CSE421: Design and Analysis of Algorithms	May 23, 2024
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P1) Write the following LP in Standard Form:

$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t.,} \quad & x_1 - x_3 = 4 \\ & 2x_2 - x_3 \geq 5 \\ & x_1, x_3 \geq 0. \end{aligned}$$

Solution:

$$\begin{aligned} \max \quad & -2x_1 + (y_2 - y_2') \\ \text{s.t.,} \quad & x_1 - x_3 \leq 4 \\ & -x_1 + x_3 \leq -4 \\ & -2(y_2 - y_2') + x_3 \leq -5 \\ & x_1, y_2, y_2', x_3 \geq 0. \end{aligned}$$

P2) Write the dual of the following program:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.,} \quad & x_1 + x_2 \leq 5 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution: To write the dual we need to choose an average, i.e., a convex combination of the constraint to get the best bound on the value of the optimum.

We have two constraint. We let y_1 denote the coefficient of the first constraint and y_2 denote the coefficient of the second constraint. Then, we need

$$2x_1 + x_3 \leq y_1(x_1 + x_2) + y_2(x_1 - 2x_2)$$

Or in other words that

$$2 \leq y_1 + y_2 \quad \text{and} \quad 1 \leq y_1 - 2y_2$$

And, we need to minimize the RHS, i.e., $5y_1 + 2y_2$. So, the dual becomes:

$$\begin{aligned} \min \quad & 5y_1 + 2y_2 \\ \text{s.t.,]} \quad & 2 \leq y_1 + y_2 \\ & 1 \leq y_1 - 2y_2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

P3) Write the dual of the LP relaxation of the min vertex cover then turn it into the standard form.

Solution: Recall the LP relaxation of the min vertex cover

$$\begin{aligned} \min \quad & \sum_v c_v x_v \\ \text{s.t.,} \quad & x_u + x_v \geq 1 \quad \forall (u,v) \in E \\ & x_v \geq 0 \qquad\qquad \forall v \in V. \end{aligned}$$

We first turn it into the standard form.

$$\begin{aligned} \max \quad & -\sum_v c_v x_v \\ \text{s.t.,} \quad & -x_u - x_v \leq -1 \quad \forall (u,v) \in E \\ & x_v \geq 0 \qquad\qquad \forall v \in V. \end{aligned}$$

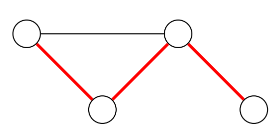
Now, we write teh dual:

$$\begin{aligned} \min \quad & -\sum_e y_e \\ \text{s.t.,} \quad & \sum_{e \sim v} -y_e \geq -c_v \quad \forall v \in V \\ & y_e \geq 0 \qquad\qquad \forall e \in E. \end{aligned}$$

Equivalently, we can write the dual in the standard form as follows:

$$\begin{aligned} \max \quad & \sum_e y_e \\ \text{s.t.,} \quad & \sum_{e \sim v} y_e \leq c_v \quad \forall v \in V \\ & y_e \geq 0 \qquad\qquad \forall e \in E. \end{aligned}$$

P4) A Hamiltonian cycle in a graph with n vertices is a cycle of length n, i.e., it is a cycle that visits all vertices exactly once and returns back to the starting point. A Hamiltonian path in a graph with n vertices is a path of length $n-1$, i.e., it is a path that visits all vertices of the graph exactly once. For example, the following graph has a Hamiltonian path marked in red but no Hamiltonian cycle.



Hamiltonian-cycle problem is defined as follows: Given a graph $G = (V, E)$, does it have a Hamiltonian cycle?

Hamiltonian-path problem is defined as follows: Given a graph $G = (V, E)$, does it have a Hamiltonian path?

Prove that Hamil-path $\leq_P$ Hamil-cycle. In other words, suppose we have a program A that solves the Hamiltonian cycle problem. Design a polynomial time algorithm that on an input graph G uses A only polynomial number of times and in polynomial time returns the solution to the Hamiltonian path problem on the given graph G.

Solution: Given an instance of Hamil-path, i.e., an undirected graph $G = (V, E)$ with n vertices, we construct the following inputs to Hamil-cycle problem: For every non-edge u, v, let $G_{u,v} = (V, E \cup \{\{u, v\}\})$ be the graph obtained by adding the edge $\{u, v\}$. We call the program Hamil-cycle on all such graphs if any of them return yes, then we output yes, and otherwise we output no. First, this reduction obviously runs in polynomial time because we call at most $O(n^2)$ many copies of Hamil-cycle and we can construct each graph $G_{u,v}$ in polynomial time. It remains to prove the correctness.

First: suppose G has a Hamiltonian path, say $v_1, \dots, v_n$. Then, the graph G_{v_1,v_n} has a Hamiltonian cycle: $v_1, \dots, v_n, v_1$. So, Hamil-cycle$(G_{v_1,v_n}) = \mathit{yes}$ and we will output yes.

Conversely, suppose Hamil-cycle$(G_{u,v}) = \mathit{yes}$ for some $u, v \in V$. This means that $G_{u,v}$ has a Hamiltonian cycle, say $v_1, \dots, v_n, v_1$. If the edge $\{u, v\}$ is not an edge of this cycle, then G also has a Hamiltonian cycle, so it also has a Hamiltonian path and we correctly output yes. Otherwise say $v_i = u, v_{i+1} = v$. Then, $v_{i+1}, \dots, v_n, v_1, \dots, v_i$ is a Hamiltonian path. So, G has a Hamiltonian path and we correctly output yes.