CSE421: Design and Analysis of Algorithms May 16, 2024

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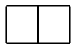
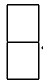
P1) A domino is shape like  or . Given an $n \times n$ table where some of the squares are removed (in the picture below removed squares are marked with an X), design a polynomial time algorithm that outputs the maximum number of dominos that can be placed on the table which are not overlapping and don't cover any X cells.

For example, given the table on the left the maximum number of dominos that can be placed is 2.

X		
		X
	X	

Solution: We construct an instance of the bipartite matching problem: First we construct a graph G: We put a vertex for every square of the table which is not marked with an X; we connect two vertices u, v with an edge if we can place a domino on them, i.e., if the corresponding two squares share a side.

We claim that G is a bipartite graph: To see that it is enough to color the graph with two colors such that any adjacent pair of vertices have distinct colors. We color the cells of the table like a chessboard, black/white. It follows that any two neighboring squares have opposite colors, therefore G is bipartite.

Algorithm: We return the size of the maximum matching in G.

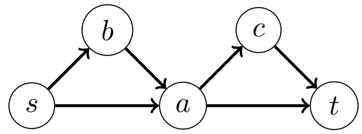
Runtime: Runtime is the time to compute the maximum matching in G, G has $O(n^2)$ vertices and $O(n^2)$ edges. So, max matching runs in time $O(n^4)$.

Correctness: We show max-matching = max number of dominos that can be placed that don't cover any X cells.

Max Matching $\geq$ Max number of dominos: Suppose the maximum number of dominos we can place on the table is k; choose the corresponding edge for every domino that is placed call this set of edges M. We claim that M is a matching in G (with k edges). This is because every cell is covered by at most one domino, so every vertex of G is adjacent to at most one edge of M. Further, since no dominos are placed on X-cells, each domino corresponds to exactly one edge of G. This implies Max matching in G is at least the max number of dominos.

Max number of dominos $\geq$ Max Matching Let M be a maximum matching with k edges in G. Since every edge in G corresponds to two adjacent cells in the table, we can put one domino on the table for every edge in M. Furthermore, since G does not have vertices for the X-cells, no domino will be placed on the X-cells and lastly since M is a matching the dominos are not overlapping. Therefore, we can put $|M|$ many dominos on the table which are not overlapping and do not cover any X's as desired.

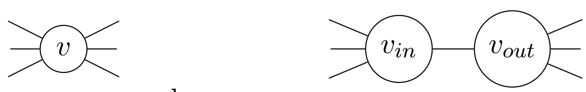
P2) Given an (unweighted) directed graph $G = (V, E)$, a pair of vertices s, t and an integer $1 \leq k \leq n$. Design an algorithm that runs in time polynomial in n, k and outputs yes if there are k *vertex* disjoint paths from s to t and no otherwise. For example, in the following graph there are two edge disjoint paths from s to t but no two vertex disjoint paths from s to t.



For this problem you can assume you have access to a polynomial time algorithm for the edge disjoint path problem defined as follows: Given a directed graph G and a pair of vertices s, t we want to find the maximum number of edge disjoint paths from s to t. Two paths P_1, P_2 from s to t are edge disjoint if they don't share an edge. We will discuss the solution to this problem in class on Friday.

Solution: Construct H from G by splitting each vertex $v \neq s, t$ to an "in" and an "out" vertex.

For any edge $u \to v$ in G we connect u_{out} to v_{in}. In the special cases of $s \to v$ or $v \to t$, we simply connect s to v_{in} and v_{out} to t, respectively, in H and



becomes

Then, we run the algorithm from class to find the maximum number of edge disjoint paths from s to t in H and we output that number. The algorithm obviously runs in time polynomial in n as it takes $O(m+n)$ to construct graph H and the algorithm to find the maximum number of edge disjoint paths runs in time $O(mn)$.

Correctness: First, observe that there is a natural bijection between paths from s to t in G and H. For any path from s to t, say $v^0 = s, v^1, \ldots, v^k = t$ the path $s, v^1_{in}, v^1_{out}, v^2_{in}, \ldots, v^{k-1}_{out}, t$ is a path from s to t in H. And, conversely a path from s to t in H is of the form $s, v^1_{in}, v^1_{out}, \ldots, v^{l-1}_{out}, v_t$; this is because the only out-going edges of s go to in-vertices and every in-vertex has a unique outgoing vertex to an out-vertex, so in/out vertices should alternate and we should end at an out-vertex before we go to t.

We claim that the maximum number of vertex disjoint paths in G is equal to the maximum number of edge disjoint paths in H.

$\leq$: suppose we have k-vertex disjoint paths in G $P_1, \ldots, P_k$ from s to t; then by the above bijection we get k-paths $P'_1, \ldots, P'_k$ from s to t in H. These paths are edge disjoint in H simply because every $v_{in} \to v_{out}$ edge can be used in at most one path, the only possible path among $P_1, \ldots, P_k$ that has vertex v.

$\geq$: Suppose we have k-edge disjoint paths $P_1, \ldots, P_k$ from s to t in H. Then, by the above bijection, they map to k paths $P'_1, \ldots, P'_k$ in G from s to t in H. Observe that each of the edges $v_{in} \to v_{out}$ can be used in at most on of $P_1, \ldots, P_k$. This implies that the paths $P'_1, \ldots, P'_k$ are vertex disjoint.