CSE 421 Spring 2025: Set 7

Instructor: Chinmay Nirkhe

Due date: May 28th, 2025 11:59pm

Instructions: Solutions should be legibly handwritten or typeset (ideally in LaTeX). Mathematically rigorous solutions are expected for all problems unless explicitly stated.

You are encouraged to collaborate on problems in small teams but everyone must individually submit solutions. Solutions for the problems may be found online or in textbooks – but do not use them.

For grading purposes, list, with each problem, the names of your collaborators. Please start each problem on a new page.

Problem 1. Linear programs are more versatile than might first meet the eye. For each of the following computations, express it as a maximation LP in standard form or minimization LP in standard dual form:

$$\begin{cases} \max c^\top x \\ \text{s.t. } Ax \le b, x \ge 0 \end{cases} \qquad \begin{cases} \min b^\top y \\ \text{s.t. } A^\top y \ge c, y \ge 0. \end{cases} \tag{1}$$

Express each of the following terms. You do not need to give a proof of correctness.

1. **[4 points]** given a constant $a \in \mathbb{R}$, a LP that evaluates to $a^{(+)}$. $a^{(+)}$ is defined as the positive component of a; if $a > 0$, $a^{(+)} = a$ and if $a \le 0$, $a^{(+)} = 0$.

2. **[4 points]** Given a constant $a \in \mathbb{R}$, a LP that evaluates to $a^{(-)}$. $a^{(-)}$ is defined as the negative component of a; if $a < 0$, $a^{(-)} = -a$ and if $a \ge 0$, $a^{(-)} = 0$.

3. **[4 points]** given constants $a, b \in \mathbb{R}$, a LP that evaluates to $|a - b|$.

4. **[4 points]** given constants $a_1, \dots, a_k \in \mathbb{R}$, a LP that evaluates to $\max(a_1, a_2, \dots, a_k)$.

Problem 2. [5 points] Express the dual of the following linear program:

$$\begin{cases} \max & a - b + c \\ & 5a + 2b \leq 3 \\ \text{s.t.} & c - a \leq -2 \\ & b + c \leq 0 \\ & a, b, c \geq 0 \end{cases} \tag{2}$$

No need to prove correctness; just show your work.

Problem 3. [10 points] Given a graph $G = (V, E)$, edge weights $w : E \to \mathbb{R}$, and vertices $s, t \in V$, write down a maximization LP that evaluates to the length of the shortest path $s \rightsquigarrow t$.

Hint: A maximization intuition was given in lecture but you still need to prove why that statement is correct. To prove correctness, we suggest you write down your LP and show that [length of shortest path] $\leq$ [value of LP] and [value of LP] $\geq$ [length of shortest path] in two separate arguments.

Problem 4. [10 points] Given a set of points (x_i, y_i) for $i = 1, \ldots, n$, write down a standard LP to to find the best line $y = ax + b$ that approximately passes through these points. The line should minimize the maximum absolute value error:

$$\max_{i \in \{1,\ldots,n\}} |y_i - ax_i - b|. \tag{3}$$

Argue why your LP is correct; you may want to use answers from previous sections.

Problem 5 (Fractional Knapsack)**.** The fractional Knapsack problem is a variant of Knapsack where fractions of items are allowed to be selected: On input $W, w_1, \dots, w_n, v_1, \dots, v_n$, the task is to calculate fractions $x_i \in [0, 1]$ such that if fraction x_i of item i is selected to include in the Knapsack, the total weight is still at most W and the total value of the Knapsack is maximized.

1. **[6 points]** Write an LP in standard form which calculates the optimal value of a fractional Knapsack instance. Argue briefly that why this LP is correct.
2. **[3 points]** Show that value of a fractional Knapsack is always $\geq$ the value of standard Knapsack with the same item weights and values and the same weight limit W.
3. **[5 points]** Give a greedy algorithm which calculates the optimal value of fractional Knapsack. Prove correctness; runtime is not required.