Midterm Exam, Wednesday, October 30, 2019

NAME: _________________________

Instructions:

• Closed book, closed notes, no calculators
• Time limit: 50 minutes
• Answer the problems on the exam paper.
• If you need extra space use the back of a page
• Problems are not of equal difficulty, if you get stuck on a problem, move on.

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Problem 1 Graph Theory (15 points):

a) True or false: Let $G = (V, E)$ be an undirected graph. If $G$ is a tree, then $G$ is bipartite. Justify your answer.

b) True or false: Let $G = (V, E)$ be an undirected graph with $n$ vertices and $m$ edges. If $m > \frac{n(n-1)}{2} - (n - 1)$ then the graph $G$ must be connected. (By definition, an undirected graph has no parallel edges and no self loops.) Justify your answer.

c) True or false: Let $G = (V, E)$ be a directed graph with $n$ vertices. If every vertex of $G$ has out degree at least one, then $G$ has a cycle. Justify your answer.
**Problem 2 Space Aliens (10 points):**

There is a new alien language which uses the Latin alphabet. However, the order among letters is unknown to you. You receive a list of sequences of letters in alphabetical order and you want to construct a total order consistent with the sequences.

For example, given the following sequences: (a,d,e), (a, c, b), (d, g, e), (d,b,e), your algorithm could return (a, d, c, g, b, e). If there are multiple valid order of letters, simply return any one of them.

Give an algorithm with runs in $O(n + m)$ time, where $n$ is the number of letters, and $m$ is the sum of the lengths of the sequences.
Problem 3 Minimum Spanning Trees (10 points):
Let $G = (V, E)$ be an undirected graph with edge weights. We will assume that the edge weights are distinct.

a) What is the edge inclusion lemma (for Minimum Spanning Trees)?

b) Let $u$ be a vertex, and suppose that $e = (u, v)$ is the minimum cost edge adjacent to $u$. Prove that the edge $e$ is in the minimum spanning tree for $G$. 
Problem 4 Interval Scheduling (10 points):
The input for an interval scheduling problem is a set of intervals $I = \{i_1, \ldots, i_n\}$ where $i_k$ has start time $s_k$, and finish time $f_k$. The problem is to find a set of non-overlapping intervals that satisfies a given criteria.

a) Suppose that you want to maximize the total length of the selected intervals. True or false: The greedy algorithm based on selecting intervals in order of decreasing length finds an optimal solution. Justify your answer.

b) The set of intervals $I' = \{i'_1, \ldots, i'_n\}$ is said to be a shrinking of the intervals $I'' = \{i''_1, \ldots, i''_n\}$ if each interval in $I'$ is contained in the corresponding interval of $I''$, in other words, for $1 \leq k \leq n$, $s'_{k} \leq s'_{k} \leq f'_{k} \leq f''_{k}$. True or false: If $I'$ is a shrinking of $I''$, then the maximum number of non-overlapping intervals in $I'$ is at least as great as the maximum number of non-overlapping intervals in $I''$. Justify your answer.
Problem 5 Bottleneck Distance (5 points):
The bottleneck length of a path $P$ is the cost of the maximum edge of $P$. The bottleneck distance between vertices $u$ and $v$ is the minimum bottleneck distance of a path between $u$ and $v$. This problem can be solved using a variant of Dijkstra’s algorithm.

Simulate Dijkstra’s bottleneck path algorithm on the graph above by filling in the table. The entries should contain the preliminary distance values.

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Problem 6 Recurrences (15 points):
Give solutions to the following recurrences. Justify your answers.

a) 
\[ T(n) = \begin{cases} 
5T\left(\frac{n}{3}\right) + n & \text{if } n > 1 \\
1 & \text{if } n \leq 1 
\end{cases} \]

b) 
\[ T(n) = \begin{cases} 
T\left(\frac{4n}{5}\right) + n & \text{if } n > 1 \\
1 & \text{if } n \leq 1 
\end{cases} \]

c) 
\[ T(n) = \begin{cases} 
16T\left(\frac{n}{4}\right) + n^2 & \text{if } n > 1 \\
1 & \text{if } n \leq 1 
\end{cases} \]