# CSE 421

## Polynomial Time Reductions
## NP Completeness

Shayan Oveis Gharan

# P, NP, EXP

P. Decision problems for which there is a poly-time algorithm.

EXP. Decision problems for which there is an exponential-time algorithm.

NP. Decision problems for which there is a poly-time certifier.

Claim. P $\subseteq$ NP.

Pf. Consider any problem X in P.

By definition, there exists a poly-time algorithm A(x) that solves X.

Certificate: t = empty string, certifier C(x, ∅) = A(x). ▪

Claim. NP $\subseteq$ EXP.

Pf. Consider any problem X in NP.

By definition, there exists a poly-time certifier C(x, t) for X.

To solve input x, run C(x, t) on all strings t with of length polyn in |x|

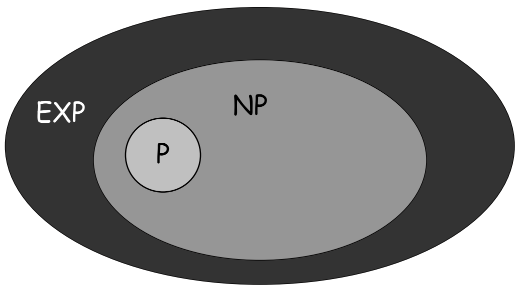
Return `yes`, if C(x, t) returns `yes` for any of these.

# The main question: P vs NP

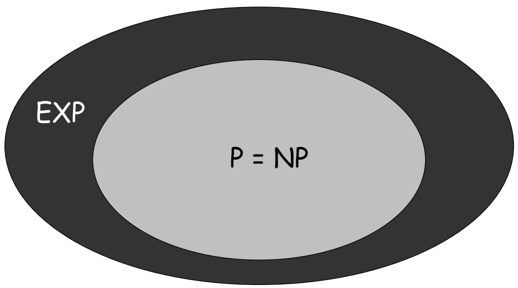
Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

Is the decision problem as easy as the certification problem?

Clay $1 million prize.




If P ≠ NP




If P = NP

If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, …

# What do we know about NP?

- Nobody knows if all problems in NP can be done in polynomial time, i.e. does P=NP?
  - one of the most important open questions in all of science.
  - Huge practical implications specially if answer is yes

- To show Hamil-cycle $\notin P$ we have to prove that there is no poly-time algorithm for it even using all mathematical theorem that will be discovered in future!

# NP Completeness

Complexity Theorists Approach: We don't know how to prove any problem in NP is hard. So, let's find hardest problems in NP.

NP-hard: A problem B is NP-hard iff for any problem $A \in NP$, we have $A \leq_p B$

NP-Completeness: A problem B is NP-complete iff B is NP-hard and $B \in NP$.

Motivations:

- If $P \neq NP$, then every NP-Complete problems is not in P. So, we shouldn't try to design Polytime algorithms
- To show $P = NP$, it is enough to design a polynomial time algorithm for just one NP-complete problem.

# Cook-Levin Theorem

Theorem (Cook 71, Levin 73): 3-SAT is NP-complete, i.e., for all problems $A \in NP$, $A \leq_p$ 3-SAT.

- So, 3-SAT is the hardest problem in NP.

What does this say about other problems of interest? Like Independent set, Vertex Cover, …

Fact: If $A \leq_p B$ and $B \leq_p C$ then, $A \leq_p C$
Pf idea: Just compose the reductions from A to B and B to C

So, if we prove 3-SAT $\leq_p$ Independent set, then Independent Set, Clique, Vertex cover, Set cover are all NP-complete

3-SAT $\leq_p$ Independent Set $\leq_p$ Vertex Cover $\leq_p$ Set Cover

# Summary

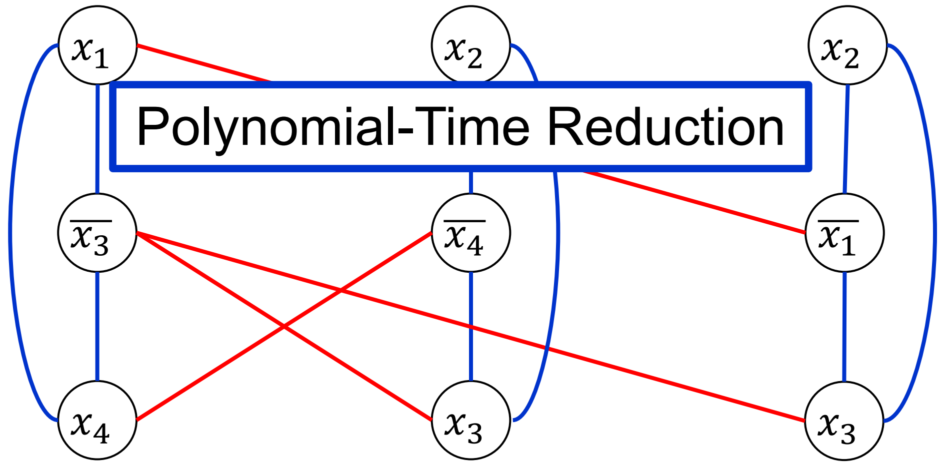
- If a problem is NP-hard it does not mean that all instances are hard, e.g., Vertex-cover has a polynomial-time algorithm on trees or bipartite graphs
- We learned the crucial idea of polynomial-time reduction. This can be even used in algorithm design, e.g., we know how to solve max-flow so we reduce image segmentation to max-flow
- NP-Complete problems are the hardest problem in NP
- NP-hard problems may not necessarily belong to NP.
- Polynomial-time reductions are transitive relations

# 3-SAT $\leq_p$ Independent Set

Map a 3-CNF to (G,k). Say m is number of clauses

- Create a vertex for each literal
- Join two literals if
  - They belong to the same clause (blue edges)
  - The literals are negations, e.g., $x_i, \overline{x_i}$ (red edges)
- Set k=m

$$(x_1 \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3) \wedge (x_2 \vee \overline{x_1} \vee x_3)$$



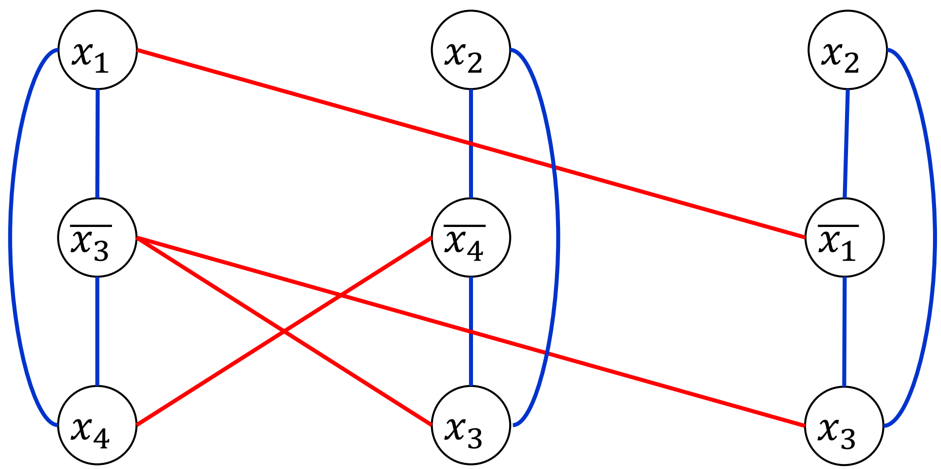
# Correctness of 3-SAT $\leq_p$ Indep Set

F satisfiable => An independent of size m

Given a satisfying assignment, Choose one node from each clause where the literal is satisfied

$$(x_1 \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3) \wedge (x_2 \vee \overline{x_1} \vee x_3)$$

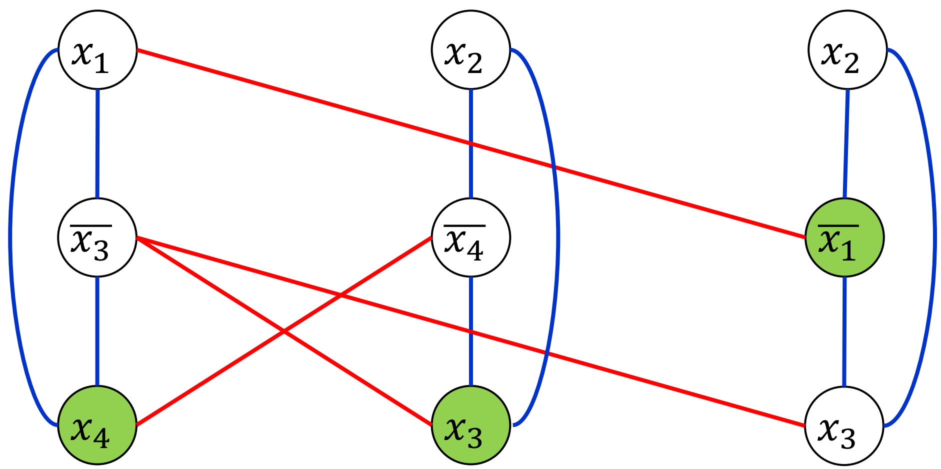
Satisfying assignment: $x_1 = T, x_2 = F, x_3 = T, x_4 = F$




- S has exactly one node per clause => No blue edges between S
- S follows a truth-assignment => No red edges between S
- S has one node per clause => |S|=m

# Correctness of 3-SAT $\leq_p$ Indep Set

An independent set of size m => A satisfying assignment
Given an independent set S of size m.
S has exactly one vertex per clause (because of blue edges)
S does not have $x_i, \overline{x_i}$ (because of red edges)
So, S gives a satisfying assignment




Satisfying assignment: $x_1 = F, x_2 =?, x_3 = T, x_4 = T$

$$(x_1 \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3) \wedge (x_2 \vee \overline{x_1} \vee x_3)$$

# Project Selection

# Project Selection

can be positive or negative

Projects with prerequisites.

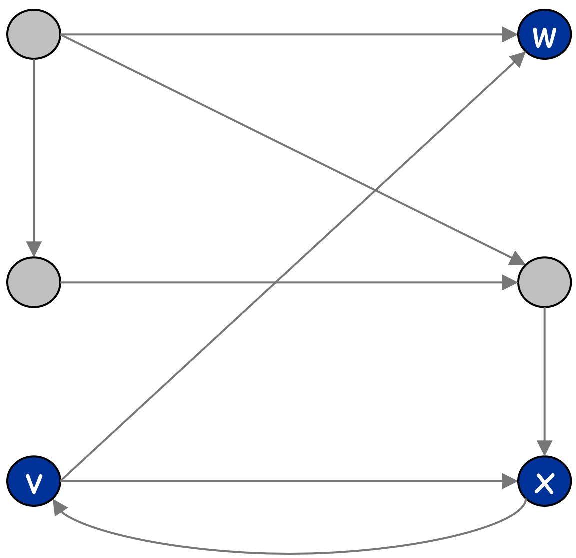
- Set P of possible projects. Project v has associated revenue $p_v$.
  - some projects generate money: create interactive e-commerce interface, redesign web page
  - others cost money: upgrade computers, get site license
- Set of prerequisites E. If $(v, w) \in E$, can't do project v and unless also do project w.
- A subset of projects $A \subseteq P$ is feasible if the prerequisite of every project in A also belongs to A.

Project selection. Choose a feasible subset of projects to maximize revenue.

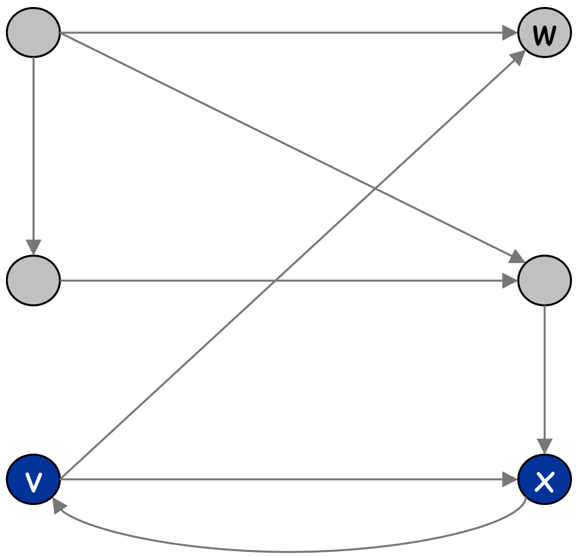
# Project Selection: Prerequisite Graph

Prerequisite graph.

- Include an edge from v to w if can't do v without also doing w.
- {v, w, x} is feasible subset of projects.
- {v, x} is infeasible subset of projects.




feasible

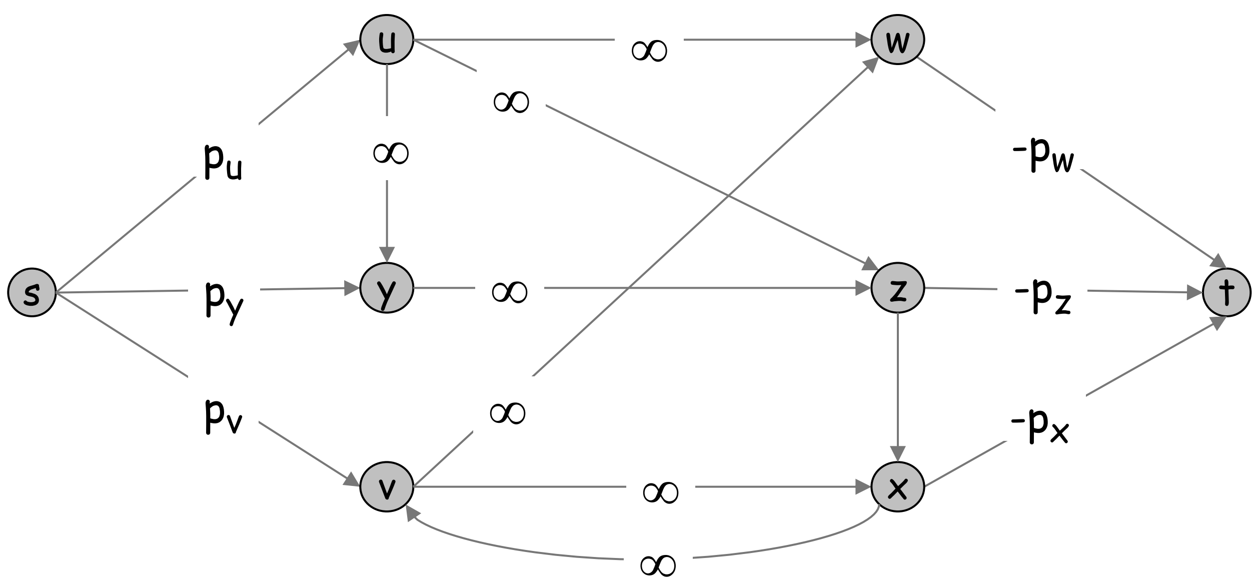



infeasible

# Project Selection: Min Cut Formulation

Min cut formulation.

- Assign capacity $\infty$ to all prerequisite edge.
- Add edge $(s, v)$ with capacity $p_v$ if $p_v > 0$.
- Add edge $(v, t)$ with capacity $-p_v$ if $p_v < 0$.
- For notational convenience, define $p_s = p_t = 0$.



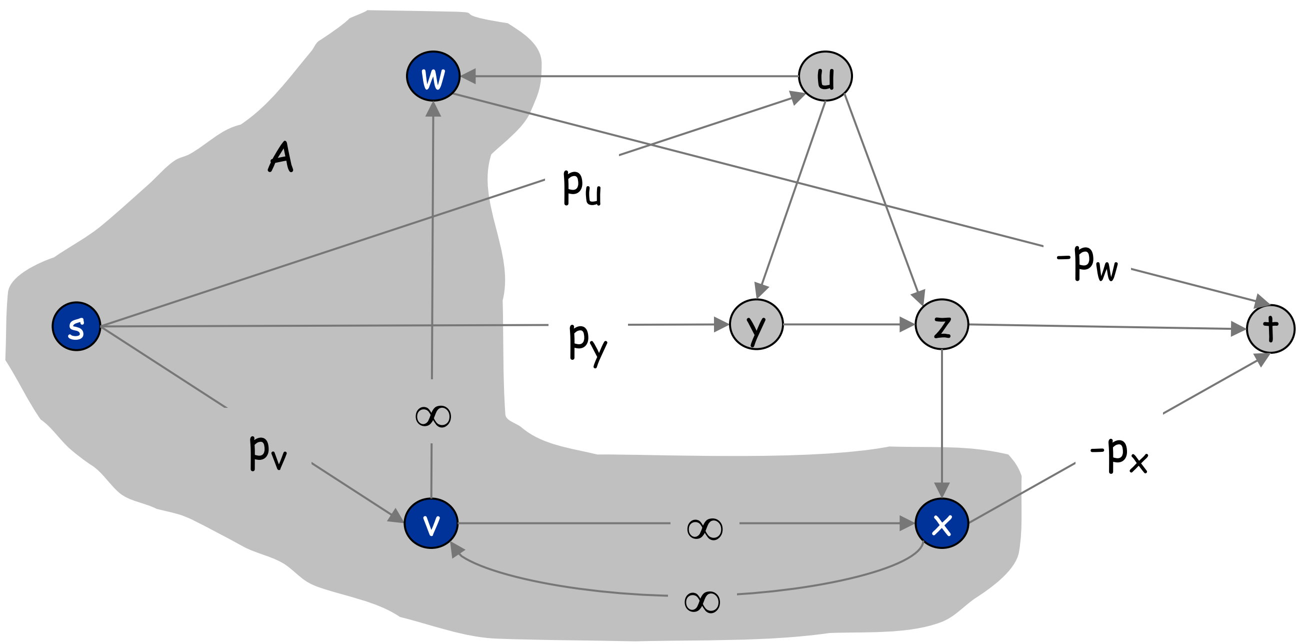
# Project Selection: Min Cut Formulation

Claim. (A, B) is min cut iff A – { s } is optimal set of projects.

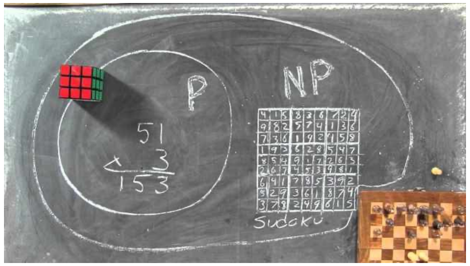
- Infinite capacity edges ensure A – { s } is feasible.
- Max revenue because:

$$cap(A, B) = \sum_{v \in B:\, p_v > 0} p_v + \sum_{v \in A:\, p_v < 0} (-p_v)$$

$$= \underbrace{\sum_{v:\, p_v > 0} p_v}_{\text{constant}} - \sum_{v \in A} p_v$$

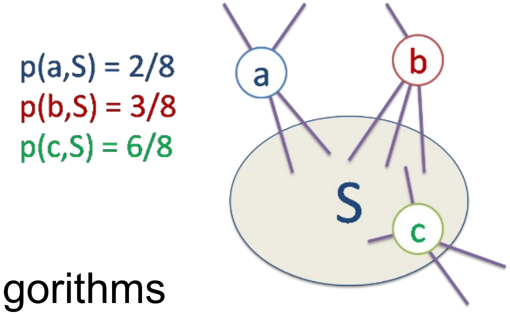


# What is next?




- CSE 431 (Complexity Course)
  - How to prove lower bounds on algorithms?
- CSE 422 (Advanced Toolkit for Modern Alg)
  - SVD, Data structures, many programming tasks
- CSE 521 (Graduate Algorithms Course)

  Prereq: 312, Math 308
  - How to design streaming algorithms?
  - How to design algorithms for high dimensional data?
  - How to use matrices/eigenvalues/eigenvectors to design algorithms
  - How to use LPs to design algorithms?




- CSE 525 (Graduate Randomized Algorithms Course)

  Prereq: CSE 521
  - How to use randomization to design algorithms?
  - How to use Markov Chains to design algorithms?

# Course Evaluations

- How can we improve this course?
- Did you like sections? Should we keep having them? Any suggestion on how to improve sections?
- Did you like topics related to linear programming? Did you like to see more of that?
- Which topic was most/least interesting to you?
- Which problem sets did you like more?