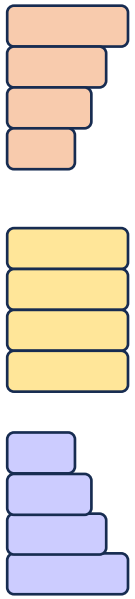
- If $\boldsymbol{a = b^k}$ then $\boldsymbol{T(n)}$ is $\boldsymbol{O(n^k \log n)}$
  - Total cost is the same for all $\boldsymbol{\log_b n}$ levels of recursion
- If $\boldsymbol{a > b^k}$ then $\boldsymbol{T(n)}$ is $\boldsymbol{O(n^{\log_b a})}$
  - Note that $\boldsymbol{\log_b a > k}$ in this case
  - Cost is dominated by total work at lowest level of recursion



Ideally: $T(n) = T\left(\frac{n}{2}\right) + n$

$\boldsymbol{a = 1, b = 2, k = 1}$ so $\boldsymbol{a < b^k}$: Solution: $\boldsymbol{O(n)}$

Worst Case: $T(n) = T(n-1) + n$

The master theorem does not apply, but the solution is $\boldsymbol{O(n^2)}$

We don't need the "ideal" for $O(n)$!

Master Theorem: Suppose that $T(n) = a \cdot T(n/b) + O(n^k)$ for $n > b$.

- If $a < b^k$ then $T(n)$ is $O(n^k)$
  - Cost is dominated by work at top level of recursion
- If $a = b^k$ then $T(n)$ is $O(n^k \log n)$
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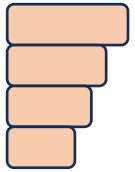

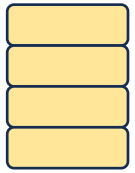

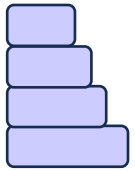

$$T(n) = T\left(\frac{3n}{4}\right) + n$$

$a = 1, b = 4/3, k = 1$ so $a < b^k$: Solution: $O(n)$

QuickSelect: Random Choice of Pivot

Consider a call to QuickSelect. We will say the pivot is "good enough" if it is among the middle half of the value

Elements of S listed in sorted order

Say p is "good enough" iff it is in the middle half

With probability $\geq 1/2$ pivot p is good enough

- For any good enough pivot the recursive call has subproblem size $\leq 3n/4$
- After 2 calls, QuickSelect has expected problem size $\leq 3n/4$

So $T(n) \leq 2T'(n)$ where

$T'(n) = T'\left(\frac{3n}{4}\right) + n$ for $b = 4/3 > 1$

$\Rightarrow$ Expected $O(n)$ time

Doing Quickselect in $O(n)$ Worst Case

- We can make adapt Quickselect by running in $O(n)$ worst case by applying some tricky extra recursion!
- Median-of-Medians:
  1. Break S into chunks of size 5
  2. Sort each chunk by its median value (i.e. value at index 2)
  3. Use Quickselect to find the median of these medians, use that as the pivot

M-o-M, Step 1: Construct sets of size 5; Step 2: sort each set

Input: 13, 15, 32, 14, 95, 5, 16, 45, 86, 65, 62, 41, 81, 52, 32, 32, 12, 73, 25, 81, 47, 8, 69, 9, 7, 81, 18, 25, 42, 91, 64, 98, 96, 91, 6, 51, 21, 12, 36, 11, 11, 9, 5, 17, 77

Group:

13	5	62	32	47	81	64	51	11
15	16	41	12	8	18	98	21	9
32	45	81	73	69	25	96	12	5
14	86	52	25	9	42	91	36	17
95	65	32	81	7	91	6	11	77

Sort each group:

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5

$O(\boldsymbol{n})$

M-o-M, Step 3: Find median of column medians

Column medians:

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5

$T(n/5)$

Choose pivot to be that median of medians

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5

This Pivot is “Good Enough”!

Column medians:

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5

$T(n/5)$

Imagining rearranging columns by columns’ medians

95	51	77	69	81	91	98	86	81
32	36	17	47	73	81	96	65	62
15	21	11	9	32	42	91	45	52
14	12	9	8	25	25	64	16	41
13	11	5	7	12	18	6	5	32

This Pivot is "Good Enough"!

Column medians:

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5

$T(n/5)$

Choose pivot to be that median of medians

All $\leq$ pivot

Not in S_{right}

Size $\geq n/4$

95	51	77	69	81	91	98	86	81
32	36	17	47	73	81	96	65	62
15	21	11	9	32	42	91	45	52
14	12	9	8	25	25	64	16	41
13	11	5	7	12	18	6	5	32

Size of S_{right} is $\leq \frac{3n}{4}$

This Pivot is "Good Enough"!

Column medians:

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
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$T(n/5)$

Choose pivot to be that median of medians

95	51	77	69	81	91	98	86	81
32	36	17	47	73	81	96	65	62
15	21	11	9	32	42	91	45	52
14	12	9	8	25	25	64	16	41
13	11	5	7	12	18	6	5	32

All $\geq$ pivot

Not in S_{left}

Size $\geq n/4$

Size of S_{left} is $\leq \frac{3n}{4}$

QuickSelect With Median-of-Medians

5	7	2	9	8	3	5	8

$i = 0$

2	5	3	5	7	8	9	8

$i = 0$

2	3	5	5

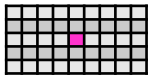
$i = i$

OR

7	8	8	9

$i = i - i_{pivot}$

- **Base Case**:
  - If $i = 0$ then return $S[0]$
- **Divide**:
  - Use median-of-medians to select an element to use as a "pivot"



    - Break S into $\frac{n}{5}$ chunks of size 5 each
    - Sort each chunk
    - Make a new list M containing all chunks' medians
    - Use QuickSelect(M, $\frac{n}{10}$) as the pivot
  - Partition
- **Conquer**:
  - Let i_{pivot} be the index of the pivot after partitioning
  - If $i < i_{pivot}$ then call QuickSelect(S_{left}, i)
  - Otherwise call Quickselect(S_{right}, $i - i_{pivot}$)
- **Combine**:
  - Nothing!

$$T\left(\frac{n}{5}\right) + n$$

$$n$$

$$T\left(\frac{3n}{4}\right)$$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{3n}{4}\right) + n$$

Solving the QS with MoM Recurrence

Master Theorem: Suppose that $T(n) = a \cdot T(n/b) + O(n^k)$ for $n > b$.

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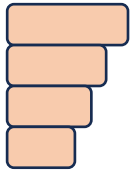
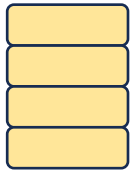
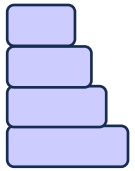
  - Note that $\log_b a > k$ in this case
  - Cost is dominated by total work at lowest level of recursion







$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{3n}{4}\right) + n$$

$$T(n) \leq T\left(\frac{19n}{20}\right) + n$$

$a = 1, b = 20/19, k = 1$ so $a < b^k$: Solution: $O(n)$