CSE421: Design and Analysis of Algorithms	March 28, 2024
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P1) Consider the following stable matching instance:

$$
\begin{array}{ll}
c_1 : a_3 > a_1 > a_2 > a_4 & \quad a_1 : c_4 > c_1 > c_3 > c_2 \\
c_2 : a_2 > a_1 > a_4 > a_3 & \quad a_2 : c_1 > c_3 > c_2 > c_4 \\
c_3 : a_2 > a_3 > a_1 > a_4 & \quad a_3 : c_1 > c_3 > c_4 > c_2 \\
c_4 : a_3 > a_4 > a_1 > a_2 & \quad a_4 : c_3 > c_1 > c_2 > c_4
\end{array}
$$

a) Run the Gale-Shapley Algorithm with companies proposing on the instance above. When choosing which free company to propose next, always choose the one with the smallest index (e.g., if c_1 and c_2 are both free, always choose c_1). The steps of the Gale-Shapley Algorithm with the companies with **highest** index proposing first:

$$
\begin{array}{ll}
c_4 \text{ chooses } a_3 & (c_4, a_3) \\
c_3 \text{ chooses } a_2 & (c_3, a_2), (c_4, a_3) \\
c_2 \text{ chooses } a_2 & (c_3, a_2), (c_4, a_3) \\
c_2 \text{ chooses } a_1 & (c_2, a_1), (c_3, a_2), (c_4, a_3) \\
c_1 \text{ chooses } a_3 & (c_1, a_3), (c_2, a_1), (c_3, a_2) \\
c_4 \text{ chooses } a_4 & (c_1, a_3), (c_2, a_1), (c_3, a_2), (c_4, a_4)
\end{array}
$$

b) Now run the algorithm with applicants proposing, breaking ties by taking the free applicant with the **smallest** index. Do you get the same result? The steps of the Gale-Shapley Algorithm with applicants proposing:

$$
\begin{array}{ll}
a_1 \text{ chooses } c_4 & (c_4, a_1) \\
a_2 \text{ chooses } c_1 & (c_1, a_2), (c_4, a_1) \\
a_3 \text{ chooses } c_1 & (c_1, a_3), (c_4, a_1) \\
a_2 \text{ chooses } c_3 & (c_1, a_3), (c_3, a_2), (c_4, a_1) \\
a_4 \text{ chooses } c_3 & (c_1, a_3), (c_3, a_2), (c_4, a_1) \\
a_4 \text{ chooses } c_1 & (c_1, a_3), (c_3, a_2), (c_4, a_1) \\
a_4 \text{ chooses } c_2 & (c_1, a_3), (c_2, a_4), (c_3, a_2), (c_4, a_1)
\end{array}
$$

No, the result is different when we have the applicants propose as opposed to the companies.

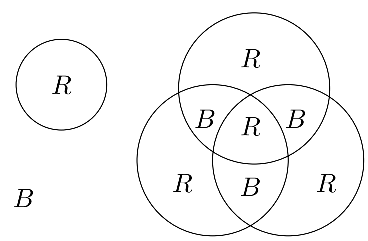
P2) Show that an instance of the stable matching problem has exactly one stable matching if and only if the company optimal matching is equal to the applicant optimal matching.

First, notice that this is an if and only if problem. That means that you have to prove both directions. In this case you have to mark each direction clearly:

Direction 1: **Suppose we are given an instance of the stable matching problem with exactly one stable matching then the company optimal matching is equal to applicant optimal.** This holds trivially since both company optimal and applicant optimal are stable; if they are not the same then there are two stable matchings

Direction 2: **Suppose we are given an instance of the stable matching problem such that the company optimal matching is equal to applicant optimal, then this instance has exactly one stable matching.** We prove by contradiction. Let M^* be the company/applicant optimal. and suppose there is another stable matching M different from M^*. So, there must be an applicant a who is matched to two different companies in M/M^* say c_1 in M and c_2 in M^*. So, both c_1 and c_2 are valid partners of a. So, the best valid partner of a is different from the worst valid partner of a. But that means that applicant optimal matching is not the same as the comapny optimal contradiction. So, we must have exactly one stable matching.

P3) Suppose we have drawn n circles on the plane. Show that we can color the regions with 2 colors (R/B) such that any two neighboring regions are colored with distinct colors. Two regions are neighbors if the share a line segment. See the following example:



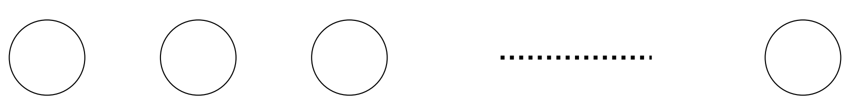
a) First explain what is wrong with the following inductive proof: We prove by induction that any n circles drawn on the plane can be colored with R/B such that any two neighboring regions have distinct colors.

The claim obviously holds for $n = 1$ we have a single circle and we color inside R and outside B.

Suppose have colored the regions with $n-1$ circles. Now, we add the n-th circle in such a way that it doesn't cross any of the previous $n-1$ circles. and we color inside of it the opposite of the outside region.

Solution: This proof is not following the framework given in class. Predicate and induction hypothesis are not clearly defined.

But the major problem is that instead of starting with an arbitrary instance with n circles and **reducing** it to an instance with $n-1$ circles it starts with an instance with $n-1$ circles and add a specifically chosen circle. If you think about it more this proof only solves instance of the following kind, **not all possible arrangements of n circles on the plane**.



b) Now, solve the problem with a correct inductive proof.

Solution: We define the following predicate:

$P(n)$ =Given **any** set of n circles on the plane we can color the regions with R/B such that any two neighboring regions are colored with distinct colors.

Base case: $P(1)$ Suppose we have one circle. Then we color inside R and outside B.

IH: Suppose $P(n-1)$ holds for some $n \geq 2$.

IS: We prove $P(n)$. Suppose we are given n circles arbitrarily drawn on the plane $C_1, \ldots, C_n$. The first step is to construct an instance of $P(n-1)$. We can delete any of the circles (here we have a choice). We delete C_n and now we are left with $n-1$ circles on the plane.

By IH we color the regions defined by $C_1, \ldots, C_{n-1}$ with R/B such that any two neighboring regions are colored differently.

Now, we add back C_n. Some regions are not crosses by C_n, but every region crossed by C_n is divided into two neighboring regions of the same color. Now, we **flip the color of every region inside** C_n. We claim that this gives a valid coloring.

First, notice by doing this, any two neighboring regions one inside and one outside of C_n will have distinct colors.

Two neighboring regions both outside of C_n don't change colors so still will have distinct colors. Two neighboring regions both inside C_n will have both of their colors flipped so still they will have distinct colors.