CSE421: Design and Analysis of Algorithms	Shayan Oveis Gharan
Lecturer:	Lecture LP relaxation for Vertex Cover

In the min cost vertex cover problem we are given an undirected graph $G = (V, E)$ with a set of non-negative weights $c_v \geq 0$ for all $v \in V$ and we want to find a vertex cover of minimum cost where the cost of a vertex cover $S \subseteq V$ is

$$\sum_{v \in S} c_v.$$

Lemma 1. *The following linear program is an LP relaxation for the min cost vertex cover problem.*

$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v & \\ s.t., \quad & x_u + x_v \geq 1 & (u, v) \in E \\ & x_v \geq 0 & \forall v \in V. \end{aligned}$$

In other words, let OPT be the optimum of the min vertex cover problem and let OPT-LP be the optimum of the above linear program then we have OPT-LP $\leq$ OPT.

Proof Given an arbitrary graph G let S be the minimum cost vertex of G, i.e., $\sum_{v \in S} c_v = OPT$. We need to show that OPT-LP $\leq \sum_{v \in S} c_v$.

We consider a feasible solution to the vertex cover problem: Let $x_v = 1$ for all $v \in S$ and $x_v = 0$ otherwise. We claim that x is a feasible solution of the LP since both constraints are satisfied: (i) Since S is a vertex cover, for any edge $(u, v) \in E$ at least one of u, v must be in S. Therefore $x_u + x_v \geq 1$ and (ii) for any $v \in V$, $x_v \in \{0, 1\}$ so it is non-negative, i.e., $x_v \geq 0$. Furthermore, by the definition of x, $\sum_{v \in S} c_v = \sum_v x_v c_v$. Finally, since x is a **candidate** for the optimum of the LP either x is the optimum or the optimum is smaller, i.e.,

$$\text{OPT-LP} \leq \sum_v c_v x_v = \text{OPT}$$

as desired. ■