Homework 7: Network Flow and Linear Programming

Be sure to read the grading guidelines and style guidelines. Especially to see the suggested format for describing algorithms.

We sometimes describe how long are justifications or proofs are. These lengths are intended to help you estimate how much detail we're expecting, you should not take those estimates as hard length-limitations.

Our solutions for any individual problem will fit in approximately one page or less.

You are allowed (and encouraged!!) to collaborate with each other. Brainstorming is much easier to do in a group than alone! But you must follow the collaboration policy (which includes needing to write your submission on your own).

You will submit to Gradescope; we will have a different box for each problem, so please give yourself extra time to submit.

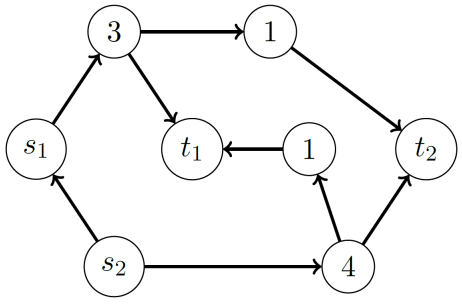
1. Dualling LPs [10 points]

Write a linear program in standard form for the following optimization problem and then write its LP dual: Maximize $4x_1 - 3x_2 + 6x_3$ such that $4x_1 + x_2 - 4x_3 = 5$, $2x_2 - 4x_3 \geq 2$, and $x_3 + 3x_1 + 4x_2 \leq 9$, and none of the x_i are negative.

2. k-Flow [25 points]

Suppose that you are given an directed graph G with k vertices $s_1, \ldots, s_k$ as source vertices and another k vertices $t_1, \ldots, t_k$ as destination vertices for some $k \geq 1$ and that no source vertex is directly connected to a sink vertex. Suppose further that each non-source, non-destination vertex v of G has an integer capacity $c_v \geq 0$ which shows the maximum amount of flow that can go through v. The source and destination vertices have no capacity limit so each vertex either has an s_i label, a t_i label or a capacity label.

Design an algorithm that runs in polynomial time that determines the maximum total flow that can be sent from all sources to all sinks combined satisfying all capacity constraints of the vertices. (The amount of flow sent from s_i does *not* have to be the same as the amount received at t_i; all we care about is the total flow.) For example, in the following picture the maximum amount of flow that can be sent from s_1, s_2 to t_1, t_2 is 7.



3. We Demand Payment! [25 points]

Write a linear program to solve the following optimization problem, which is somewhat like Circulation with Demands but with more constraints and an optimization goal. Like Circulation with Demands, the input is a directed graph $G = (V, E)$ where each edge e has a capacity $c(e) \geq 0$, each vertex v has a demand $d(v)$, and we have $\sum_v d(v) = 0$. However, this graph also has a capacity $c(v) \geq 0$ for each vertex v on the amount of flow that goes through v without being part of the supply/demand at v. In addition, each edge e has a toll payment $p(e) \geq 0$ indicating the cost of sending a unit of flow through that edge, and each vertex v has a toll payment $p(v) \geq 0$ per unit of flow that is not involved in the supply/demand at v.

The objective is to find a circulation $f : E \to \mathbb{R}_{\geq 0}$ in this network (if one exists) that meets all the demands, doesn't exceed any capacity, and incurs a total of all toll payments required that is as small as possible.

Clearly explain what every variable of your LP represents and what every constraint of your LP represents.

4. Allocating Cell Bandwidth [25 points]

A service provider has deployed a wireless network in a city via of a number of base station towers at fixed locations in the city. The stations were installed at different times so they have a variety of power/reach of their signal and can each support a variety numbers of clients. Each client they wish to serve is also assumed to be served via an antenna at a fixed and known location. The goal of the provider is to serve as many of their clients as possible by allocating their clients to talk with with specific based stations.

Suppose that the locations of each of the B base station towers and each of the n client antennas are at known coordinates in the xy-plane. Suppose that base stations 1 through B have signal radius $r_1, \ldots, r_B$, respectively (that is, base station b can communicate with any client within Euclidean distance r_b of its location) and, respectively, can handle a load of up to $L_1, \ldots, L_B$ clients at the same time,

Give a polynomial-time algorithm that will figure out the maximum number of clients that the service provider can communicate with at a time, and an allocation of clients to base station towers that will achieve this maximum.

5. Even More Efficient Matchmaking [Extra credit]

We saw how repeatedly augmenting along shortest paths (given by a BFS each time) produced a better runtime analysis for general network flow. In this problem you will do something similar in the special case of finding a maximum matching in a bipartite graph $G = (V, E)$ where the two sides of V are X and Y, $|V| = n$ and $|E| = m$. The idea here will be to augment along many shortest paths at the same time. In order for this to work, the paths better not share any edges. Your algorithm will add the slightly stronger condition that the paths do not share any vertices and will include as many shortest paths as possible at each step.

More precisely, at each round the new algorithm will simultaneously augment along all paths in some *maximal set* P_{short} of vertex-disjoint shortest augmenting paths in the flow graph associated with the bipartite matching problem. (In other words, if $d = d_f(s, t)$ is the length of the shortest augmenting path in the residual graph G_f then every path in P_{short} has length d, no two paths in P_{short} share any vertices other than s or t and any other st-path of length d in G_f shares at least one vertex other than s and t with some path in P_{short}.)

(a) Show how to find a set P_{short} and do all the augmentations on its paths to get a new flow f' in time $O(m+n)$.

(b) Prove that $d_{f'}(s, t) \geq d_f(s, t) + 2$.

(c) Given two matchings M' and M on graph G we can define the symmetric difference, $M' \oplus M$, of M' and M to be the graph consisting of edges that occur in exactly one of the two matchings. $M' \oplus M$ consists of a collection of vertex-disjoint paths and cycles (why?).
Show that if M' is a maximum matching and flow f corresponds to a matching of M then

i. $M' \oplus M$ contains exactly $|M'| - |M|$ vertex-disjoint odd-length paths,

ii. Any path of odd length k in $M' \oplus M$ corresponds to an augmenting path of length $k + 2$ in G_f.

iii. Use these two properties to prove that once all augmenting paths in G_f are of length at least $k + 2$ then at most n/k additional augmentations will produce a maximum flow (and hence maximum matching).

(d) Use the above analysis with a suitable value of k to show that this algorithm computes a maximum matching in $O(m\sqrt{n})$ time