CSE 421

LP Duality

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Intro to Duality

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.}, \quad & x_1 + 3x_2 \leq 2 \\ & 2x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimum solution: $x_1 = 5/4$ and $x_2 = 1/4$ with value $x_1 + 2x_2 = 7/4$
How can you prove an upper-bound on the optimum?

First attempt: Since $x_1, x_2 \geq 0$

$$x_1 + 2x_2 \leq x_1 + 3x_2 \leq 2$$

Second attempt:

$$x_1 + 2x_2 \leq \frac{2}{3}(x_1 + 3x_2) + \frac{1}{3}(2x_1 + 2x_2) \leq \frac{2}{3}(2) + \frac{1}{3}(3) = \frac{7}{3}$$

Third attempt:

$$x_1 + 2x_2 \leq \frac{1}{2}(x_1 + 3x_2) + \frac{1}{4}(2x_1 + 2x_2) \leq \frac{1}{2}(2) + \frac{1}{4}(3) = \frac{7}{4}$$

Dual Certificate

$$\begin{aligned} max \quad & x_1 + 2x_2 \\ s.t., \quad & x_1 + 3x_2 \leq 2 \quad && y_1 \\ & 2x_1 + 2x_2 \leq 3 \quad && y_2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Goal: Minimize $2y_1 + 3y_2$

But, we must make sure the sum of the LHS is least the objective, i.e.,

$$x_1 + 2x_2 \ \leq y_1(x_1 + 3x_2) + y_2(2x_1 + 2x_2)$$

In other words,

$$1 \leq 1 \cdot y_1 + 2 \cdot y_2$$
$$2 \leq 3 \cdot y_1 + 2 \cdot y_2$$

Finally, $y_1, y_2 \geq 0$ (else the direction of inequalities change)

Dual Program

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.,} \quad & x_1 + 3x_2 \le 2 \\ & 2x_1 + 2x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

OPT: $x_1 = 5/4$ and $x_2 = 1/4$
Value $7/4$

$$\begin{aligned} \min \quad & 2y_1 + 3y_2 \\ \text{s.t.,} \quad & y_1 + 2y_2 \ge 1 \\ & 3y_1 + 2y_2 \ge 2 \\ & y_1, y_2 \ge 0 \end{aligned}$$

OPT: $y_1 = 1/2$ and $y_2 = 1/4$
Value $7/4$

Dual of Standard LP

$$\begin{aligned} \max \quad & \langle c, x \rangle \\ \text{s.t.,} \quad & \langle a_1, x \rangle \le b_1 \qquad {\color{red} y_1} \\ & \langle a_2, x \rangle \le b_2 \qquad {\color{red} y_2} \\ & \vdots \\ & \langle a_m, x \rangle \le b_m \qquad {\color{red} y_m} \\ & x_1, \dots, x_n \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \langle b, y \rangle \\ \text{s.t.,} \quad & a_{1,1} y_1 + \cdots + a_{m,1} y_m \ge c_1 \\ & a_{1,2} y_1 + \cdots + a_{m,2} y_m \ge c_2 \\ & \vdots \\ & a_{1,n} y_1 + \cdots + a_{m,n} y_m \ge c_n \\ & y_1, \dots, y_m \ge 0 \end{aligned}$$

$$\begin{aligned} max \quad & \langle c, x \rangle \\ \text{s.t.,} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Primal

$$\begin{aligned} min \quad & \langle b, y \rangle \\ \text{s.t.,} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

Dual

Facts About Linear Programs

Lem: Dual of Dual = Primal

Thm (weak duality): Every solution to the primal is at most every solution to the dual

$$\langle c, x\rangle \leq \langle b, y\rangle$$

Thm (strong duality): If primal has a solution and dual has a solution then optimum of primal is equal to optimum of dual

Dual of Max-Flow

$$\max \quad \sum_{e \text{ out of } s} x_e$$

$$s.t. \quad \sum_{e\ out\ of\ v} x_e = \sum_{e\ in\ to\ v} x_e \qquad \forall v \neq s,t \qquad {\color{red} b_v}$$

$$x_e \leq c(e) \qquad \forall e \qquad {\color{red} a_e}$$

$$x_e \geq 0 \qquad \forall e$$

$$\min \quad \langle c, a \rangle$$

$$\text{s.t.}, \quad a_e + b_v \geq 1 \qquad e = (s,v)$$

$$a_e - b_v \geq 0 \qquad e = (v,t)$$

$$a_e + b_u - b_v \geq 0 \qquad \text{other } e = (u,v)$$

$$a_e \geq 0 \qquad \forall e$$

$$
\begin{array}{lll}
\min & \langle c, a \rangle & \\
\text{s.t.,} & a_e + b_v \geq 1 & e = (s, v) \\
& a_e - b_v \geq 0 & e = (v, t) \\
& a_e + b_u - b_v \geq 0 & \text{other } e = (u, v) \\
& a_e \geq 0 & \forall e
\end{array}
$$

⇩

$$
\begin{array}{lll}
\min & \langle c, a \rangle & \\
\text{s.t.,} & a_e = \max(0, 1 - b_v) & e = (s, v) \\
& a_e = \max(0, b_v) & e = (v, t) \\
& a_e = \max(0, b_v - b_u) & \text{other } e = (u, v)
\end{array}
$$

$$\min \quad \langle c, a \rangle$$

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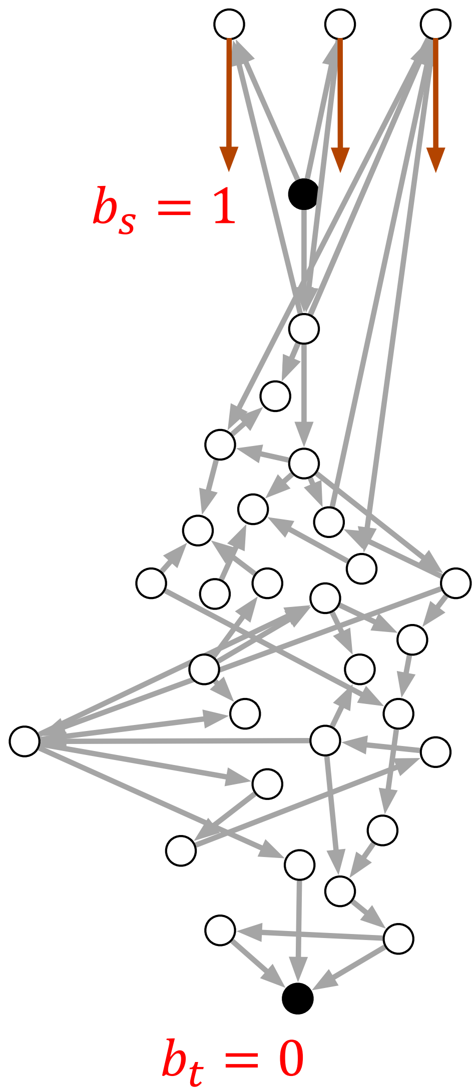
$$a_e = \max(0, b_v - b_u) \quad \text{other } e = (u, v)$$

Lem: In OPT $0 \leq b_v \leq 1$ for all v

Pf: If not, move up/down the value only decreases

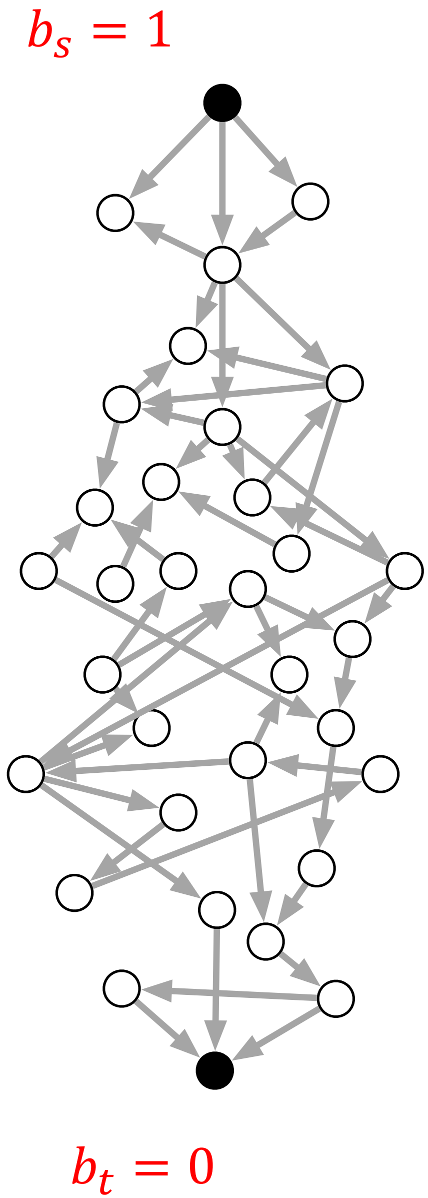


$$\begin{aligned} \min \quad & \langle c, a \rangle \\ \text{s.t.,} \quad & b_s = 1, b_t = 0 \\ & 0 \le b_v \le 1 \\ & a_e = \max(0, b_v - b_u) \quad e = (u, v) \end{aligned}$$

Lem: In OPT $0 \le b_v \le 1$ for all v

Pf: If not, move up/down the value only decreases

Lem: In OPT $b_v \in \{0,1\}$ for all v
Pf: If not, choose a u.r. $0 \le t \le 1$
If $b_v \ge t$ set $b_v = 1$ else set $b_v = 0$.
Then, the expected value of resulting solution sames as OPT.



$$\min \quad \langle c, a \rangle$$
$$\text{s.t.,} \quad b_s = 1, b_t = 0$$
$$0 \le b_v \le 1$$
$$a_e = \max(0, b_v - b_u) \quad \text{other } e = (u, v)$$

Lem: In OPT $0 \le b_v \le 1$ for all v

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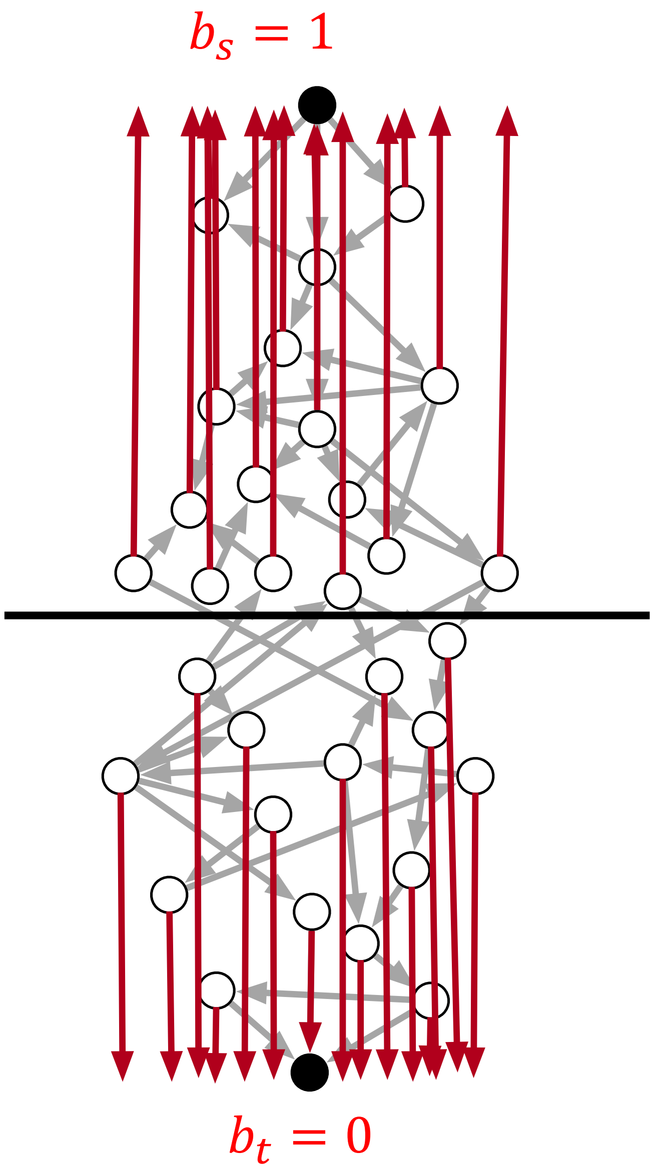


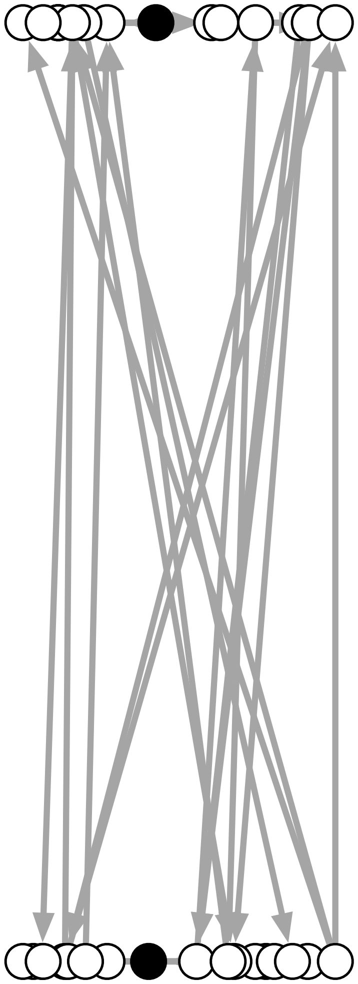
$$\min \quad \langle c, a \rangle$$

$$\text{s.t.,} \quad b_s = 1, b_t = 0$$

$$b_v \in \{0,1\}$$

$$a_e = \max(0, b_v - b_u) \quad \text{other } e = (u, v)$$

Min Cut!



Beyond LP: Convex Programming

A function $f: \mathbb{R} \to \mathbb{R}$ is convex if $f'' \geq 0$.

e.g., $f(x) = x^2$.

A function $f: \mathbb{R}^d \to \mathbb{R}$ is convex if $\nabla^2 f \succcurlyeq 0$

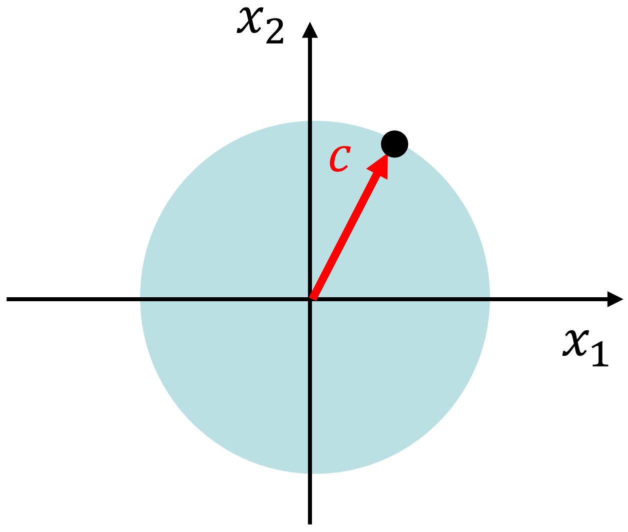
Convex Program

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.,} \quad & g_1(x) \leq b_1 \\ & g_2(x) \leq b_2 \\ & \vdots \\ & g_m(x) \leq b_m \end{aligned}$$

f and $g_1, \ldots, g_m$ must be convex.
$\geq$ and $=$ are not allows!

Example

$$\begin{aligned} \max \quad & c_1 x_1 + c_2 x_2 \\ \text{s.t.,} \quad & x_1^2 + x_2^2 \leq 1 \end{aligned}$$



Summary (Linear Programming)

- Linear programming is one of the biggest advances in 20th century

- It is being used in many areas of science: Mechanics, Physics, Operations Research, and in CS: AI, Machine Learning, Theory, …

- Almost all problems that we talked can be solved with LPs, Why not use LPs?
  - Combinatorial algorithms are typically faster
  - They exhibit a better understanding of worst case instances of a problem
  - They give certain structural properties, e.g., Integrality of Max-flow when capacities are integral

- There is rich theory of LP-duality which generalizes max-flow min-cut theorem

Reductions & NP-Completeness

Computational Complexity

Goal: Classify problems according to the amount of computational resources used by the best algorithms that solve them

Here we focus on time complexity

Recall: worst-case running time of an algorithm

- **max** # steps algorithm takes on any input of size **n**

Computational Complexity and Reduction

In most cases, we cannot characterize the true hardness of a computational problem

So?

We only reduce the number of problems

Want to be able to make statements of the form

- "If we could solve problem B in polynomial time then we can solve problem A in polynomial time"
- "Problem B is at least as hard as problem A"

Polynomial Time Reduction

Def A $\leq_P$ B: if there is an algorithm for problem A using a 'black box' (subroutine) that solve problem B s.t.,

- Algorithm uses only a polynomial number of steps
- Makes only a polynomial number of calls to a subroutine for **B**

So,

B is Polynomial time solvable → A is Polynomial time solvable

Conversely,

No efficient Algorithm for A → No efficient Algorithm for B

In words, B is as hard as A (it can be even harder)

$\leq_p^1$ Reductions

In this lecture we see a restricted form of polynomial-time reduction often called Karp or many-to-one reduction

$A \leq_p^1 B$: if and only if there is an algorithm for A given a black box solving B that on input x

- Runs for polynomial time computing an input f(x) of B
- Makes one call to the black box for B for input f(x)
- Returns the answer that the black box gave

We say that the function f(.) is the reduction

Decision Problems

A decision problem is a computational problem where the answer is just yes/no

Here, we study computational complexity of decision Problems.

Why?

- much simpler to deal with
- Decision version is not harder than Search version, so it is easier to lower bound Decision version
- Less important, usually, you can use decider multiple times to find an answer .

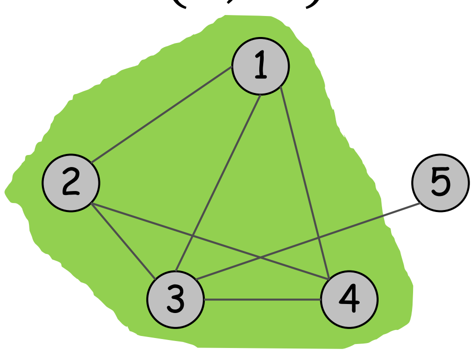
Example 1: Indep Set $\leq_p$ Clique

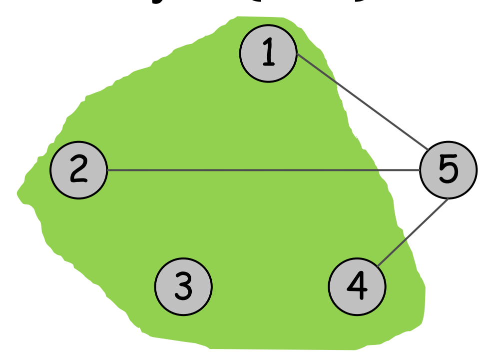
Indep Set: Given G=(V,E) and an integer k, is there $S \subseteq V$ s.t. $|S| \geq k$ an no two vertices in S are joined by an edge?

Clique: Given a graph G=(V,E) and an integer k, is there $S \subseteq V$, |U| ≥ k s.t., every pair of vertices in S is joined by an edge?

Claim: Indep Set $\leq_p$ Clique

Pf: Given $G = (V, E)$ and instance of indep Set. Construct a new graph $G' = (V, E')$ where $\{u, v\} \in E'$ if and only if $\{u, v\} \notin E$.





S is an independent set in G ⟷ S is an Clique in G'

Example 2: Vertex Cover $\leq_p$ Indep Set

Vertex Cover: Given a graph G=(V,E) and an integer k, is there a vertex cover of size at most k?

Claim: For any graph $G = (V, E)$, S is an independent set iff $V - S$ is a vertex cover

Pf:

=> Let S be a independent set of G

Then, S has at most one endpoint of every edge of G

So, $V - S$ has at least one endpoint of every edge of G

So, $V - S$ is a vertex cover.

<= Suppose $V - S$ is a vertex cover

Then, there is no edge between vertices of S (otherwise, $V - S$ is not a vertex cover)

So, S is an independent set.

Example 3: Vertex Cover $\leq_p$ Set Cover

Set Cover: Given a set U, collection of subsets $S_1, \ldots, S_m$ of U and an integer k, is there a collection of k sets that contain all elements of U?

Claim: Vertex Cover $\leq_p$ Set Cover

Pf:

Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set S_v of all edges connected to v

This clearly is a polynomial-time reduction

So, we need to prove it gives the right answer

Example 3: Vertex Cover $\leq_p$ Set Cover

Claim: Vertex Cover $\leq_p$ Set Cover
Pf: Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set S_v of all edges connected to v

Vertex-Cover (G,k) is yes => Set-Cover f(G,k) is yes
If a set $W \subseteq V$ covers all edges,, just choose S_v for all $v \in W$, it covers all U.

Set-Cover f(G,k) is yes => Vertex-Cover (G,k) is yes
If $(S_{v_1}, \dots, S_{v_k})$ covers all U, the set $\{v_1, \dots, v_k\}$ covers all edges of G.