CSE 421, Introduction to Algorithms

Lecture 15, Winter 2024

Dynamic Programming

Longest Common Subsequence

Announcements

- Dynamic Programming Reading:
  - Weighted Interval Scheduling, Segmented Least Squares, Knapsack and Subset Sum
  - 6.6 String Alignment
  - 6.8 Shortest Paths (Bellman-Ford)
- Midterm, Friday, Feb 9
  - Material through 6.3 and HW 5
  - Feb 8 Section will be Midterm review
    - Old exam problems on course homepage
  - Homework 6 due Feb 14

Longest Common Subsequence

- $C=c_1\ldots c_g$ is a subsequence of $A=a_1\ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

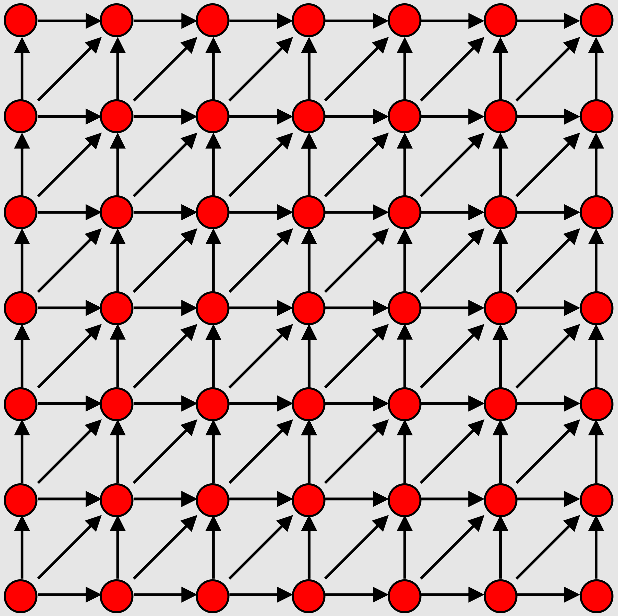
LCS(BARTHOLEMEWSIMPSON, KRUSTYTHECLOWN)
= RTHOWN

Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If $a_j \neq b_k$, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

Dynamic Programming Computation

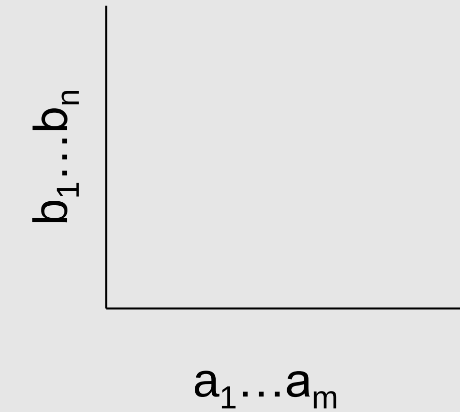


Code to compute Opt[n, m]

```
for (int i = 0; i < n; i++)
  for (int j = 0; j < m; j++)
    if (A[ i ] == B[ j ] )
      Opt[ i,j ] = Opt[ i-1, j-1 ] + 1;
    else if (Opt[ i-1, j ] >= Opt[ i, j-1 ])
       Opt[ i, j ] := Opt[ i-1, j ];
    else
       Opt[ i, j ] := Opt[ i, j-1];
```

Storing the path information

```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else    {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```



Reconstructing Path from Distances

```
LCS Arguments
 21103132110203321212000032130 2
 100222010121310130323233121011
_0 _0 _0 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1 _1
_1 _1 _1 _1 _1 _1 _1 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2 _2
_1 _1 _1 _1 _1 _1 _1 _2 _2 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3 _3
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _3 _3 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4 _4
_1 _2 _2 _2 _2 _2 _2 _2 _3 _3 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _4 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6
_1 _2 _2 _2 _2 _2 _2 _3 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6 _6
_1 _2 _2 _3 _3 _3 _3 _3 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7
_1 _2 _2 _3 _3 _3 _3 _4 _4 _4 _5 _6 _6 _6 _6 _6 _6 _6 _6 _7 _7 _7 _7 _7 _8 _8 _8 _8 _8 _8
_1 _2 _2 _3 _3 _3 _3 _4 _4 _5 _5 _6 _6 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _7 _8 _8 _9 _9 _9 _9
_1 _2 _3 _3 _3 _3 _4 _4 _5 _5 _5 _6 _6 _7 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _8 _9 10 10 10
_1 _2 _3 _4 _4 _4 _4 _4 _5 _5 _6 _6 _6 _7 _8 _8 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _6 _7 _8 _8 _8 _9 _9 _9 _9 _9 _9 _9 _9 _9 _9 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 10 10 10 10 10
_1 _2 _3 _4 _4 _4 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 10 11 11 11 11 11 11 11 11 11 11
_1 _2 _3 _4 _5 _5 _5 _5 _5 _5 _6 _6 _7 _7 _8 _8 _9 _9 10 11 11 12 12 12 12 12 12 12 12 12
_1 _2 _3 _4 _5 _5 _5 _6 _6 _6 _6 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 13 13 13 13 13
_1 _2 _3 _4 _5 _6 _6 _6 _6 _6 _7 _7 _7 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 14 14 14 14
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _7 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _6 _7 _7 _7 _8 _8 _8 _8 _8 _9 _9 _9 10 11 11 12 12 12 13 14 15 15 15 15
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _8 _8 _9 _9 _9 10 10 11 11 12 12 12 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _8 _8 _9 _9 _9 _9 10 10 11 11 12 12 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _7 _8 _8 _9 _9 _9 _9 _9 _9 10 10 11 12 12 13 13 13 13 14 15 16 16 16
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 10 10 10 10 10 10 11 12 12 13 13 13 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _8 _9 _9 10 11 11 11 11 11 11 11 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 _9 10 11 11 12 12 12 12 12 12 13 13 14 14 14 14 15 16 17 17
_1 _2 _3 _4 _5 _6 _7 _8 _9 _9 10 10 11 11 12 12 12 12 12 13 13 14 14 14 14 15 15 16 17 17
```

How good is this algorithm?

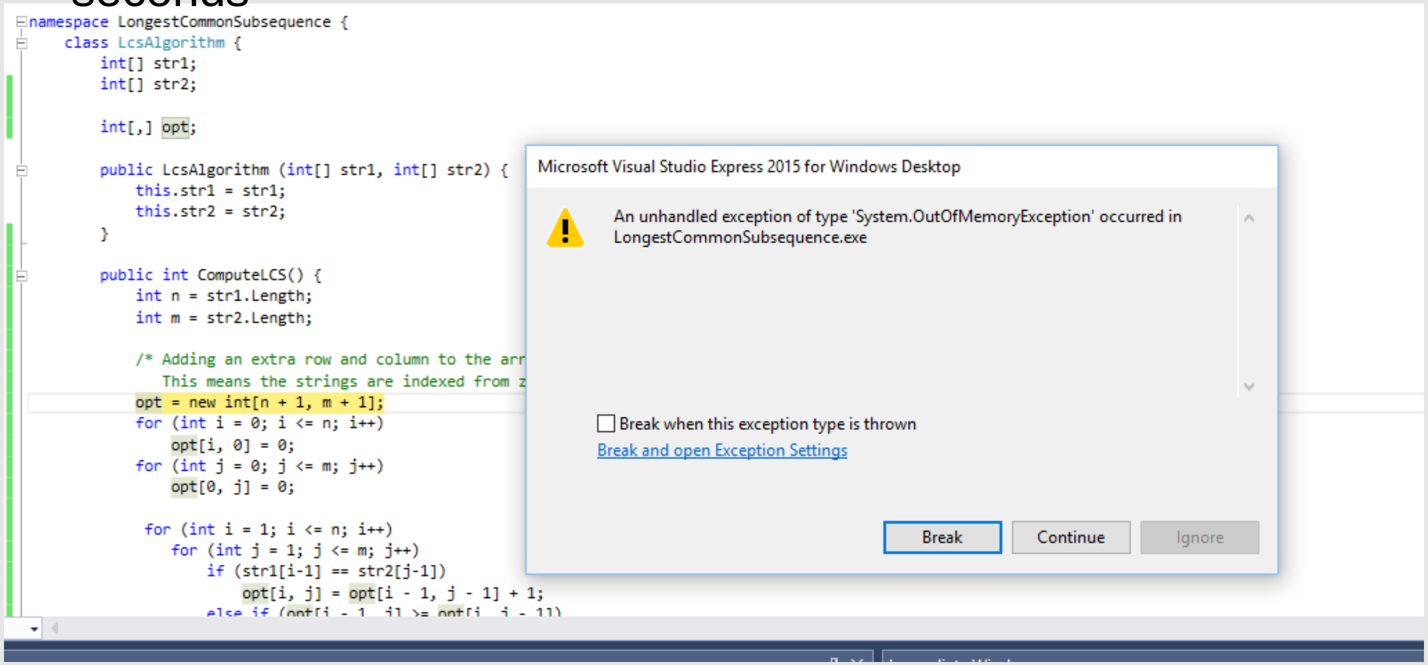
- Is it feasible to compute the LCS of two strings of length 300,000 on a standard desktop PC? Why or why not.

Implementation 1

```
public int ComputeLCS() {
           int n = str1.Length;
           int m = str2.Length;

           int[,] opt = new int[n + 1, m + 1];
           for (int i = 0; i <= n; i++)
               opt[i, 0] = 0;
           for (int j = 0; j <= m; j++)
               opt[0, j] = 0;

            for (int i = 1; i <= n; i++)
               for (int j = 1; j <= m; j++)
                   if (str1[i-1] == str2[j-1])
                       opt[i, j] = opt[i - 1, j - 1] + 1;
                   else if (opt[i - 1, j] >= opt[i, j - 1])
                       opt[i, j] = opt[i - 1, j];
                   else
                       opt[i, j] = opt[i, j - 1];

           return opt[n,m];
       }
```

N = 17000

Runtime should be about 5 seconds*



* Personal PC, 10 years old

Manufacturer:	Dell
Model:	Optiplex 990
Processor:	Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz 3.10 GHz
Installed memory (RAM):	8.00 GB (7.88 GB usable)
System type:	64-bit Operating System, x64-based processor

Implementation 2

```
public int SpaceEfficientLCS() {
          int n = str1.Length;
          int m = str2.Length;
          int[] prevRow = new int[m + 1];
          int[] currRow = new int[m + 1];

          for (int j = 0; j <= m; j++)
              prevRow[j] = 0;

          for (int i = 1; i <= n; i++) {
              currRow[0] = 0;
              for (int j = 1; j <= m; j++) {
                  if (str1[i - 1] == str2[j - 1])
                      currRow[j] = prevRow[j - 1] + 1;
                  else if (prevRow[j] >= currRow[j - 1])
                      currRow[j] = prevRow[j];
                  else
                      currRow[j] = currRow[j - 1];
              }
              for (int j = 1; j <= m; j++)
                  prevRow[j] = currRow[j];
          }

          return currRow[m];
      }
```

N = 300000

N: 10000 Base 2 Length: 8096	Gamma: 0.8096	Runtime:00:00:01.86
N: 20000 Base 2 Length: 16231	Gamma: 0.81155	Runtime:00:00:07.45
N: 30000 Base 2 Length: 24317	Gamma: 0.8105667	Runtime:00:00:16.82
N: 40000 Base 2 Length: 32510	Gamma: 0.81275	Runtime:00:00:29.84
N: 50000 Base 2 Length: 40563	Gamma: 0.81126	Runtime:00:00:46.78
N: 60000 Base 2 Length: 48700	Gamma: 0.8116667	Runtime:00:01:08.06
N: 70000 Base 2 Length: 56824	Gamma: 0.8117715	Runtime:00:01:33.36

N: 300000 Base 2 Length: 243605 Gamma: 0.8120167
Runtime:00:28:07.32

Chvatal-Sankoff Constant: γ_k

Observations about the Algorithm

- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values

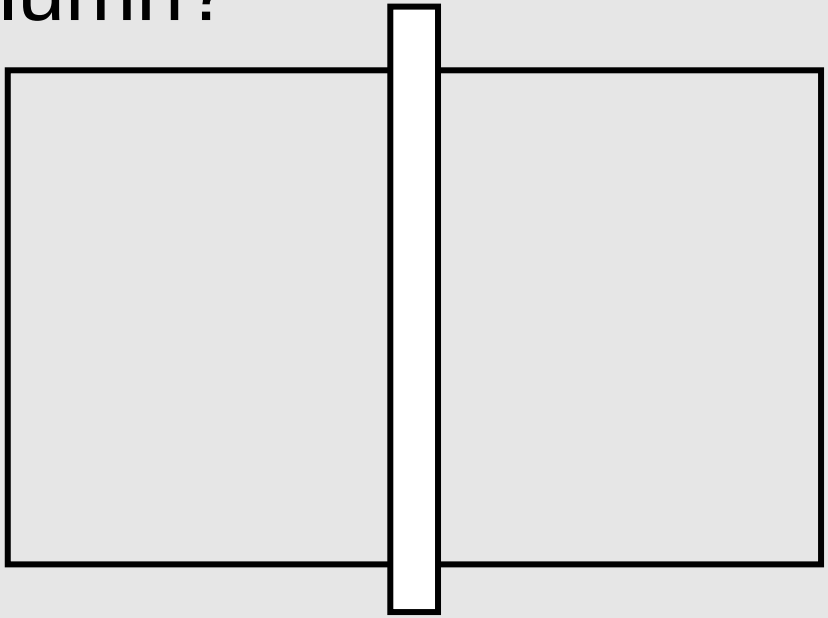
- The algorithm can be run from either end of the strings

Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$

- $LCS_{4,3}$(abbacbb, cbbaa)

$A = RRSSRTTRTS$

$B = RTSRRRSTST$

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

A = RRSSRTTRTS

B=RTSRRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

j	left	right
0	0	4
1	1	4
2	1	3
3	2	3
4	3	3
5	3	2
6	3	2
7	3	1
8	4	1
9	4	0

Computing the middle column

- From the left, compute LCS($a_1 \ldots a_{m/2}, b_1 \ldots b_j$)
- From the right, compute LCS($a_{m/2+1} \ldots a_m, b_{j+1} \ldots b_n$)
- Add values for corresponding j's

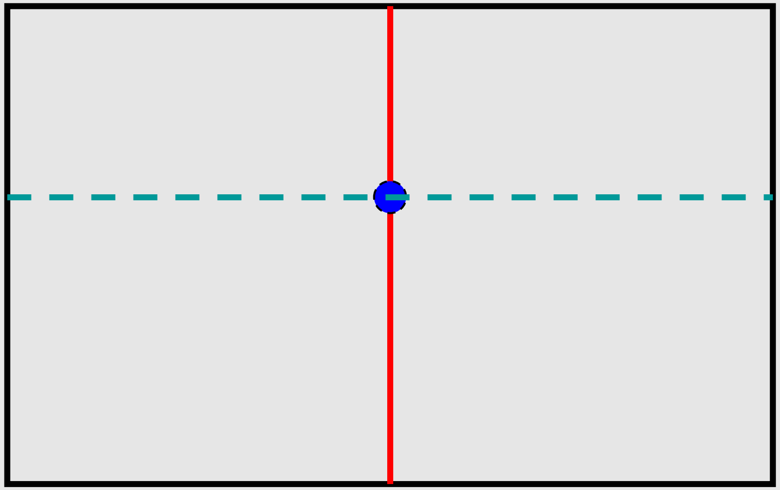
- Note – this is space efficient

Divide and Conquer

- $A = a_1,\ldots,a_m$ $B = b_1,\ldots,b_n$
- Find j such that
  - $LCS(a_1\ldots a_{m/2}, b_1\ldots b_j)$ and
  - $LCS(a_{m/2+1}\ldots a_m, b_{j+1}\ldots b_n)$ yield optimal solution
- Recurse

Algorithm Analysis

- $T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm$



Prove by induction that $T(m,n) <= 2cmn$

$T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm$

Memory Efficient LCS Summary

- We can afford O(nm) time, but we can’t afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes

String Alignment Problem

- Align sequences with gaps

```
CAT TGA  AT

CAGAT AGGA
```

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Note: the problem is often expressed as a minimization problem, with $\gamma_{xx} = 0$ and $\delta_x > 0$

Give the Optimization Recurrence for the String Alignment Problem

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Opt[j, k] =

Let $a_j = x$ and $b_k = y$
Express as minimization

String edit with Typo Distance

- Find closest dictionary word to typed word
- Dist(‘a’, ‘s’) = 1
- Dist(‘a’, ‘u’) = 6
- Capture the likelihood of mistyping characters