Lecture07

CSE 421
Introduction to Algorithms

Winter 2024
Dijkstra's algorithm

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Readings

- Topics
  - Dijkstra's Algorithm (Section 4.4)
  - Monday: Minimum Spanning Trees
  - Wednesday: Divide and Conquer
- Reading
  - 4.4, 4.5, 4.7, 4.8

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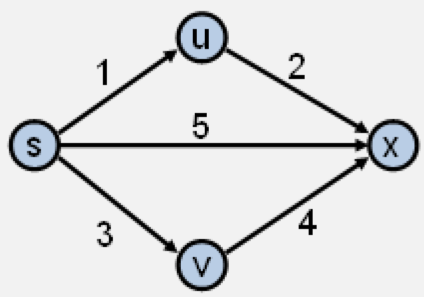
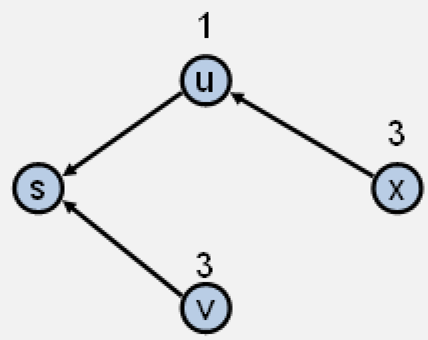
Single Source Shortest Path Problem

- Given a graph and a start vertex s
  - Determine distance of every vertex from s
  - Identify shortest paths to each vertex
    - Express concisely as a “shortest paths tree”
    - Each vertex has a pointer to a predecessor on shortest path





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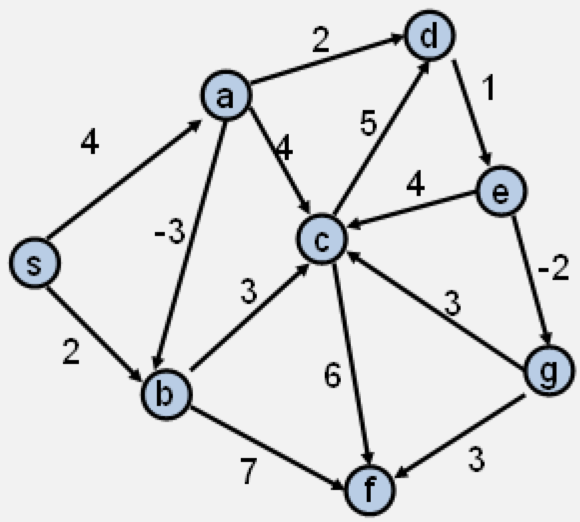
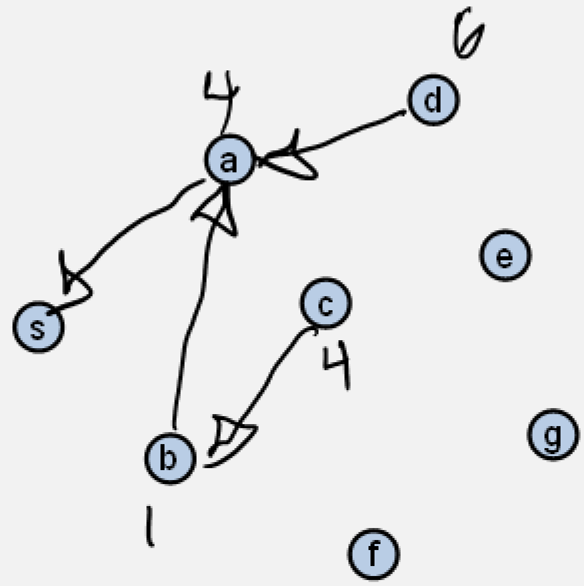
Construct Shortest Path Tree from s





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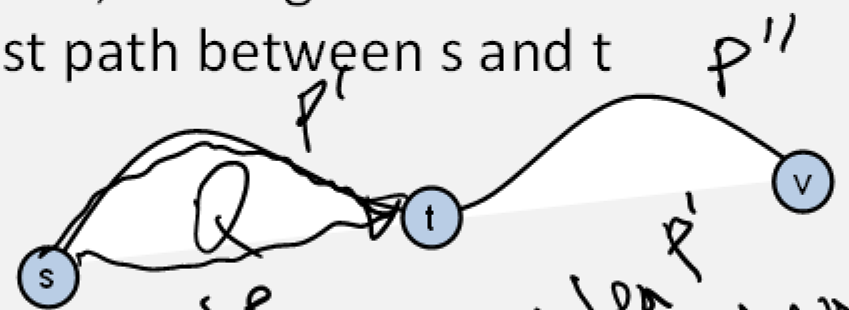
Warmup

$P = P' P''$

Claim P' is a SP s to t

- If P is a shortest path from s to v, and if t is on the path P, the segment from s to t is a shortest path between s and t



- WHY?

If P' is not a SP let Q be a SP

len Q < len P'

len QP'' < len P

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Assume all edges have non-negative cost

Dijkstra's Algorithm

S = { }; d[s] = 0; d[v] = infinity for v != s

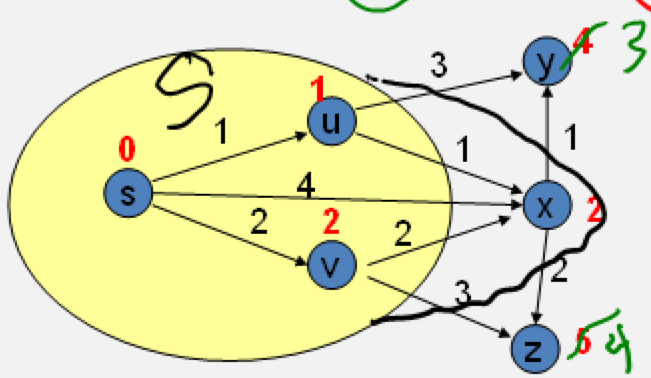
While S != V

  Choose v in V-S with minimum d[v]

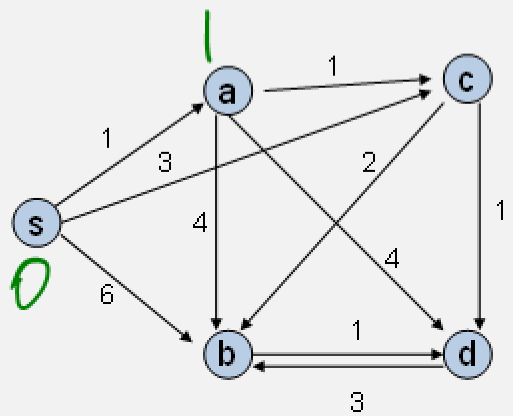
  Add v to S

  For each w in the neighborhood of v

    d[w] = min(d[w], d[v] + c(v, w))



Simulate Dijkstra's algorithm (starting from s) on the graph



Round	Vertex Added	s	a	b	c	d
0		0	∞	∞	∞	∞
1	s	0	1	6	3	∞
2	a	0	1	5	2	5
3						
4						
5						

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Who was Dijkstra?

- What were his major contributions?

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http://www.cs.utexas.edu/users/EWD/

- Edsger Wybe Dijkstra was one of the most influential members of computing science's founding generation. Among the domains in which his scientific contributions are fundamental are
  - algorithm design
  - programming languages
  - program design
  - operating systems
  - distributed processing
  - formal specification and verification
  - design of mathematical arguments

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Dijkstra's Algorithm as a greedy algorithm

- Elements committed to the solution by order of minimum distance

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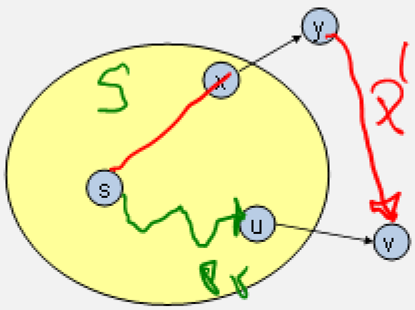
Correctness Proof

Base – start vertex s $d[v] = dist(s,v)$ for $v \in S$

- Elements in S have the correct label
- Key to proof: when v is added to S, it has the correct distance label. $dist(s,v) = d[v]$



suppose P goes s to v

$$\text{len } P = dist(s,x) + c_{xy} + \text{len } P'$$
$$\geq d[y] + \text{len } P'$$
$$\geq d[y] \geq d[v]$$

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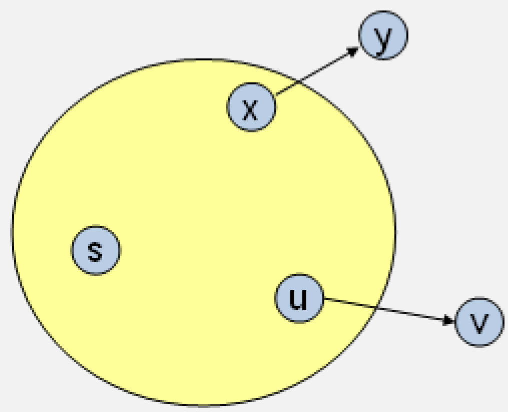
Proof

- Let v be a vertex in V-S with minimum d[v]
- Let P_v be a path of length d[v], with an edge (u,v)
- Let P be some other path to v. Suppose P first leaves S on the edge (x, y)
  - $P = P_{sx} + c(x,y) + P_{yv}$
  - $Len(P_{sx}) + c(x,y) >= d[y]$
  - $Len(P_{yv}) >= 0$
  - Len(P) >= d[y] + 0 >= d[v]



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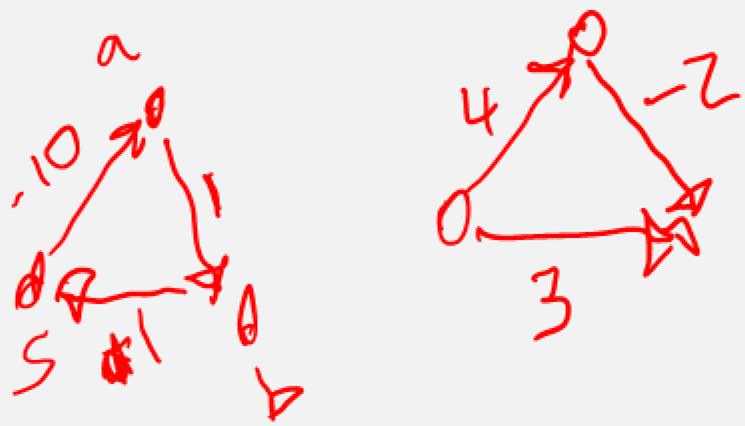
Negative Cost Edges

- Draw a small example a negative cost edge and show that Dijkstra's algorithm fails on this example



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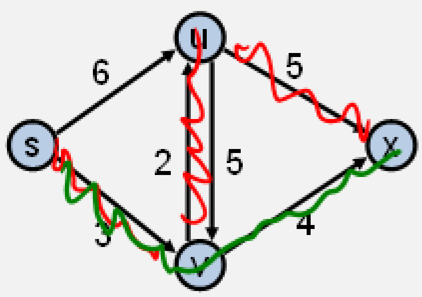
Bottleneck Shortest Path

- Define the bottleneck distance for a path to be the maximum cost edge along the path



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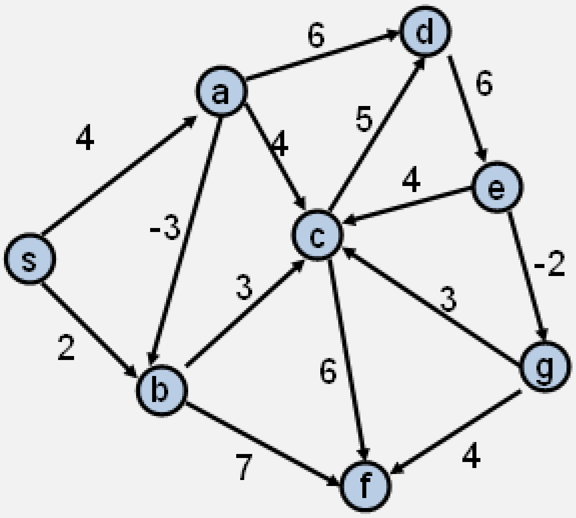
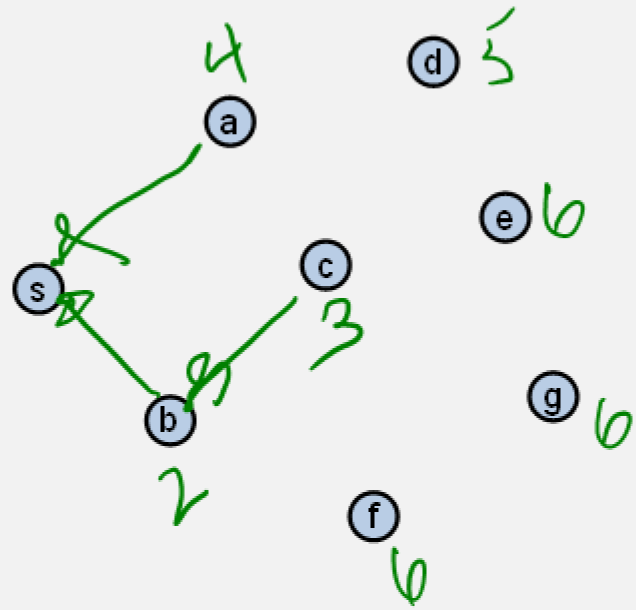
Compute the bottleneck shortest paths





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How do you adapt Dijkstra's algorithm to handle bottleneck distances

- Does the correctness proof still apply?

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