CSE 421

Linear Programming

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Linear Programming

System of Linear Equations

Find a solution to

$$
\begin{aligned}
x_3 - x_1 &= 4\\
x_3 - 2x_2 &= 3\\
x_1 + 2x_2 + x_3 &= 7
\end{aligned}
$$

Can be solved by Gaussian elimination method in $O(n^3)$ when we have n variables/n constraints

Linear Algebra Premier

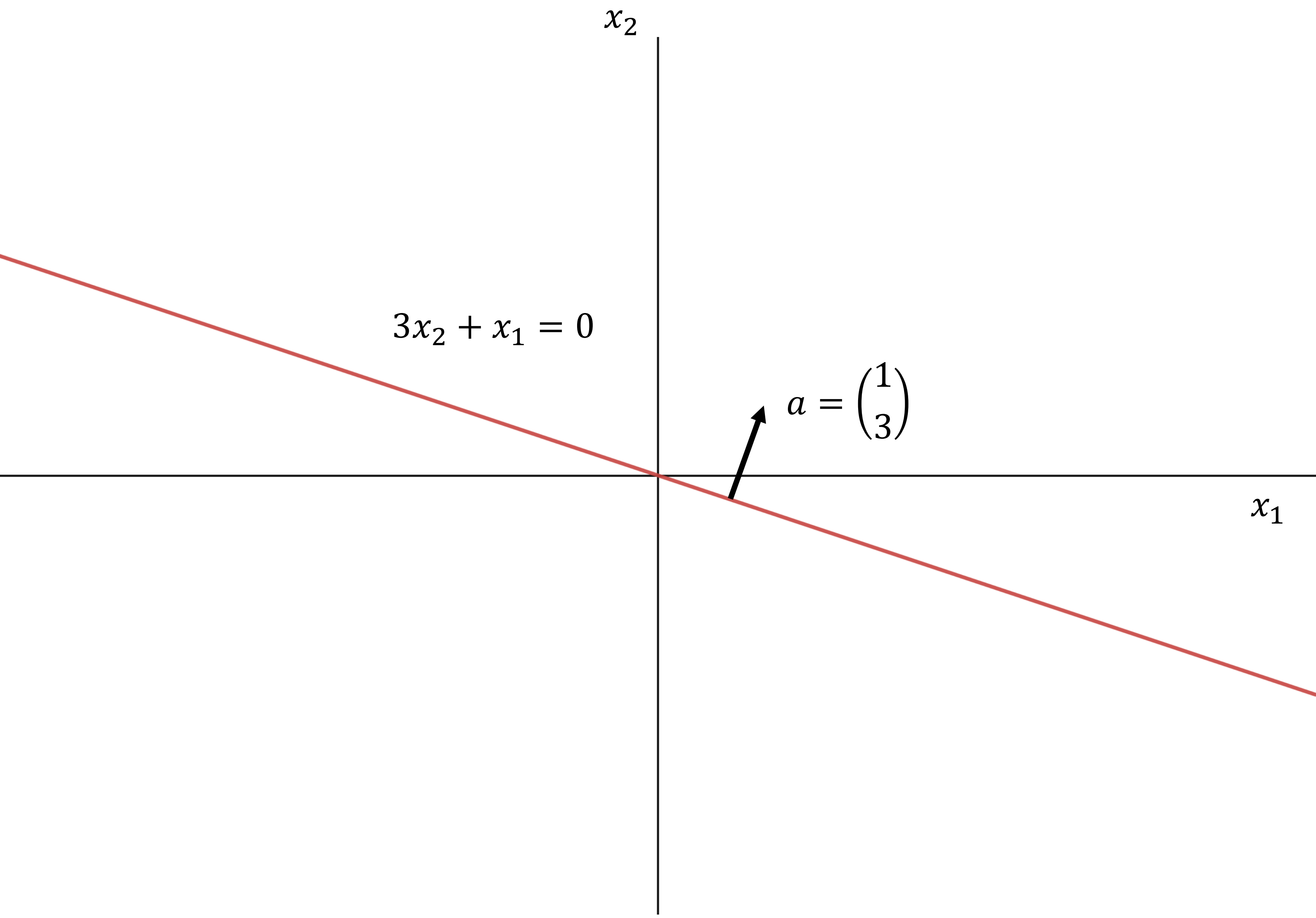
Let a be a column vector in $\mathbb{R}^d$ and x a column vector of d variables.

$$\langle a, x \rangle = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d$$

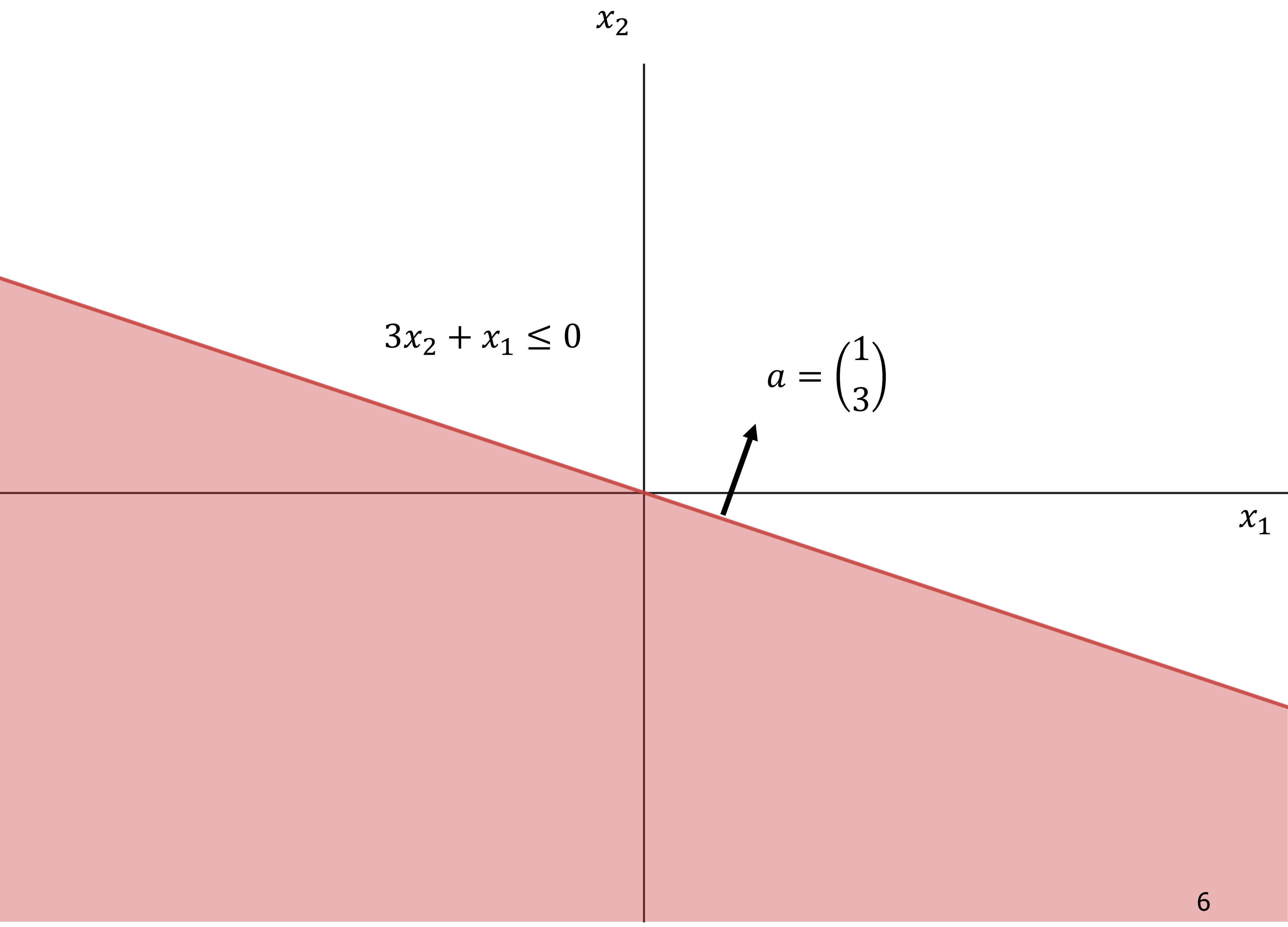
Hyperplane: A hyperplane is the set of points x such that $\langle a, x \rangle = b$ for some $b \in \mathbb{R}$

Halfspace: A halfspace is the set of points on one side of a hyperplane.

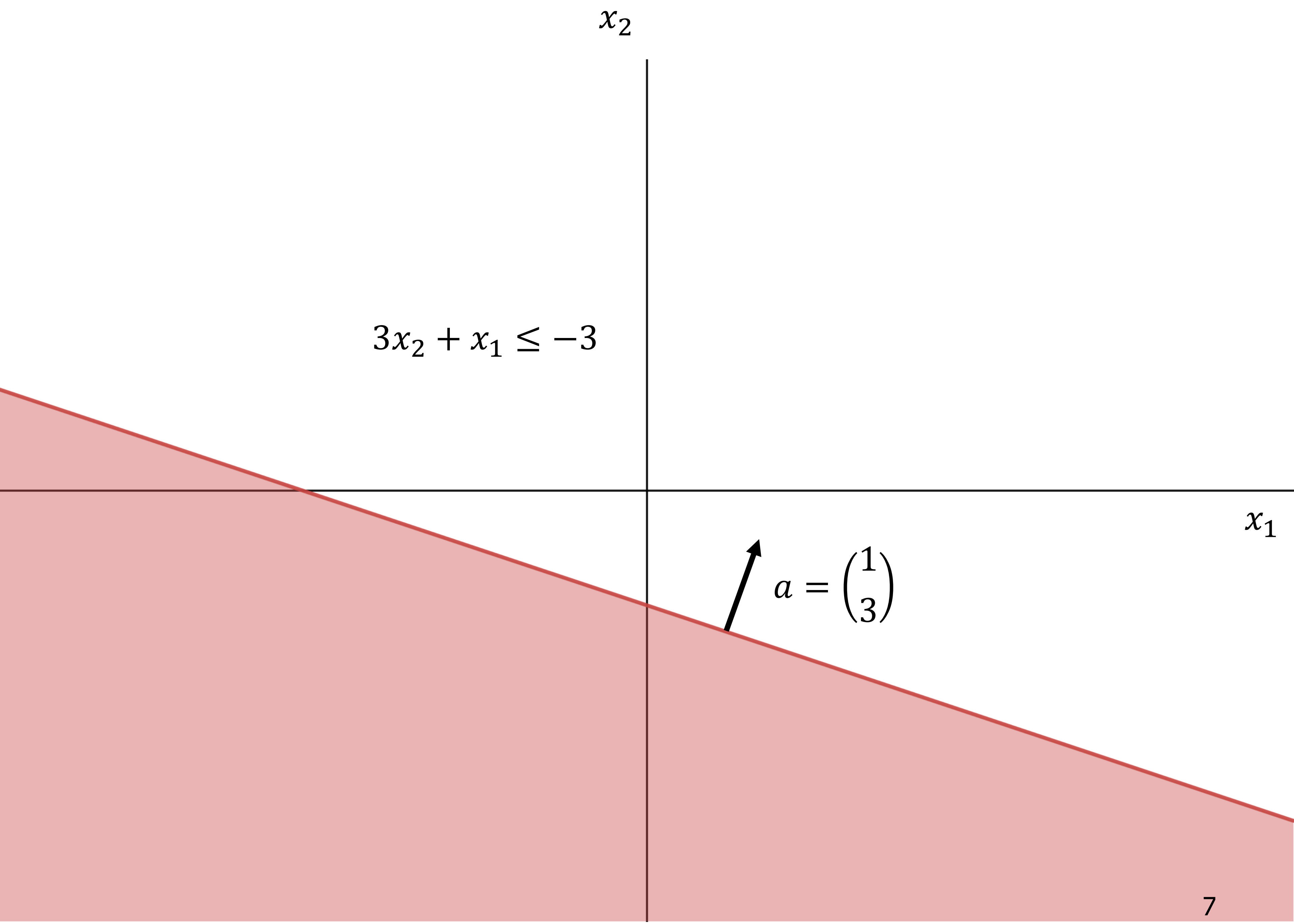
$$\{x : \langle a, x \rangle \leq b\} \quad or \quad \{x : \langle a, x \rangle \geq b\}$$



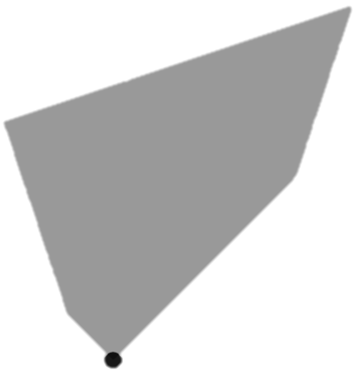
x_2
$3x_2 + x_1 = 0$
$a = \binom{1}{3}$
x_1

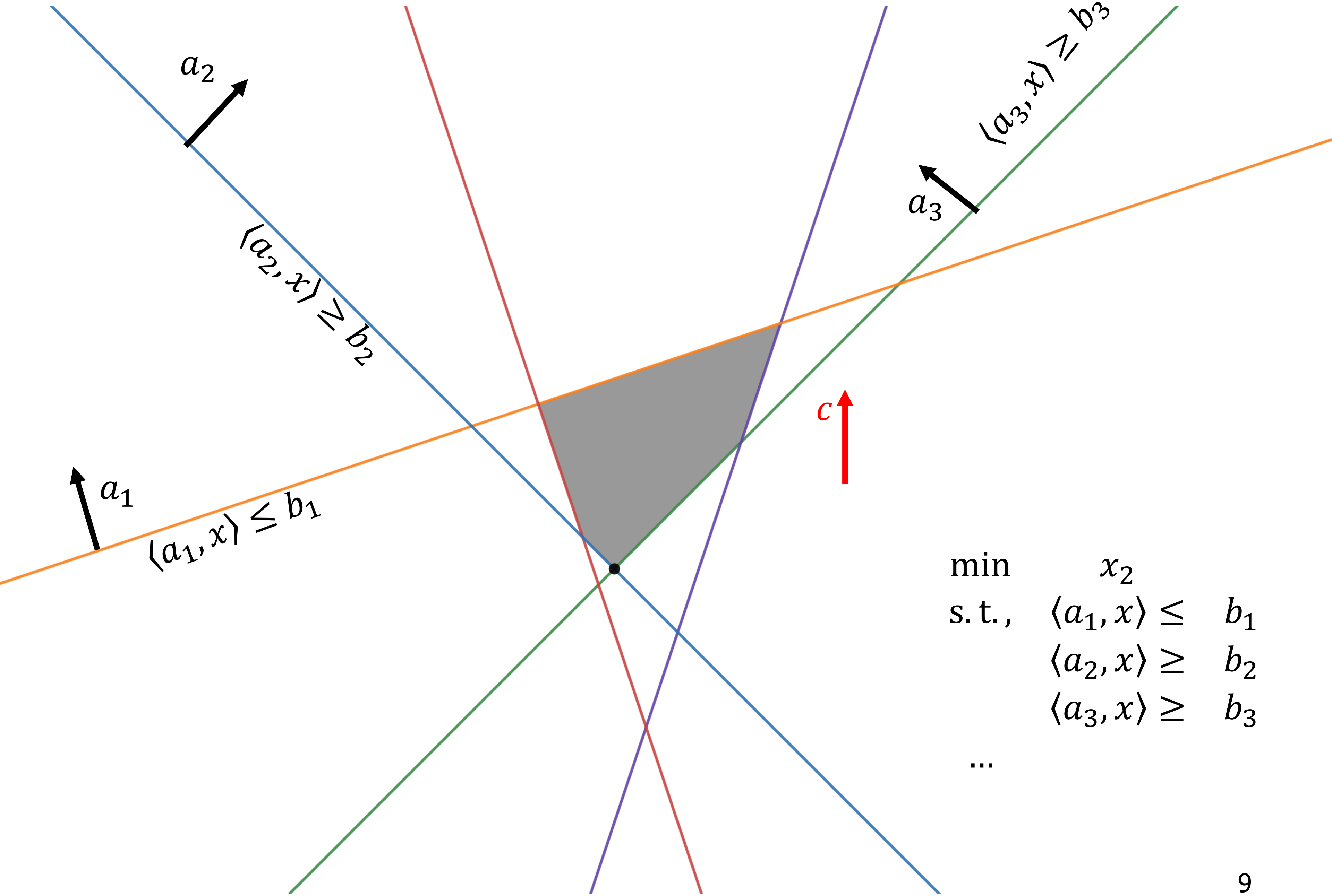


x_2
$3x_2 + x_1 \leq 0$
$a = \binom{1}{3}$
x_1


x_2
$3x_2 + x_1 \leq -3$
x_1
$a = \binom{1}{3}$

Find the smallest point in a polytope





$$
\begin{array}{lll}
\min & x_2 & \\
\text{s.t.,} & \langle a_1, x\rangle \leq & b_1 \\
 & \langle a_2, x\rangle \geq & b_2 \\
 & \langle a_3, x\rangle \geq & b_3 \\
\ldots & &
\end{array}
$$

Linear Programming

Optimize a linear function subject to linear inequalities

min

is linear $\langle c, x \rangle$

↳ cost vector

obj

$$
\begin{aligned}
\max\quad & \boxed{3x_1 - 4x_3} \\
s.t.,\quad & x_1 + x_2 \le 5 \\
& \boxed{x_3 - x_1 = 4} \rightarrow \text{equalities} \\
& \boxed{x_3 - x_2 \ge -5} \rightarrow \text{ineq} \\
& \underbrace{x_1, x_2, x_3 \ge 0}_{\text{non-neg const}}
\end{aligned}
$$

- We can have equalities and inequalities,
- We can have a linear objective functions

Linear Algebra Premier

Let a be a column vector in $\mathbb{R}^d$ and x a column vector of d variables.

$$\langle a, x \rangle = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \quad \Rightarrow \quad Ax = \begin{pmatrix} \langle a_1, x \rangle \\ \langle a_2, x \rangle \\ \vdots \\ \langle a_m, x \rangle \end{pmatrix}$$

$$Ax \le b \quad \Rightarrow \quad \begin{matrix} \langle a_1, x \rangle \le b_1 \\ \langle a_2, x \rangle \le b_2 \\ \vdots \\ \langle a_m, x \rangle \le b_m \end{matrix}$$

Linear Programming Standard Form

$$\begin{aligned} \max \quad & \langle c, x \rangle \\ s.t., \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

no equalities

you can turn any LP into standard form.

Transforming to Standard Form

$$\begin{aligned} &\min && y_1 - 2y_2 \\ &s.t., && y_1 + 2y_2 = 3 \\ &&& y_1 - y_2 \geq 1 \\ &&& y_1, y_2 \geq 0 \end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned} &\max && -y_1 + 2y_2 \\ &s.t., && y_1 + 2y_2 \leq 3 \\ &&& -(y_1 + 2y_2) \leq -3 \\ &&& -(y_1 - y_2) \leq -1 \\ &&& y_1, y_2 \geq 0 \end{aligned}$$

$$\begin{aligned} &\max && y_1 \\ &s.t., && y_1 + y_2 \leq 3 \\ &&& y_2 \geq 0 \end{aligned}$$

Replace y_1 with $z_1 - z_1'$

$$\Rightarrow$$

$$\begin{aligned} &\max && z_1 - z_1' \\ &s.t., && (z_1 - z_1') + y_2 \leq 3 \\ &&& z_1, z_1', y_2 \geq 0 \end{aligned}$$

Applications of Linear Programming

Generalizes: Ax=b, 2-person zero-sum games, shortest path, max-flow, matching, multicommodity flow, MST, min weighted arborescence, …

Why significant?

- We can solve linear programming in polynomial time.
- Useful for approximation algorithms
- We can model many practical problems with a linear model and solve it with linear programming

Linear Programming in Practice:

- There are very fast implementations: IBM CPLEX, Gorubi in Python, CVX in Matlab, ….
- CPLEX can solve LPs with millions of variables/constraints in minutes

Example 1: Diet Problem

Suppose you want to schedule a diet for yourself. There are four category of food: veggies, meat, fruits, and dairy. Each category has its own (p)rice, (c)alory and (h)appiness per pound:

	veggies	meat	fruits	dairy
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

Linear Modeling: Consider a linear model: If we eat 0.5lb of meat, 0.2lb of fruits we will be $0.5\ h_m + 0.2\ h_f$ happy

- You should eat 1500 calories to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

Diet Problem by LP

- You should eat 1500 calories to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

	veggies	meat	fruits	dairy
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

$$\begin{aligned} \max \quad & x_v h_v + x_m h_m + x_f h_f + x_d h_d \\ s.t. \quad & x_v p_v + x_m p_m + x_f p_f + x_d p_d \leq 20 \\ & x_v c_v + x_m c_m + x_f c_f + x_d c_d \leq 1500 \\ & x_v, x_m, x_f, x_d \geq 0 \end{aligned}$$

#pounds of veggies, meat, fruits, dairy to eat per day

Components of a Linear Program

- Set of variables
- Bounding constraints on variables,
  - Are they nonnegative?
- Objective function
- Is it a minimization or a maximization problem
- LP Constraints, make sure they are linear
  - Is it an equality or an inequality?

Example 2: Max Flow

Define the set of variables

- For every edge e let x_e be the flow on the edge e

Put bounding constraints on your variables

- $x_e \geq 0$ for all edge e (The flow is nonnegative)

Write down the constraints

- $x_e \leq c(e)$ for every edge e, (Capacity constraints)
- $\sum_{e \text{ out of } v} x_e = \sum_{e \text{ in to } v} x_e \quad \forall v \neq s, t$ (Conservation constraints)

Write down the objective function

- $\sum_{e \text{ out of } s} x_e$

Decide if it is a minimize/maximization problem

- max

Example 2: Max Flow

$$\begin{aligned}
\max \quad & \sum_{e \text{ out of } s} x_e \\
s.t. \quad & \sum_{e\ out\ of\ v} x_e = \sum_{e\ in\ to\ v} x_e && \forall v \neq s, t \\
& x_e \leq c(e) && \forall e \\
& x_e \geq 0 && \forall e
\end{aligned}$$

Q: Do we get exactly the same properties as Ford Fulkerson?
A: Not necessarily, the max-flow may not be integral

Example 3: Min Cost Max Flow

Suppose we can route 100 gallons of water from s to t.
But for every pipe edge e we have to pay $p(e)$ for each gallon of water that we send through e.

Goal: Send 100 gallons of water from s to t with minimum possible cost

$$
\begin{aligned}
\min \quad & \sum_{e \in \mathrm{E}} p(e) \cdot x_e \\
s.t. \quad & \sum_{e \text{ out of } v} x_e = \sum_{e\ in\ to\ v} x_e \qquad && \forall v \neq s, t \\
& \sum_{e \text{ out of } s} x_e = 100 \\
& x_e \leq c(e) && \forall e \\
& x_e \geq 0 && \forall e
\end{aligned}
$$

Linear Programming and Approximation Algorithms

Integer Program for Vertex Cover

Given a graph G=(V,E) with costs c_v on the vertices. Find a vertex cover of G with minimum cost, i.e., $\min \sum_{v \in S} c_v$

Write LP with Integrality Constraint:

- Variables: One variable x_v for each vertex v
- Bound: $x_v \in \{0,1\}$
- Edge cover Constraints: $x_u + x_v \geq 1$ for every edge $(u, v) \in E$
- Obj: $\min \sum_v c_v x_v$

IP for Vertex Cover

$$
\begin{aligned}
&min && \sum_{v} c_v x_v \\
&s.t., && x_v + x_u \geq 1 \quad \forall (u,v) \in E \\
& && x_v \in \{0,1\} \qquad \forall v \in V
\end{aligned}
$$

IP is NP-complete general!
But there are fast algorithms in practice that often work

Fact: T
min vertex cover.

Pf:

- First, any vertex cover S, $x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{o.w.} \end{cases}$ is feasible
- For any feasible solution x, the $S = \{v : x_v = 1\}$ is a vertex cover

LP Relaxation Vertex Cover

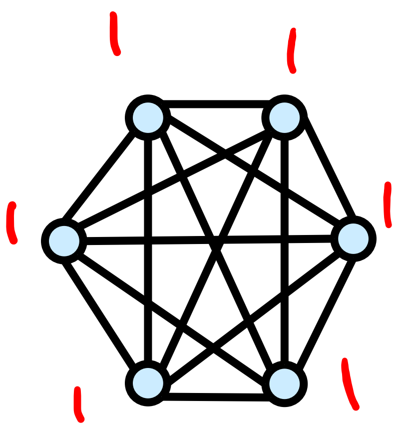
$$
\begin{aligned}
min \quad & \sum_v c_v x_v \\
s.t., \quad & x_v + x_u \geq 1 \quad && \forall (u,v) \in E \\
& 0 \leq x_v \leq 1 && \forall v \in V
\end{aligned}
$$

Fact: OPT-LP ≤ Min Vertex Cover
Pf: Min vertex cover is a feasible solution of the LP

Q: Can we hope to get an integer solution?

Bad Optimum solutions

$$
\begin{aligned}
min \quad & \sum_{v} c_v x_v \\
s.t., \quad & x_v + x_u \geq 1 \quad \forall (u,v) \in E \\
& 0 \leq x_v \leq 1 \qquad \forall v \in V
\end{aligned}
$$



K_n complete graph

A feasible solution:
Set $x_v = 0.5$ for all v
in the complete graph

If $c_v = 1$ for all v, then
Min vertex cover=$n - 1$
But OPT LP=n/2.

Approximation Alg for Vertex Cover

Given a graph G=(V,E) with costs c_v on the edges. Find a vertex cover of G with minimum cost, i.e., $\min \sum_{v \in S} c_v$

Thm: There is a 2-approximation Alg for weighted vertex cover.

ALG: Solve LP. Let $S = \{v : x_v \geq 0.5\}$. Output S.

Pf: First, for every edge (u, v), $x_u + x_v \geq 1$ So at least one is in S. So, S is a vertex cover.

Second,

$$\sum_{v \in S} c_v \leq \sum_{v \in S} c_v (2x_v) \leq 2\mathrm{OPTLP} \leq 2 \text{ Min Vertex Cov}$$

Duality

Intro to Duality

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.,} \quad & x_1 + 3x_2 \le 2 \\ & 2x_1 + 2x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Optimum solution: $x_1 = 5/4$ and $x_2 = 1/4$ with value $x_1 + 2x_2 = 7/4$
How can you prove an upper-bound on the optimum?

First attempt: Since $x_1, x_2 \ge 0$

$$x_1 + 2x_2 \le x_1 + 3x_2 \le 2$$

Second attempt:

$$x_1 + 2x_2 \le \frac{2}{3}(x_1 + 3x_2) + \frac{1}{3}(2x_1 + 2x_2) \le \frac{2}{3}(2) + \frac{1}{3}(3) = \frac{7}{3}$$

Third attempt:

$$x_1 + 2x_2 \le \frac{1}{2}(x_1 + 3x_2) + \frac{1}{4}(2x_1 + 2x_2) \le \frac{1}{2}(2) + \frac{1}{4}(3) = \frac{7}{4}$$

Dual Certificate

$$\begin{array}{lcr} max & x_1 + 2x_2 & \\ s.t., & x_1 + 3x_2 \leq 2 & y_1 \\ & 2x_1 + 2x_2 \leq 3 & y_2 \\ & x_1, x_2 \geq 0 & \end{array}$$

Goal: Minimize $2y_1 + 3y_2$

But, we must make sure the sum of the LHS is at most objective, i.e.,

$$x_1 + 2x_2 \leq y_1(x_1 + 3x_2) + y_2(2x_1 + 2x_2)$$

In other words,

$$1 \leq 1 \cdot y_1 + 2 \cdot y_2$$
$$2 \leq 3 \cdot y_1 + 2 \cdot y_2$$

Finally, $y_1, y_2 \geq 0$ (else the direction of inequalities change)

Dual Program

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.,} \quad & x_1 + 3x_2 \le 2 \\ & 2x_1 + 2x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

OPT: $x_1 = 5/4$ and $x_2 = 1/4$
Value $7/4$

$$\begin{aligned} \min \quad & 2y_1 + 3y_2 \\ \text{s.t.,} \quad & y_1 + 2y_2 \ge 1 \\ & 3y_1 + 2y_2 \ge 2 \\ & y_1, y_2 \ge 0 \end{aligned}$$

OPT: $y_1 = 1/2$ and $y_2 = 1/4$
Value $7/4$