CSE 421

Network Flow/Linear Programming

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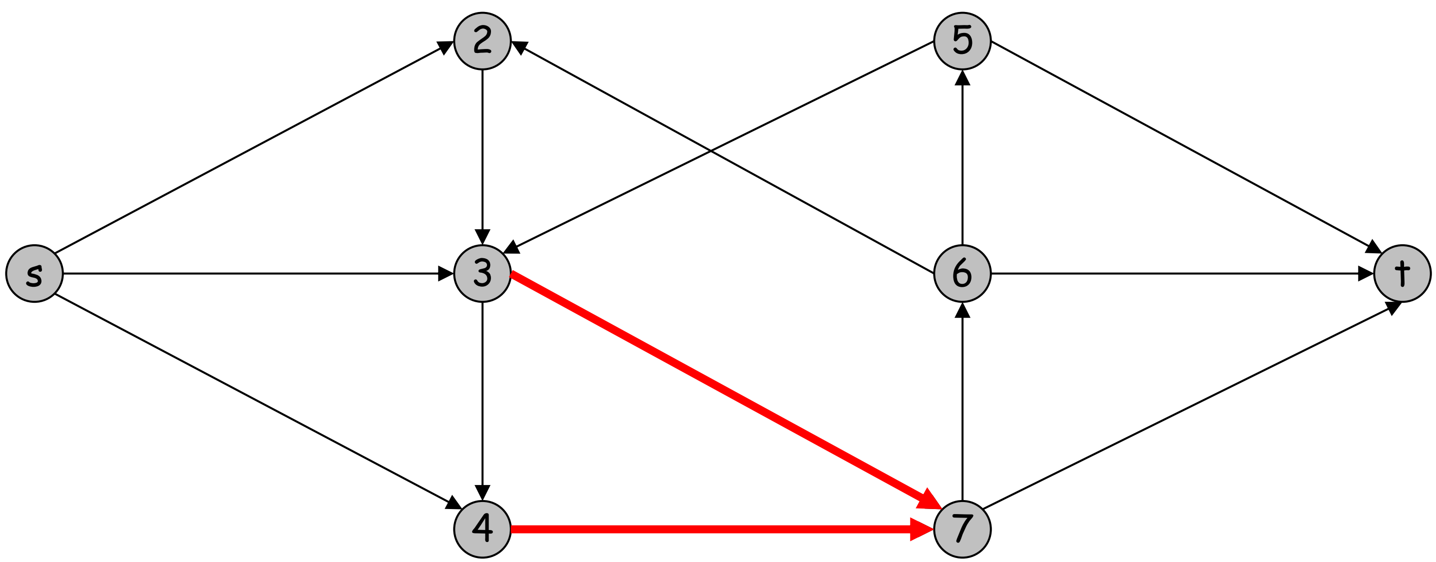
Network Connectivity

Network Connectivity

Given a digraph G = (V, E) and two nodes s and t, find min number of edges whose removal disconnects t from s.

Def. A set of edges $F \subseteq E$ disconnects t from s if all s-t paths uses at least one edge in F.

Ex: In testing network reliability



Network Connectivity using Min Cut

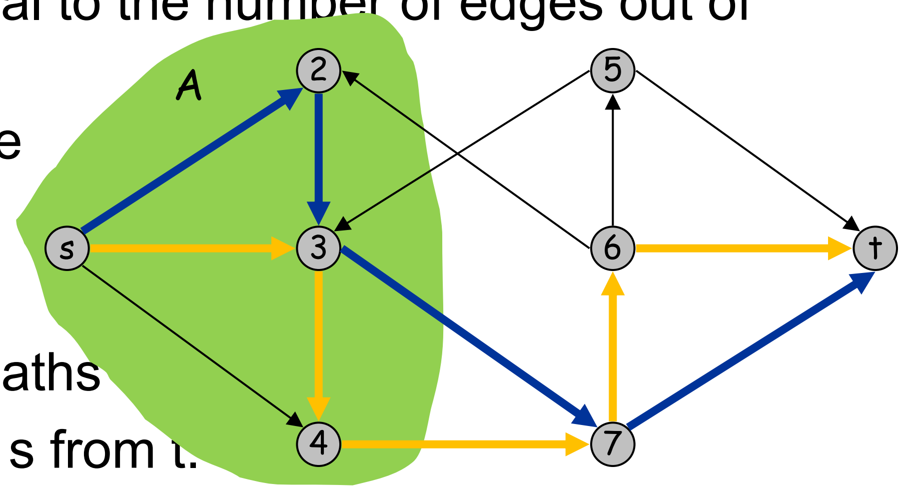
Thm. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf.

i) We showed that max number edge disjoint s-t paths = max flow.

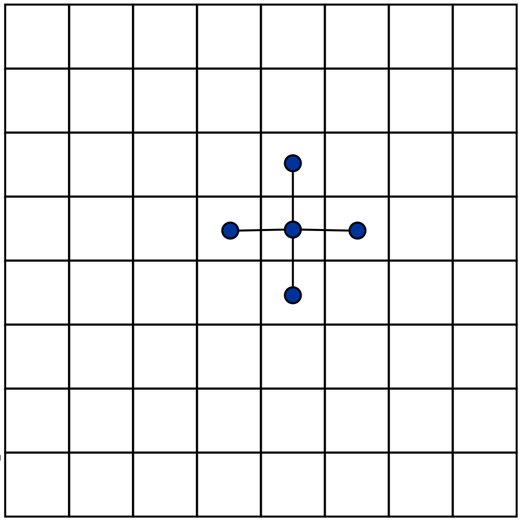
ii) Max-flow Min-cut theorem => min s-t cut = max-flow

iii) For a s-t cut (A,B), cap(A,B) is equal to the number of edges out of A. In other words, every s-t cut (A,B) corresponds to cap(A,B) edges whose removal disconnects s from t.



So, max number of edge disjoint s-t paths = min number of edges to disconnect s from t.

Foreground / background segmentation



Label each pixel as foreground/background.

- V = set of pixels, E = pairs of neighboring pixels.
- $a_i \geq 0$ is likelihood pixel i in foreground.
- $b_i \geq 0$ is likelihood pixel i in background.
- $p_{i,j} \geq 0$ is separation penalty for labeling one of i and j as foreground, and the other as background.

Goals.

Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.

Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground.

Find partition (A, B) that maximizes:

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Foreground

Background

Image Seg: Min Cut Formulation

Difficulties:

- Maximization (as opposed to minimization)
- No source or sink
- Undirected graph

Step 1: Turn into Minimization

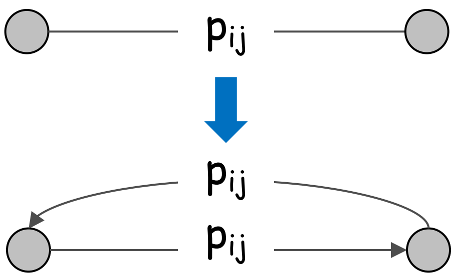
Maximizing
$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

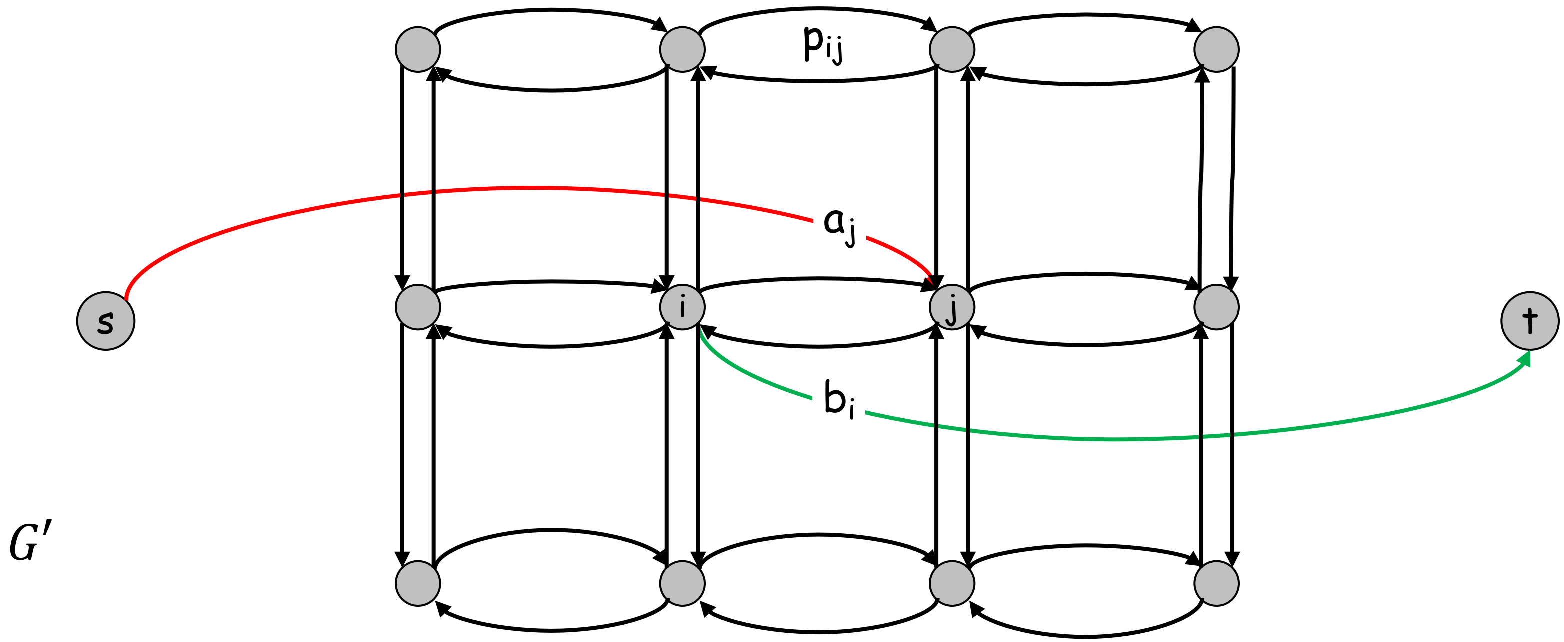
Equivalent to minimizing
$$+\sum_{i \in V} a_i + \sum_{j \in V} b_j - \sum_{i \in A} a_i - \sum_{j \in B} b_j + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Equivalent to minimizing
$$+\sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Min cut Formulation (cont'd)

G' = (V', E').
Add s to correspond to foreground;
Add t to correspond to background
Use two anti-parallel edges
instead of undirected edge.



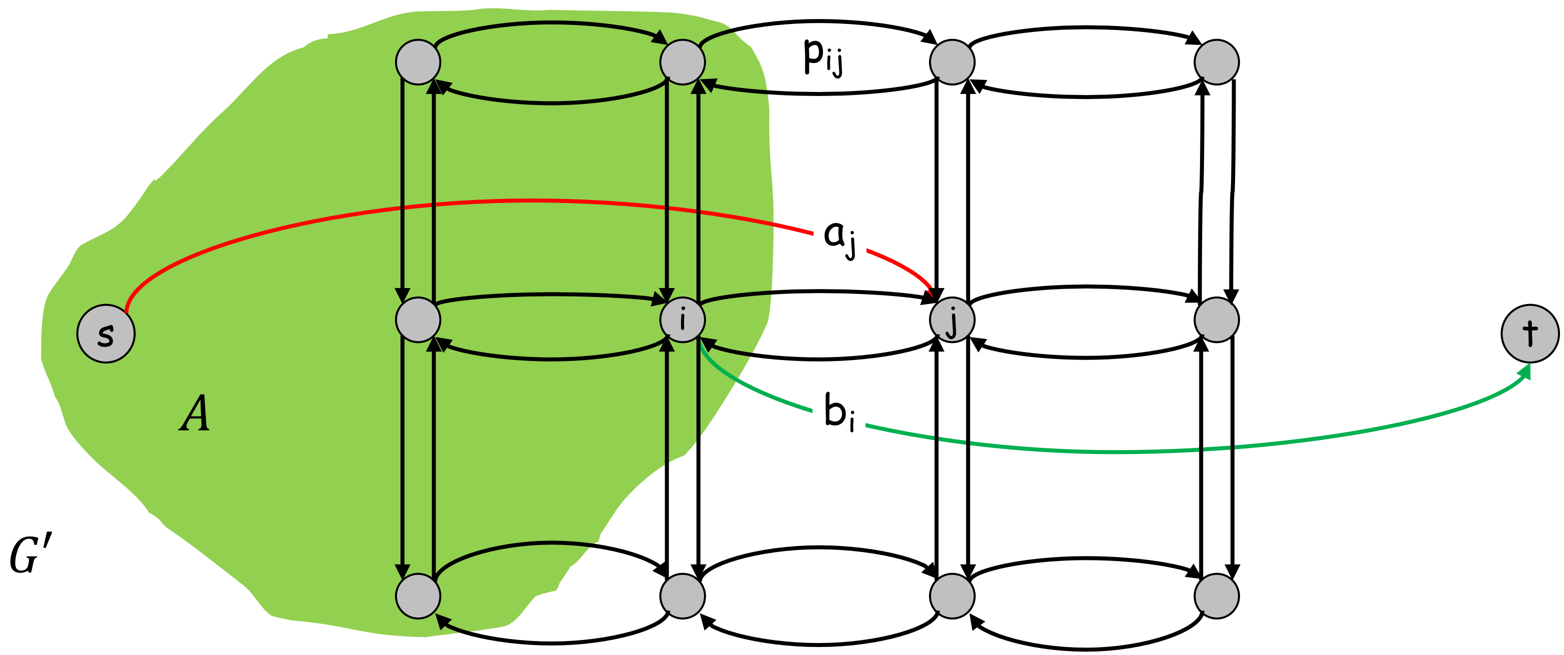


Min cut Formulation (cont'd)

Consider min cut (A, B) in G'. (A = foreground.)

$$cap(A,B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Precisely the quantity we want to minimize.



Linear Programming

System of Linear Equations

Find a solution to

$$x_3 - x_1 = 4$$
$$x_3 - 2x_2 = 3$$
$$x_1 + 2x_2 + x_3 = 7$$

Can be solved by Gaussian elimination method in $O(n^3)$ when we have n variables/n constraints

Linear Algebra Premier

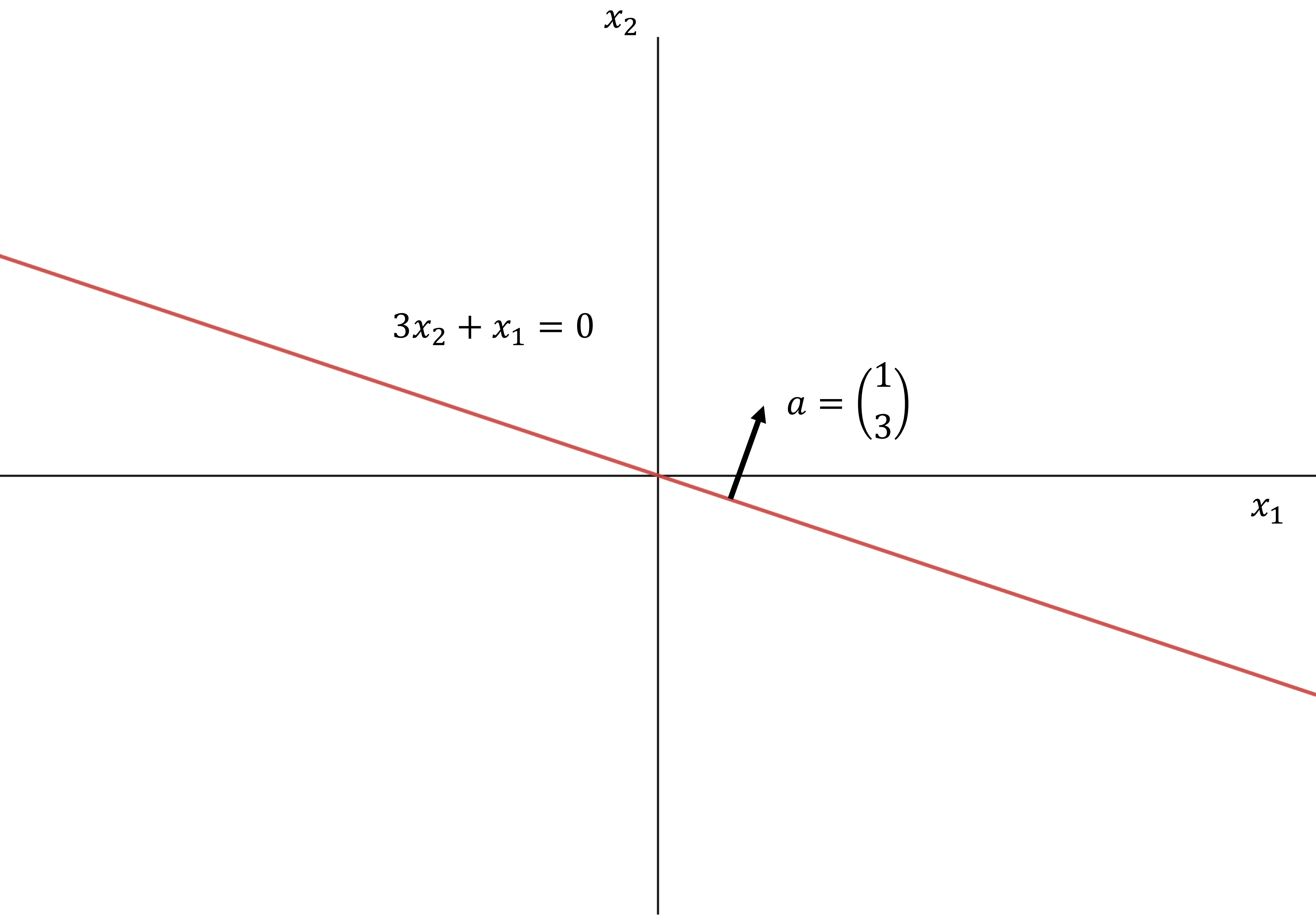
Let a be a column vector in $\mathbb{R}^d$ and x a column vector of d variables.

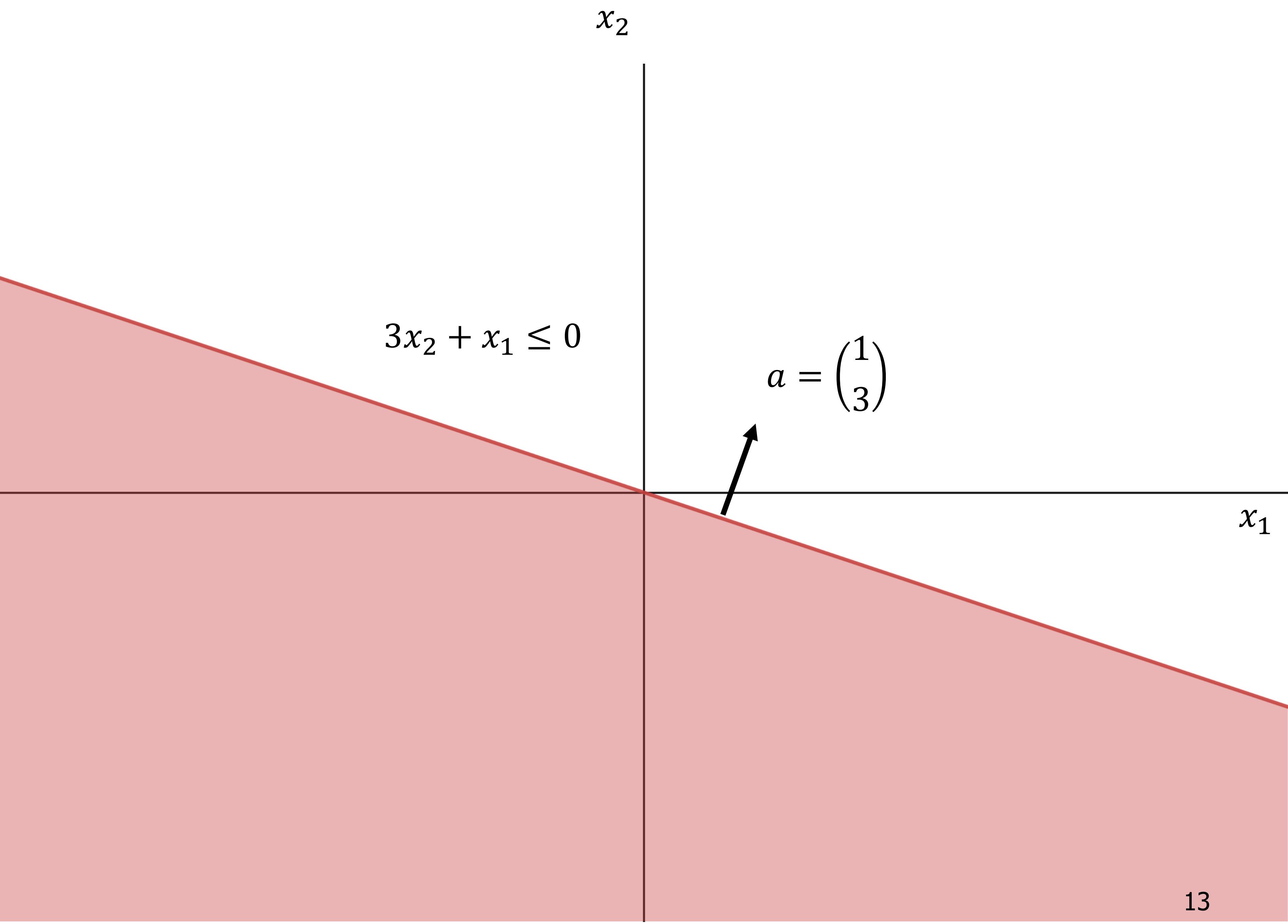
$$\langle a, x \rangle = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d$$

Hyperplane: A hyperplane is the set of points x such that $\langle a, x \rangle = b$ for some $b \in \mathbb{R}$

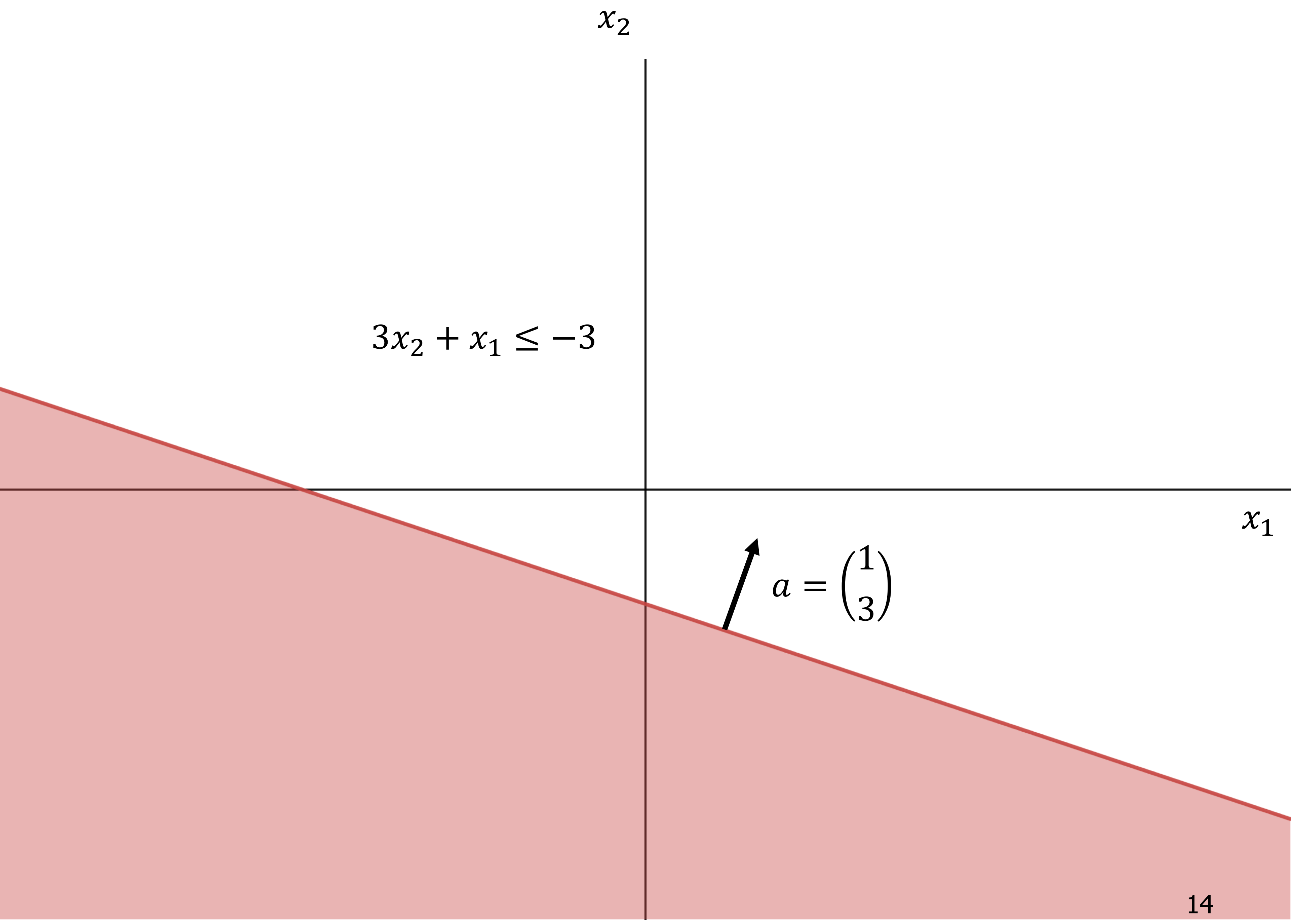
Halfspace: A halfspace is the set of points on one side of a hyperplane.

$$\{x: \langle a, x \rangle \leq b\} \quad or \quad \{x: \langle a, x \rangle \geq b\}$$

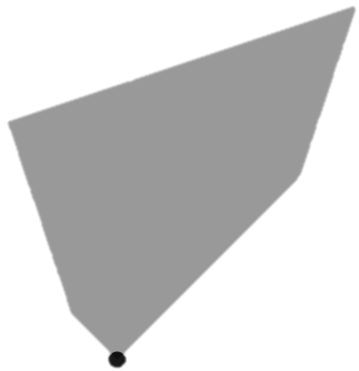

x_2
$3x_2 + x_1 = 0$
$a = \binom{1}{3}$
x_1

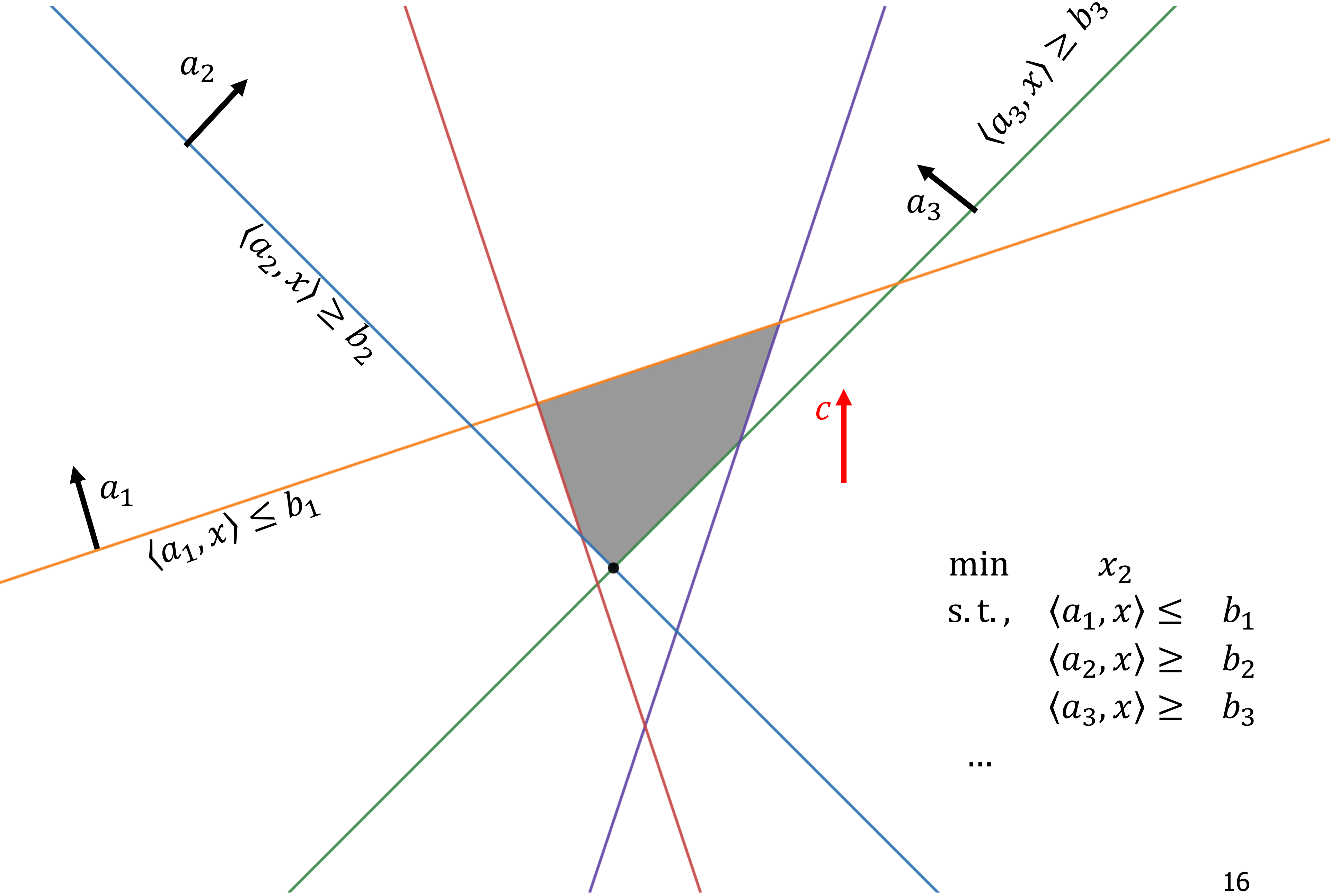


x_2
$3x_2 + x_1 \leq 0$
$a = \binom{1}{3}$
x_1


x_2
$3x_2 + x_1 \leq -3$
x_1
$a = \binom{1}{3}$

Find the smallest point in a polytope





$$
\begin{aligned}
\min \quad & x_2 \\
\text{s.t.,} \quad & \langle a_1, x \rangle \leq b_1 \\
& \langle a_2, x \rangle \geq b_2 \\
& \langle a_3, x \rangle \geq b_3 \\
& \dots
\end{aligned}
$$

Linear Programming

Optimize a linear function subject to linear inequalities

$$
\begin{aligned}
\max \quad & 3x_1 - 4x_3 \\
s.t., \quad & x_1 + x_2 \le 5 \\
& x_3 - x_1 = 4 \\
& x_3 - x_2 \ge -5 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

- We can have equalities and inequalities,
- We can have a linear objective functions

Linear Algebra Premier

Let a be a column vector in $\mathbb{R}^d$ and x a column vector of d variables.

$$\langle a, x \rangle = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \quad \Rightarrow \quad Ax = \begin{pmatrix} \langle a_1, x \rangle \\ \langle a_2, x \rangle \\ \vdots \\ \langle a_m, x \rangle \end{pmatrix}$$

$$Ax \leq b \quad \Rightarrow \quad \begin{matrix} \langle a_1, x \rangle \leq b_1 \\ \langle a_2, x \rangle \leq b_2 \\ \vdots \\ \langle a_m, x \rangle \leq b_m \end{matrix}$$

Linear Programming Standard Form

$$\begin{aligned} max \quad & \langle c, x \rangle \\ s.t., \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Any linear program can be translated into the standard form.

$$\begin{aligned} min \quad & y_1 - 2y_2 \\ s.t., \quad & y_1 + 2y_2 = 3 \\ & y_1 - y_2 \ge 1 \\ & y_1 \ge 0 \end{aligned}$$

Replace y_2 with $z_2 - z_2'$

$$\begin{aligned} max \quad & -y_1 + 2(z_2 - z_2') \\ s.t., \quad & y_1 + 2(z_2 - z_2') \le 3 \\ & -\left(y_1 + 2(z_2 - z_2')\right) \le -3 \\ & -\left(y_1 - (z_2 - z_2')\right) \le -1 \\ & y_1, z_2, z_2' \ge 0 \end{aligned}$$

Applications of Linear Programming

Generalizes: Ax=b, 2-person zero-sum games, shortest path, max-flow, matching, multicommodity flow, MST, min weighted arborescence, …

Why significant?

- We can solve linear programming in polynomial time.
- Useful for approximation algorithms
- We can model many practical problems with a linear model and solve it with linear programming

Linear Programming in Practice:

- There are very fast implementations: IBM CPLEX, Gorubi in Python, CVX in Matlab, ….
- CPLEX can solve LPs with millions of variables/constraints in minutes

Example 1: Diet Problem

Suppose you want to schedule a diet for yourself. There are four category of food: veggies, meat, fruits, and dairy. Each category has its own (p)rice, (c)alory and (h)appiness per pound:

	veggies	**meat**	**fruits**	**dairy**
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

Linear Modeling: Consider a linear model: If we eat 0.5lb of meat, 0.2lb of fruits we will be $0.5\, h_m + 0.2\, h_f$ happy

- You should eat 1500 calories to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

Diet Problem by LP

- You should eat 1500 calaroies to be healthy
- You can spend 20 dollars a day on food.

Goal: Maximize happiness?

	veggies	meat	fruits	dairy
price	p_v	p_m	p_f	p_d
calorie	c_v	c_m	c_f	c_d
happiness	h_v	h_m	h_f	h_d

$$\begin{aligned} \max \quad & x_v h_v + x_m h_m + x_f h_f + x_d h_d \\ s.t. \quad & x_v p_v + x_m p_m + x_f p_f + x_d p_d \leq 20 \\ & x_v c_v + x_m c_m + x_f c_f + x_d c_d \leq 1500 \\ & x_v, x_m, x_f, x_d \geq 0 \end{aligned}$$

#pounds of veggies, meat, fruits, dairy to eat per day

Components of a Linear Program

- Set of variables
- Bounding constraints on variables,
  - Are they nonnegative?
- Objective function
- Is it a minimization or a maximization problem
- LP Constraints, make sure they are linear
  - Is it an equality or an inequality?

Example 2: Max Flow

Define the set of variables

- For every edge e let x_e be the flow on the edge e

Put bounding constraints on your variables

- $x_e \geq 0$ for all edge e (The flow is nonnegative)

Write down the constraints

- $x_e \leq c(e)$ for every edge e, (Capacity constraints)
- $\sum_{e \text{ out of } v} x_e = \sum_{e \text{ in to } v} x_e \quad \forall v \neq s, t$ (Conservation constraints)

Write down the objective function

- $\sum_{e \text{ out of } s} x_e$

Decide if it is a minimize/maximization problem

- max

Example 2: Max Flow

$$\begin{aligned} \max \quad & \sum_{e \text{ out of } s} x_e \\ s.t. \quad & \sum_{e \text{ out of } v} x_e = \sum_{e \text{ in to } v} x_e & \forall v \neq s, t \\ & x_e \leq c(e) & \forall e \\ & x_e \geq 0 & \forall e \end{aligned}$$

Q: Do we get exactly the same properties as Ford Fulkerson?
A: Not necessarily, the max-flow may not be integral

Example 3: Min Cost Max Flow

Suppose we can route 100 gallons of water from s to t.
But for every pipe edge e we have to pay $p(e)$ for each gallon of water that we send through e.

Goal: Send 100 gallons of water from s to t with minimum possible cost

$$
\begin{aligned}
\min \quad & \sum_{e \in \mathrm{E}} p(e) \cdot x_e \\
s.t. \quad & \sum_{e \text{ out of } v} x_e = \sum_{e\ in\ to\ v} x_e \qquad \forall v \neq s, t \\
& \sum_{e \text{ out of } s} x_e = 100 \\
& x_e \leq c(e) \qquad \forall e \\
& x_e \geq 0 \qquad \forall e
\end{aligned}
$$

Linear Programming and Approximation Algorithms

Integer Program for Vertex Cover

Given a graph G=(V,E) with costs c_v on the vertices. Find a vertex cover of G with minimum cost, i.e., $\min \sum_{v \in S} c_v$

Write LP with Integrality Constraint:

- Variables: One variable x_v for each vertex v
- Bound: $x_v \in \{0,1\}$
- Edge cover Constraints: $x_u + x_v \geq 1$ for every edge $(u, v) \in E$
- Obj: $\min \sum_v c_v x_v$

IP for Vertex Cover

$$\min \quad \sum_v c_v x_v$$

$$s.t., \quad x_v + x_u \geq 1 \quad \forall (u,v) \in E$$

$$x_v \in \{0,1\} \quad \forall v \in V$$

IP is NP-complete general!
But there are fast algorithms in practice that often work

Fact: T

min vertex cover.

Pf:

- First, any vertex cover S, $x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{o.w.} \end{cases}$ is feasible
- For any feasible solution x, the $S = \{v: x_v = 1\}$ is a vertex cover

LP Relaxation Vertex Cover

$$
\begin{aligned}
min \quad & \sum_v c_v x_v \\
s.t., \quad & x_v + x_u \geq 1 \quad \forall (u,v) \in E \\
& 0 \leq x_v \leq 1 \qquad \forall v \in V
\end{aligned}
$$

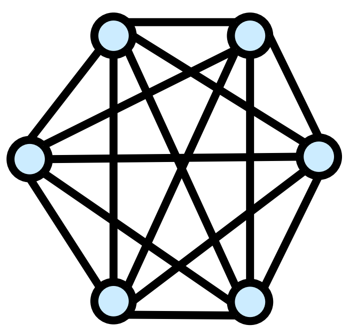
Fact: OPT-LP ≤ Min Vertex Cover

Pf: Min vertex cover is a feasible solution of the LP

Q: Can we hope to get an integer solution?

Bad Optimum solutions

$$\min \quad \sum_v c_v x_v$$
$$s.t., \quad x_v + x_u \geq 1 \quad \forall (u,v) \in E$$
$$0 \leq x_v \leq 1 \quad \forall v \in V$$



K_n complete graph

A feasible solution:
Set $x_v = 0.5$ for all v
in the complete graph

If $c_v = 1$ for all v, then
Min vertex cover=$n - 1$
But OPT LP=n/2.

Approximation Alg for Vertex Cover

Given a graph G=(V,E) with costs c_v on the edges. Find a vertex cover of G with minimum cost, i.e., $\min \sum_{v \in S} c_v$

Thm: There is a 2-approximation Alg for weighted vertex cover.

ALG: Solve LP. Let $S = \{v : x_v \geq 0.5\}$. Output S.

Pf: First, for every edge (u, v), $x_u + x_v \geq 1$ So at least one is in S. So, S is a vertex cover.

Second,

$$\sum_{v \in S} c_v \leq \sum_{v \in S} c_v (2x_v) \leq 2\mathrm{OPTLP} \leq \text{Min Vertex Cov}$$