CSE 421

Network Flows

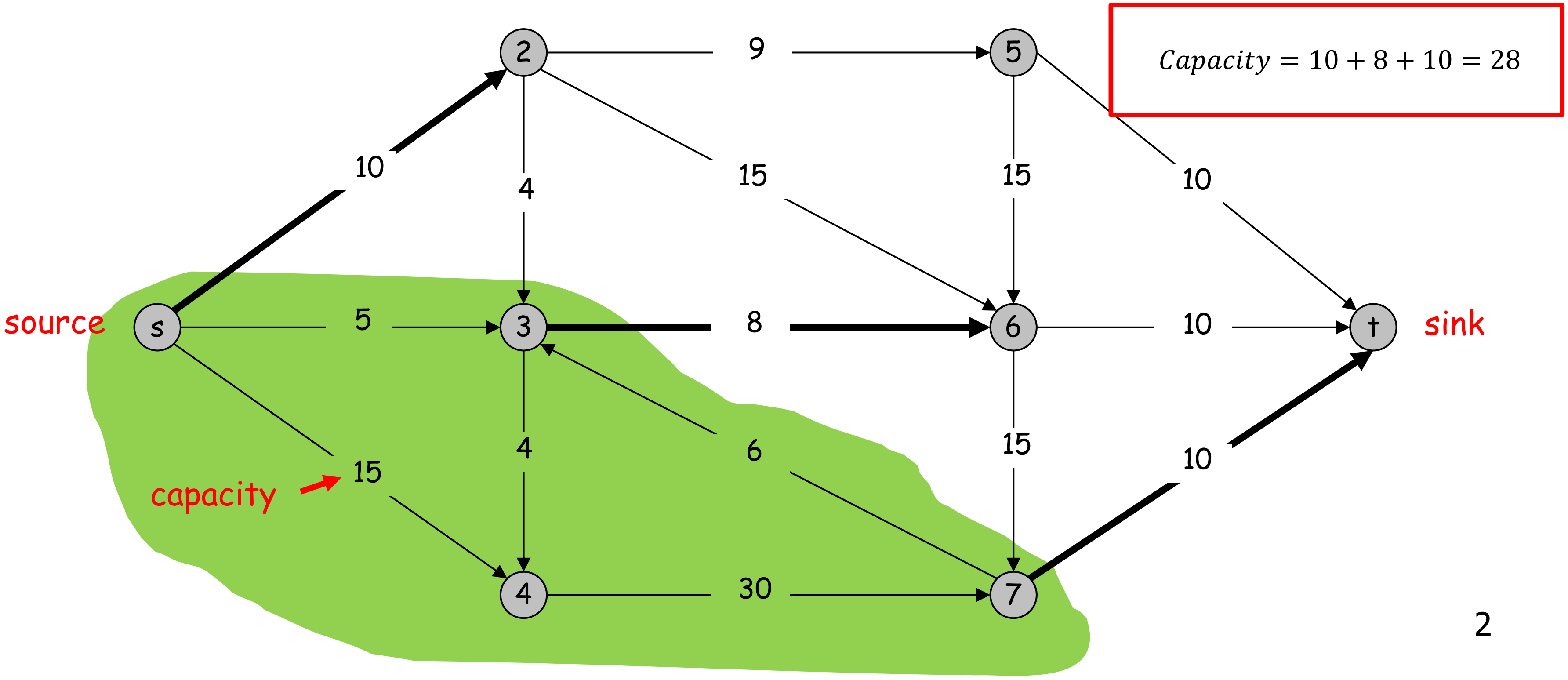
Shayan Oveis Gharan

Minimum s-t Cut Problem

Given a directed graph G = (V, E) = directed graph and two distinguished nodes: s = source, t = sink.

Suppose each directed edge e has a nonnegative capacity $c(e)$

Goal: Find a s-t cut of minimum capacity

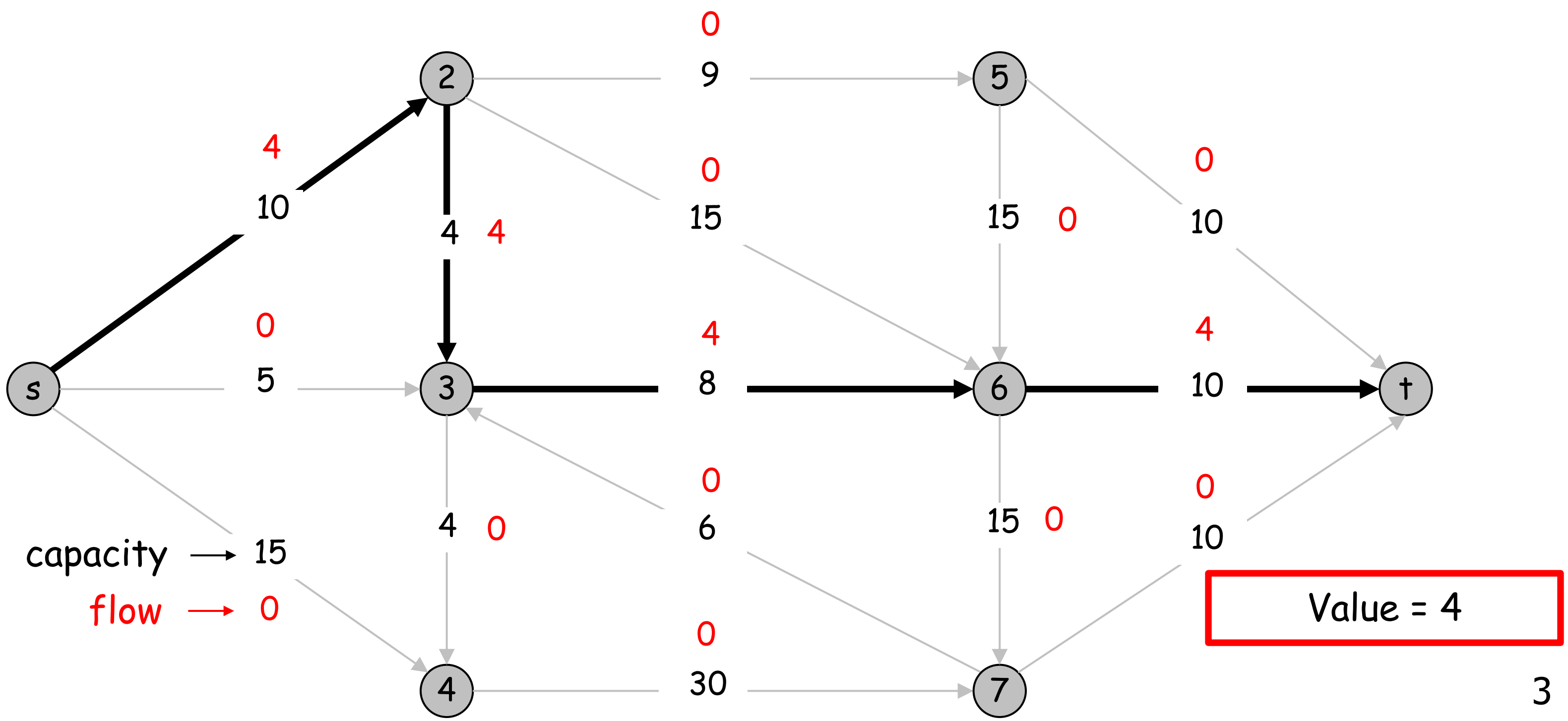


s-t Flows

Def. An s-t flow is a function that satisfies:

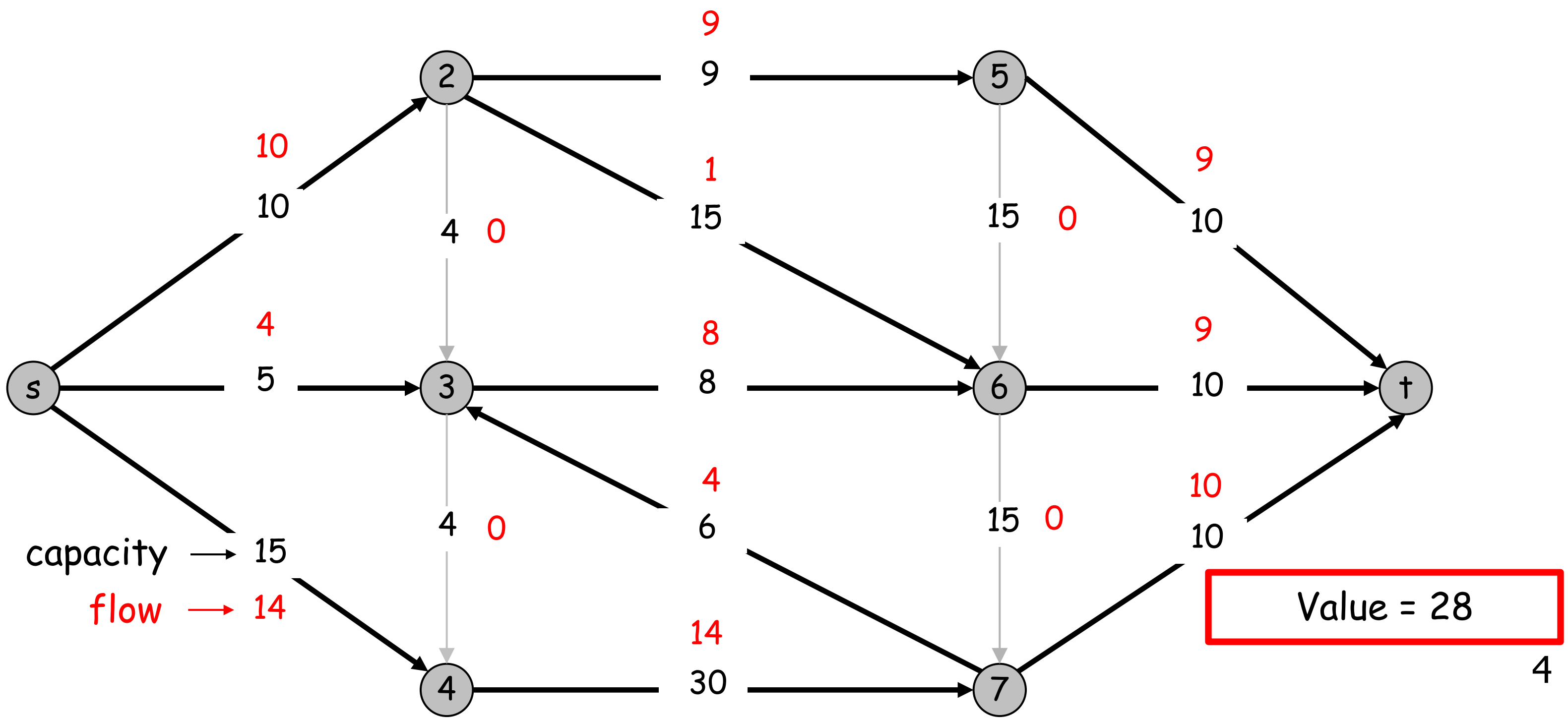
- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$



Maximum s-t Flow Problem

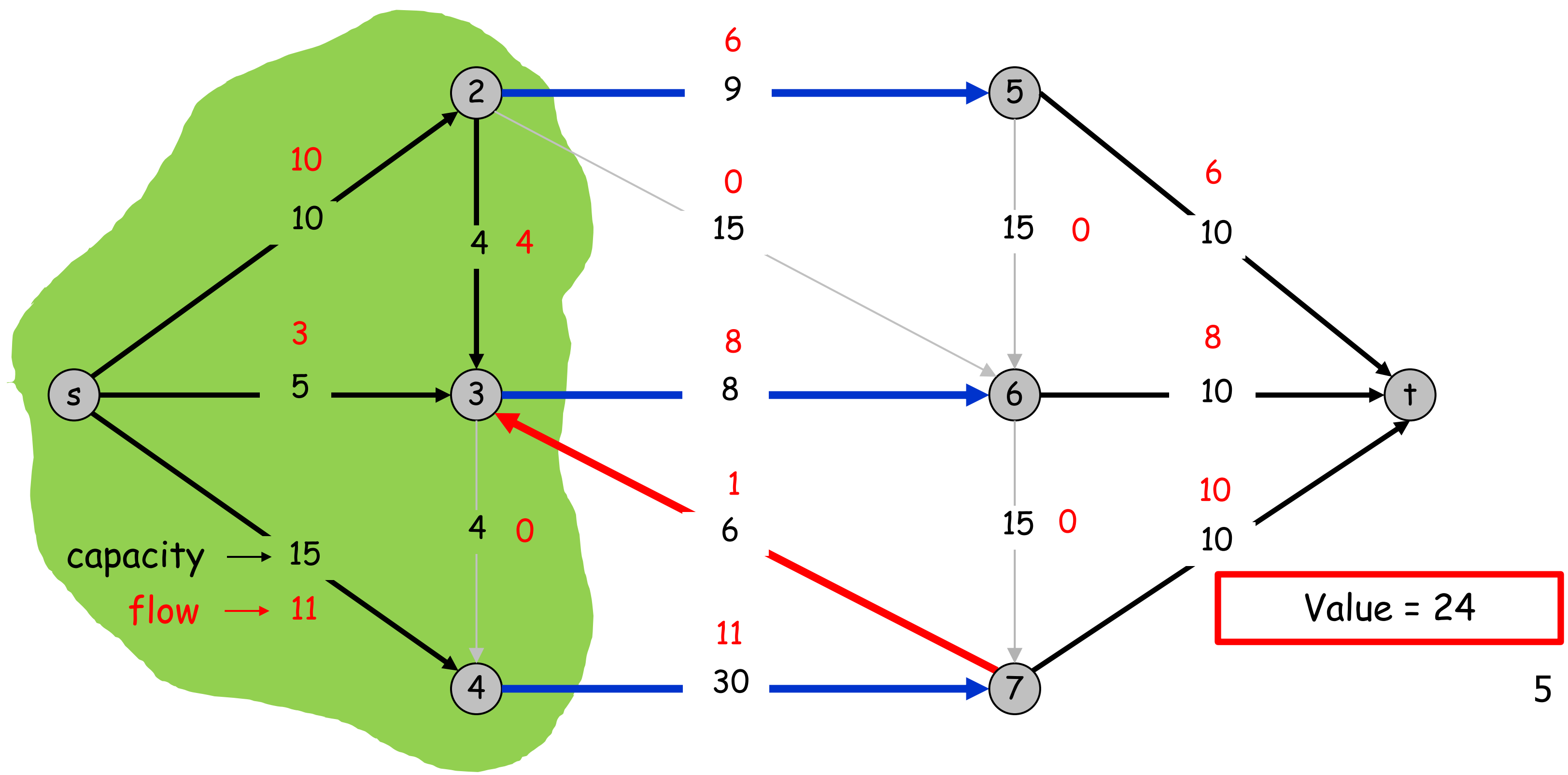
Goal: Find a s-t flow of largest value.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



Pf of Flow value Lemma

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$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

By conservation of flow, all terms except v=s are0

$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

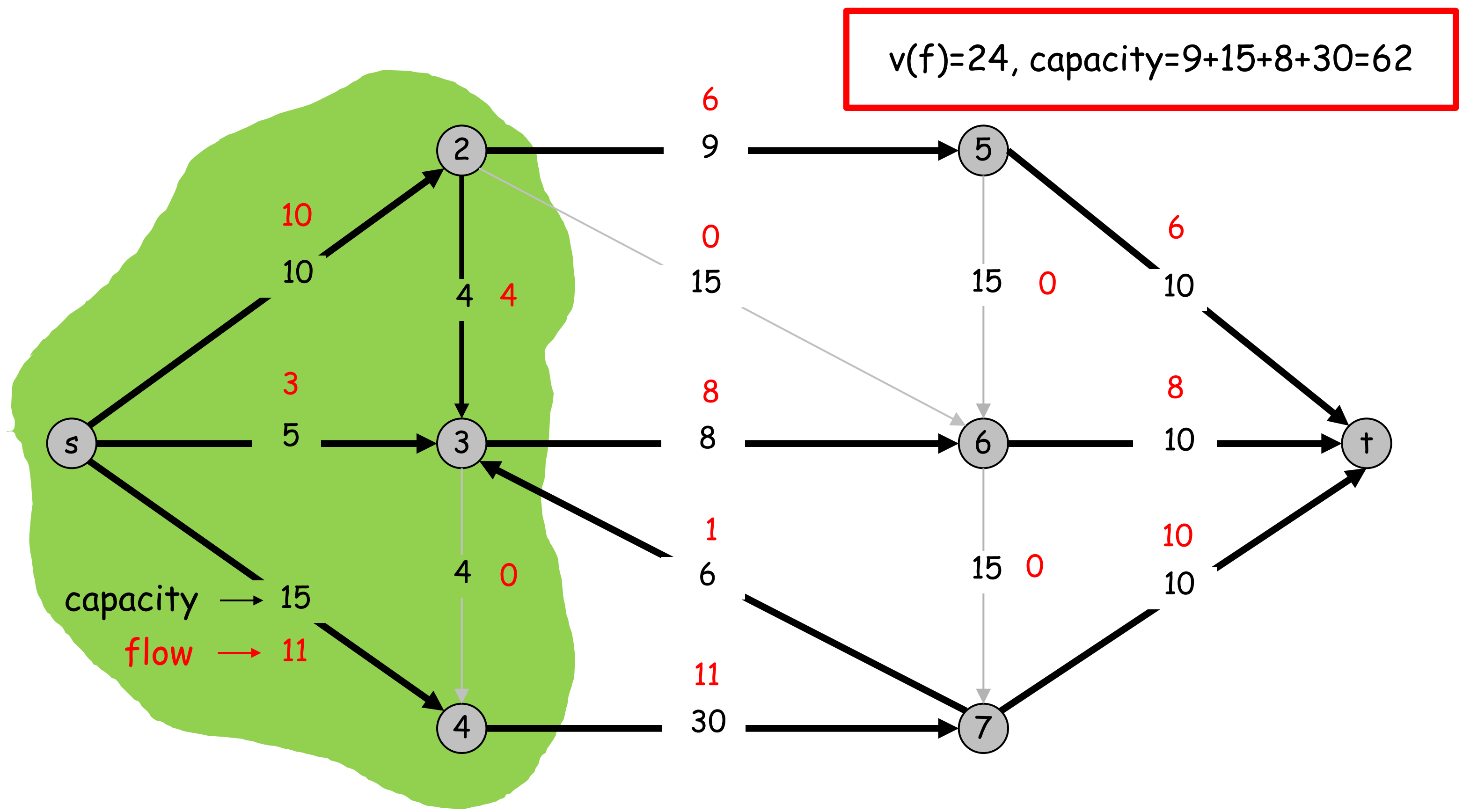
All contributions due to internal edges cancel out

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

Weak Duality of Flows and Cuts

Cut Capacity lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

$$v(f) \le cap(A, B)$$



Weak Duality of Flows and Cuts

Cut capacity lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

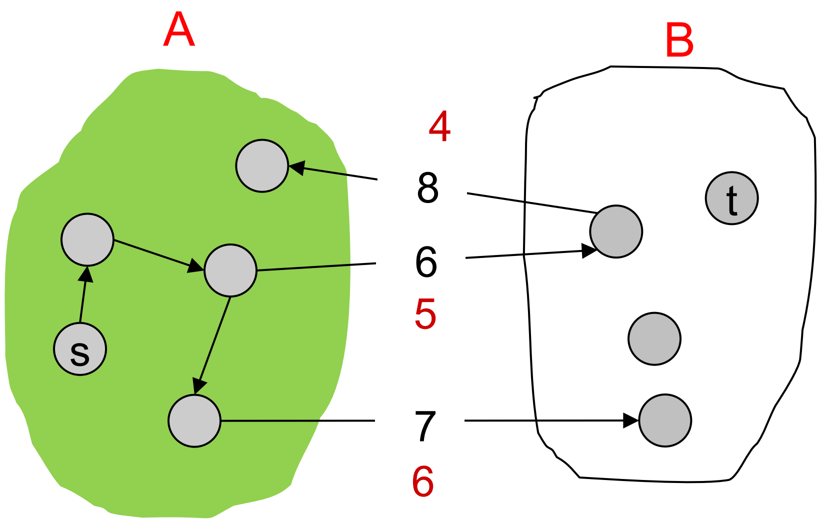
$$v(f) \leq cap(A,B)$$

Pf.

$$v(f) = \sum_{e\ out\ of\ A} f(e) - \sum_{e\ in\ to\ A} f(e)$$

$$\leq \sum_{e\ out\ of\ A} f(e)$$

$$\leq \sum_{e\ out\ of\ A} c(e) = cap(A,B)$$



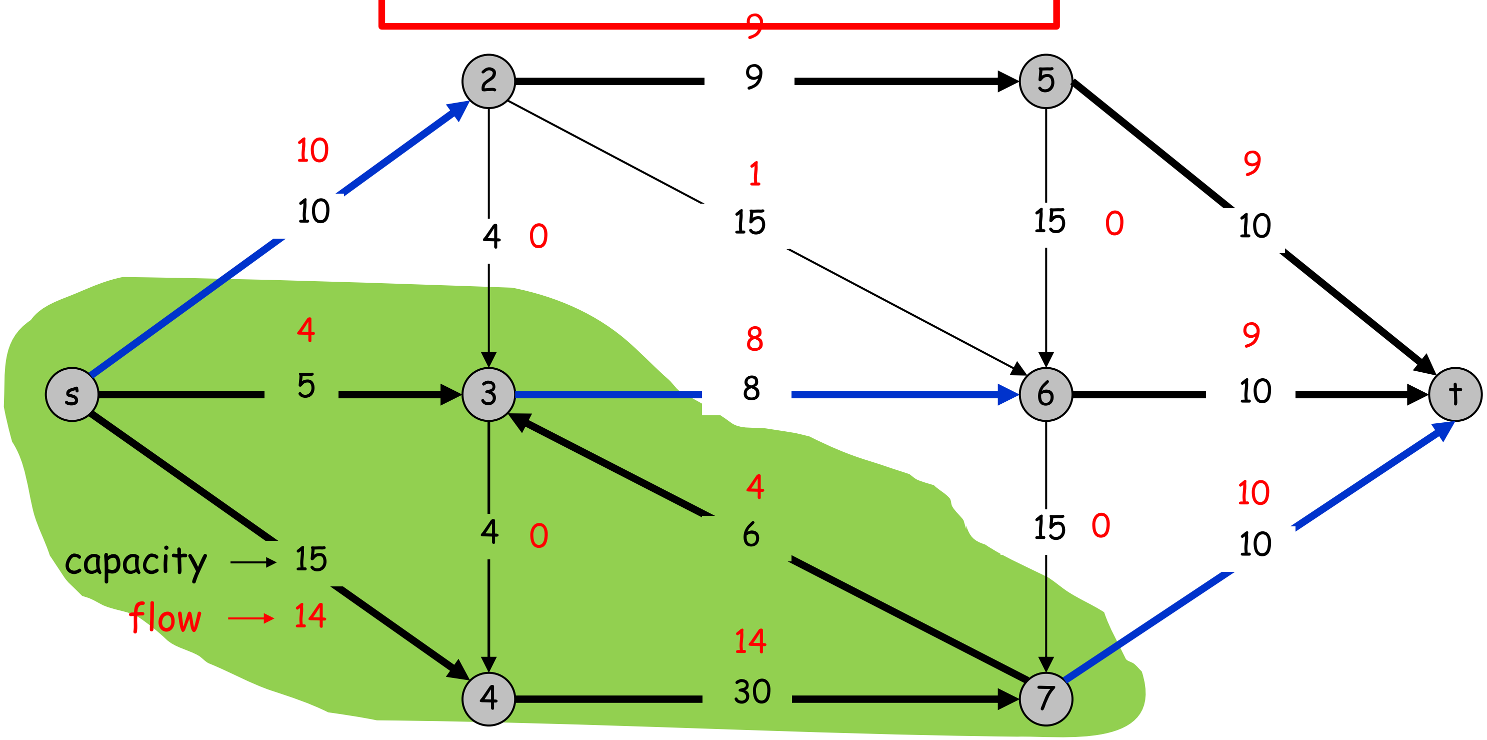
Certificate of Optimality

Corollary: Suppose there is a s-t cut (A,B) such that

$$v(f) = cap(A, B)$$

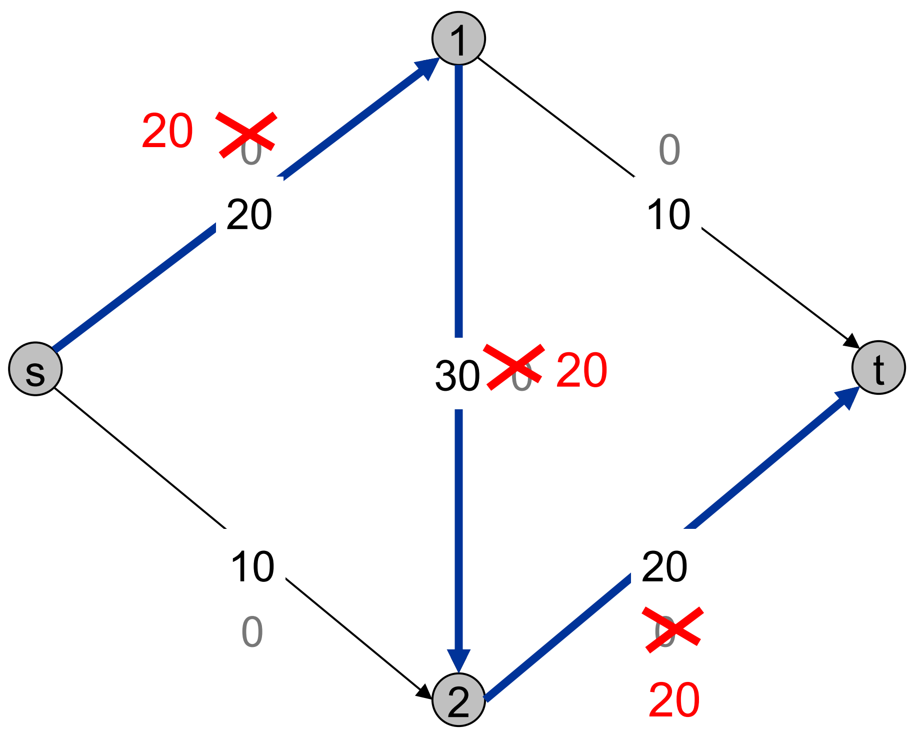
Then, f is a maximum flow and (A,B) is a minimum cut.

v(f)=28, cap(A,B)=28



A Greedy Algorithm for Max Flow

- Start with f(e) = 0 for all edge $e \in E$.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.



A Greedy Algorithm for Max Flow

- Start with f(e) = 0 for all edge e $\in$ E.
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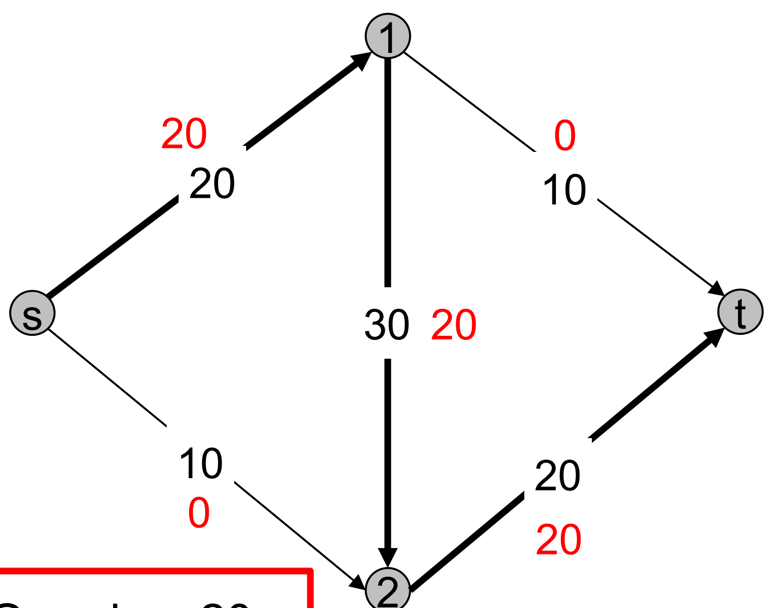
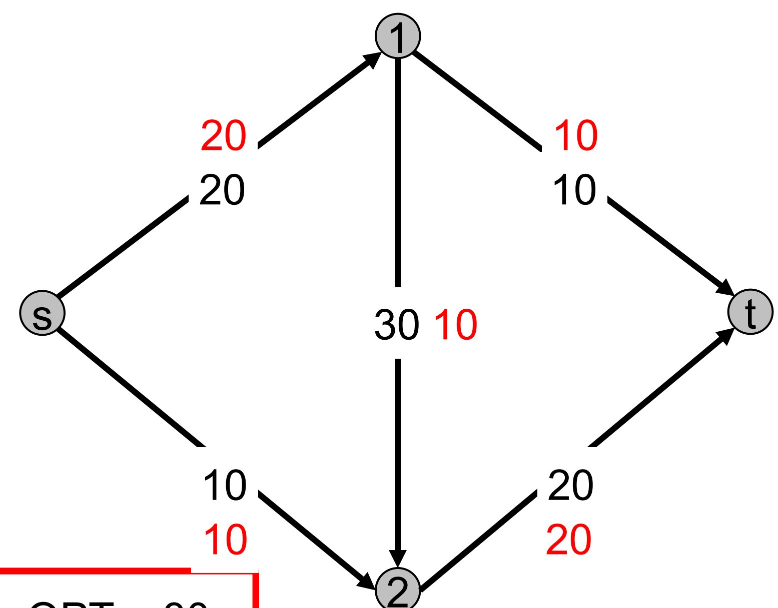
Local Optimum $\neq$ Global Optimum

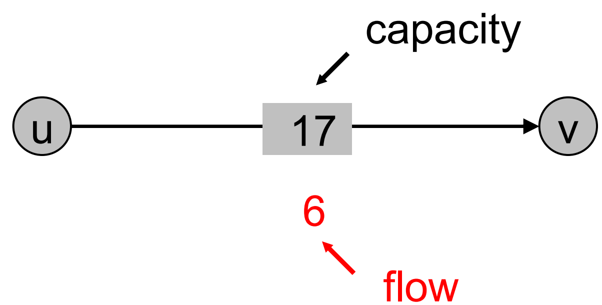


Greedy = 20



OPT = 30

Residual Graph



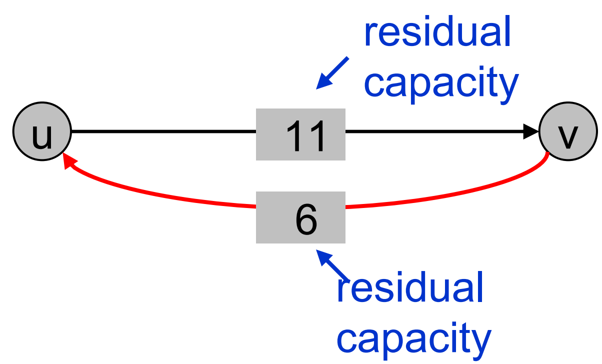
Original edge: e = (u, v) ∈ E.

- Flow f(e), capacity c(e).

Residual edge.

- "Undo" flow sent.
- e = (u, v) and e^R = (v, u).
- Residual capacity:

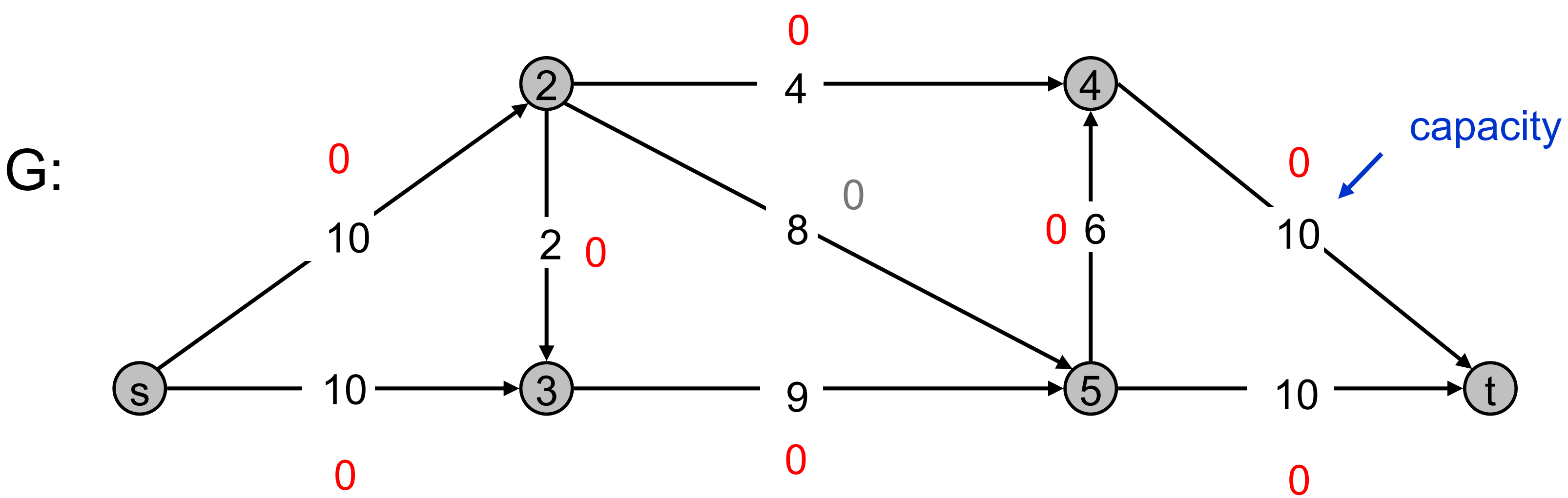
$$c_f(e) = \begin{cases} c(e) - f(e) & if\ e \in E \\ f(e) & if\ e^R \in E \end{cases}$$

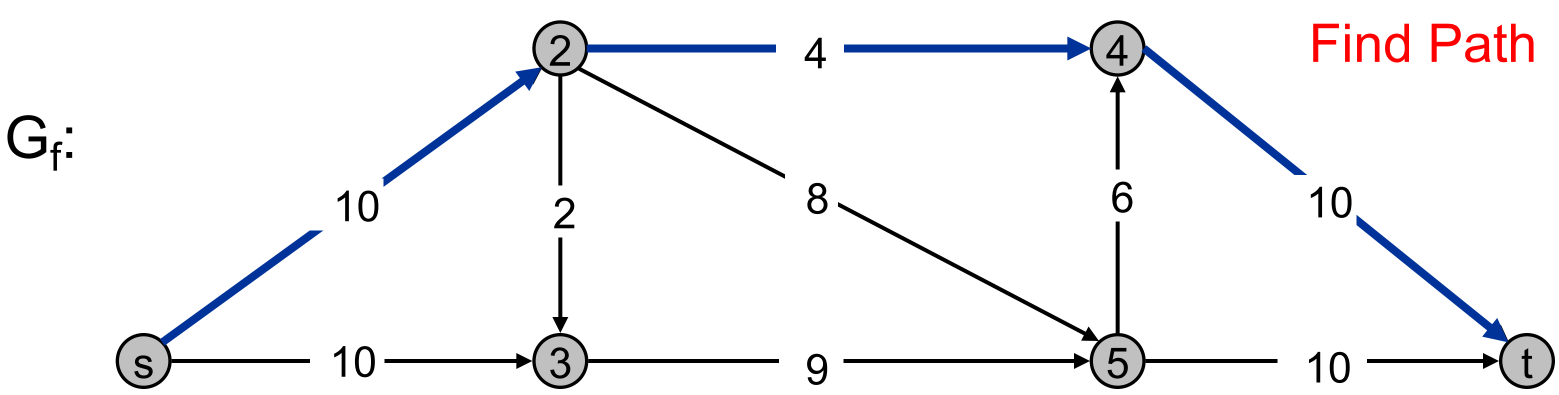


Residual graph: G_f = (V, E_f).

- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e : f(e^R) > 0\}$.

Ford-Fulkerson Alg: Greedy on G_f

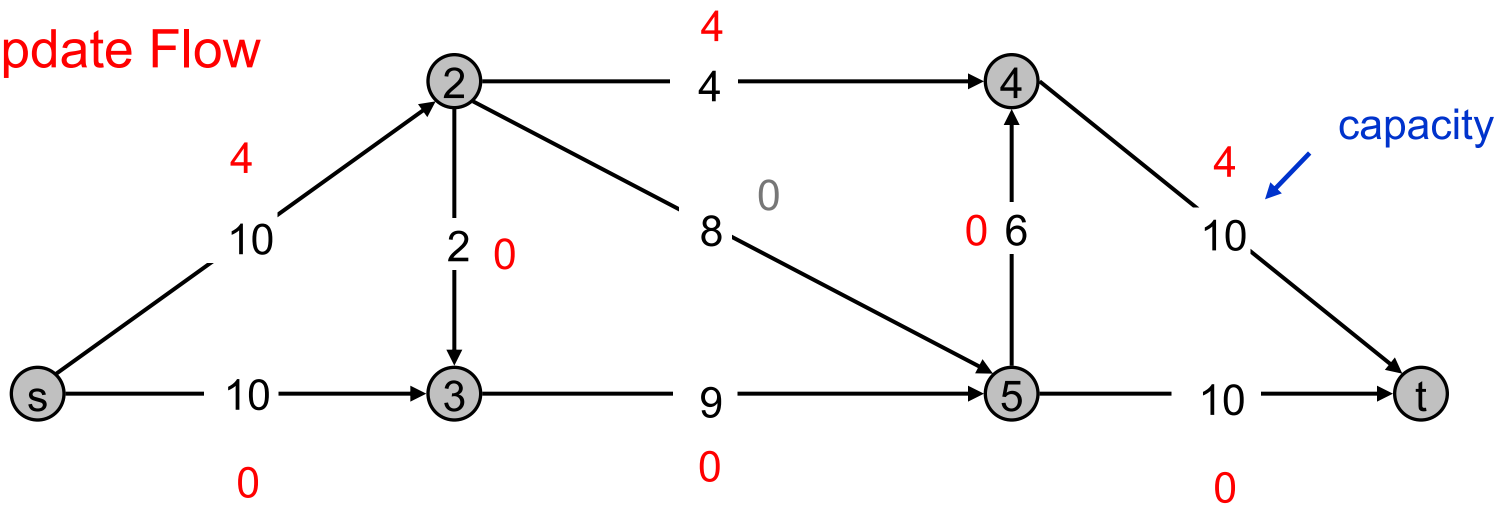




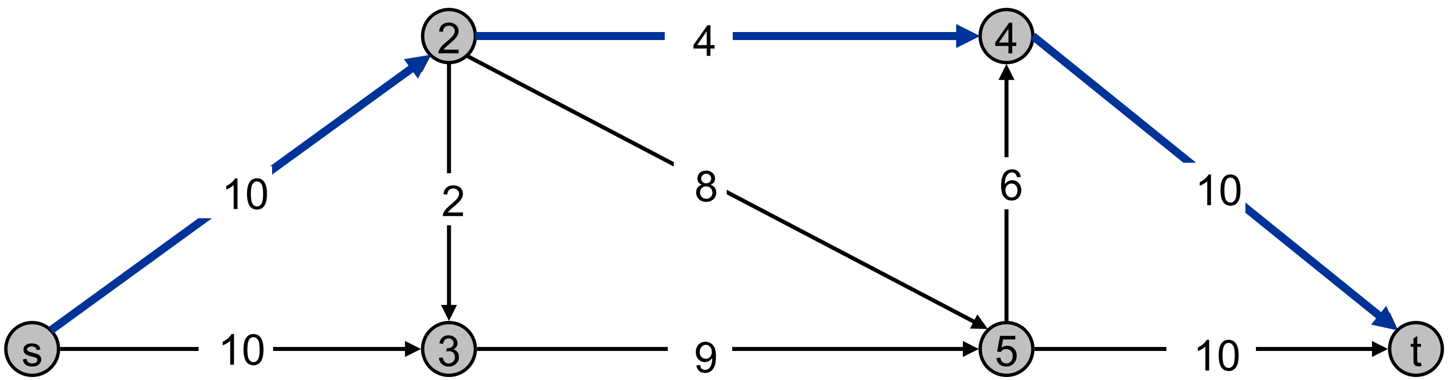
Ford-Fulkerson Alg: Greedy on G_f

Update Flow

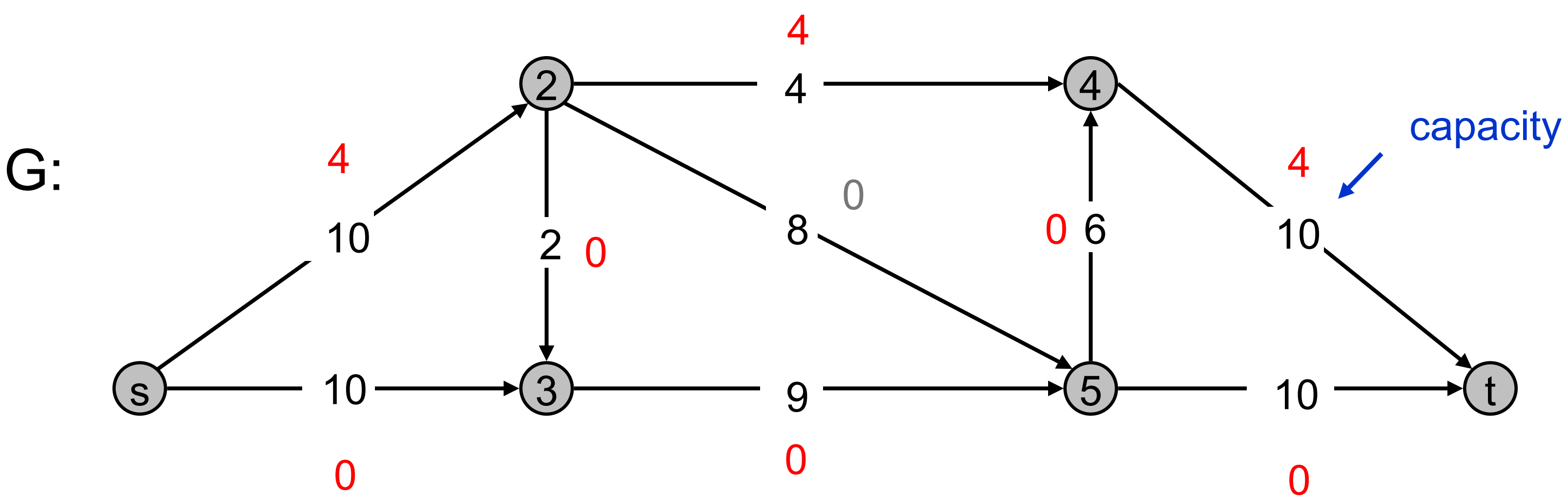
G:

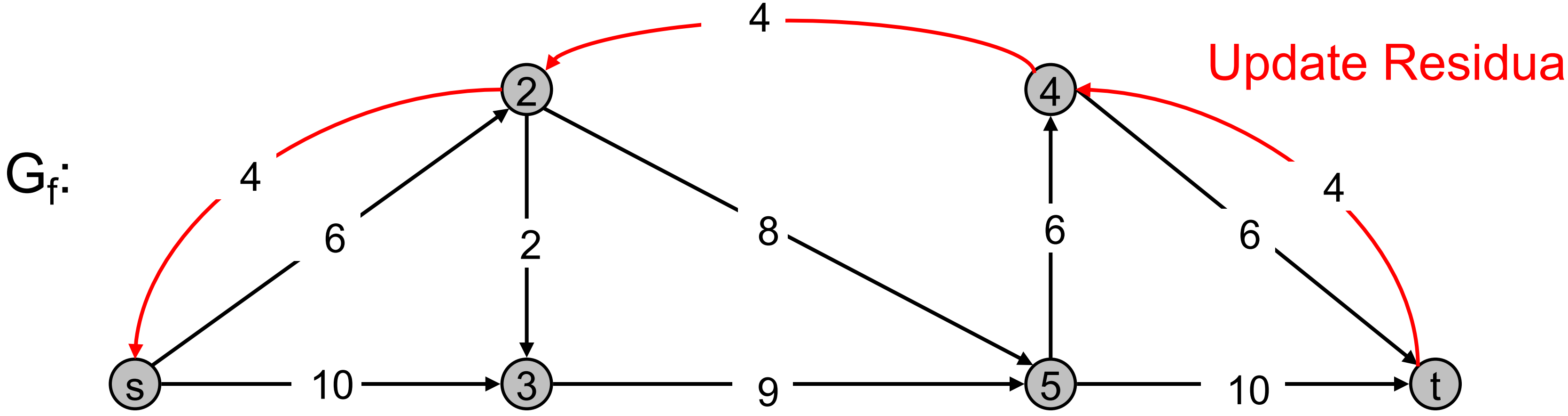


G_f:



Ford-Fulkerson Alg: Greedy on G_f





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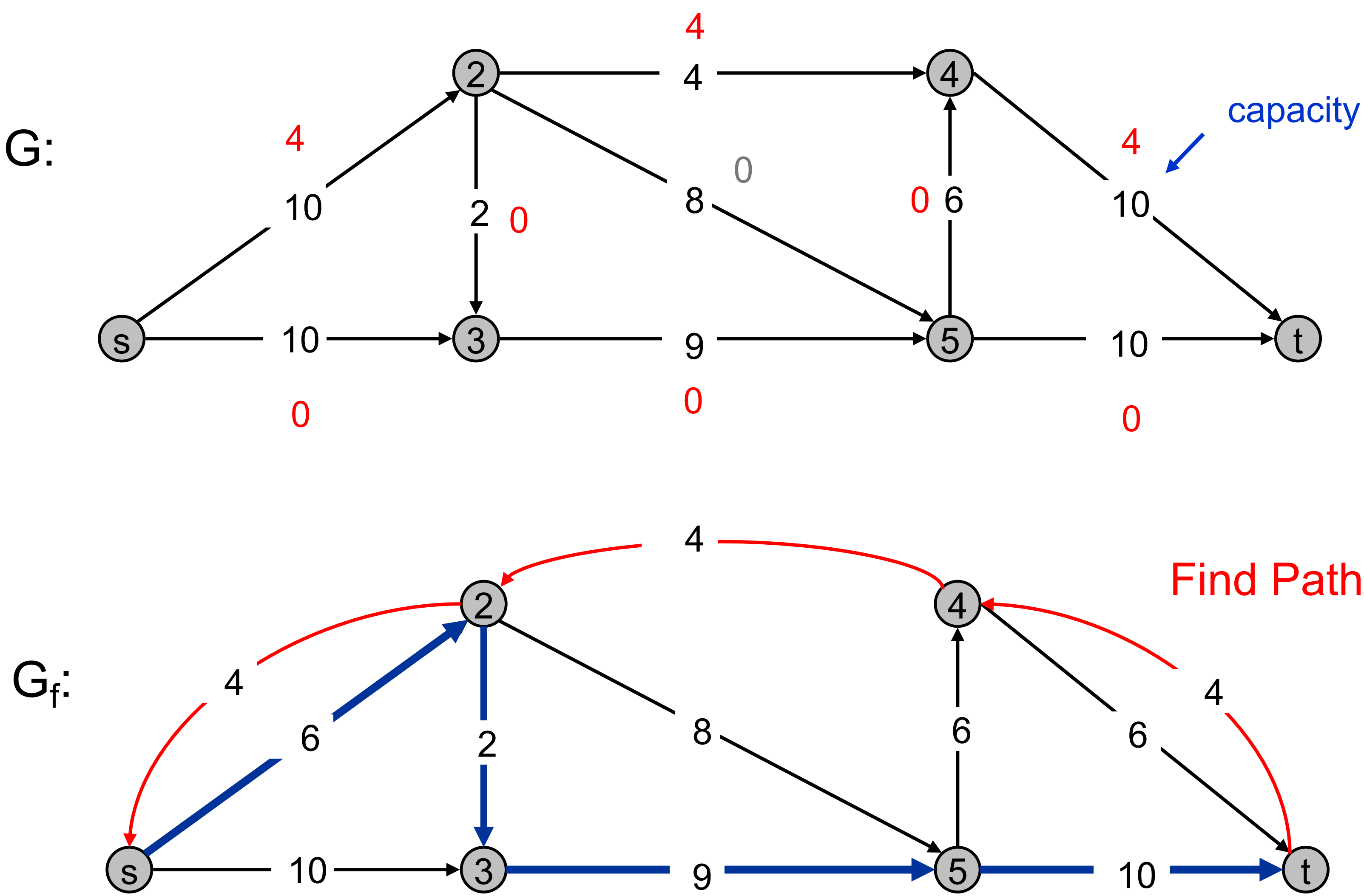

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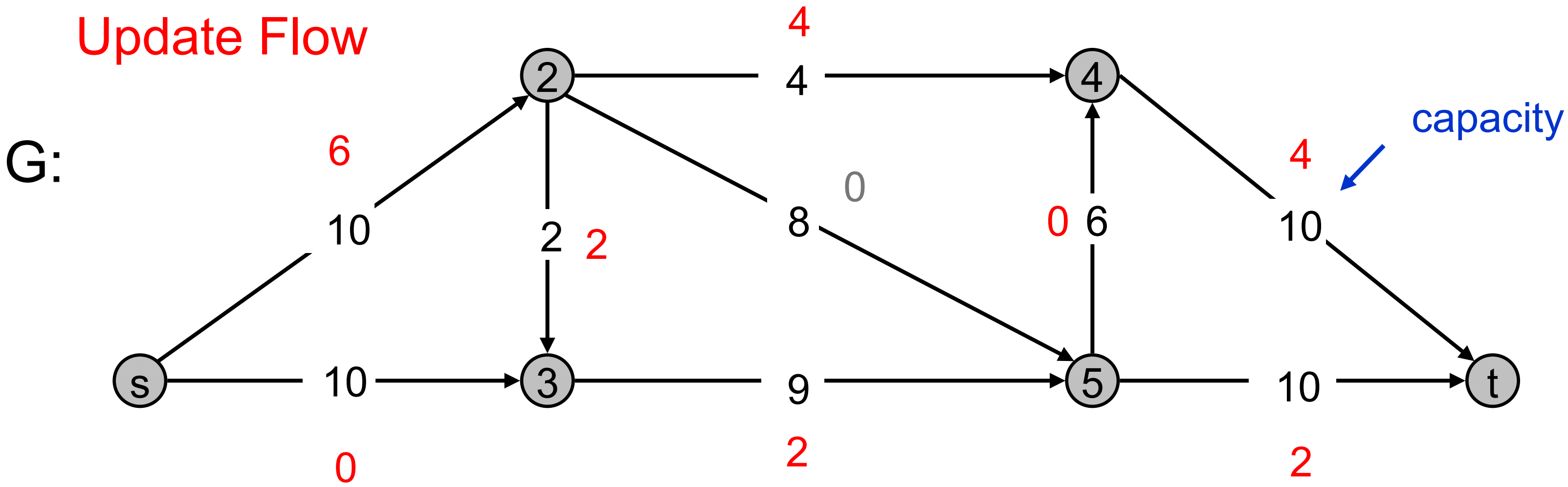

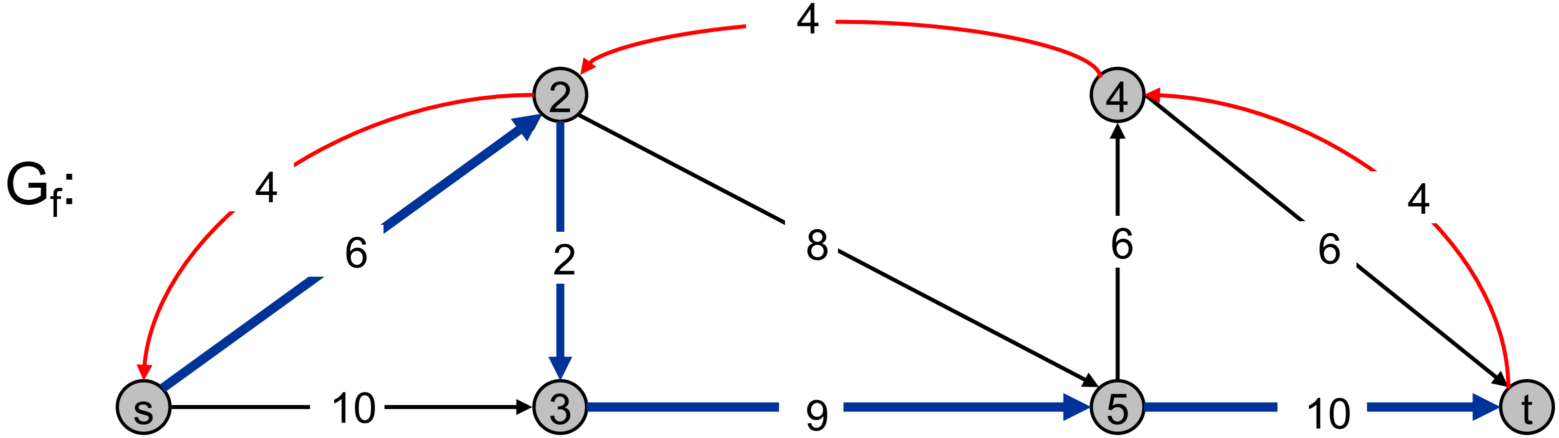

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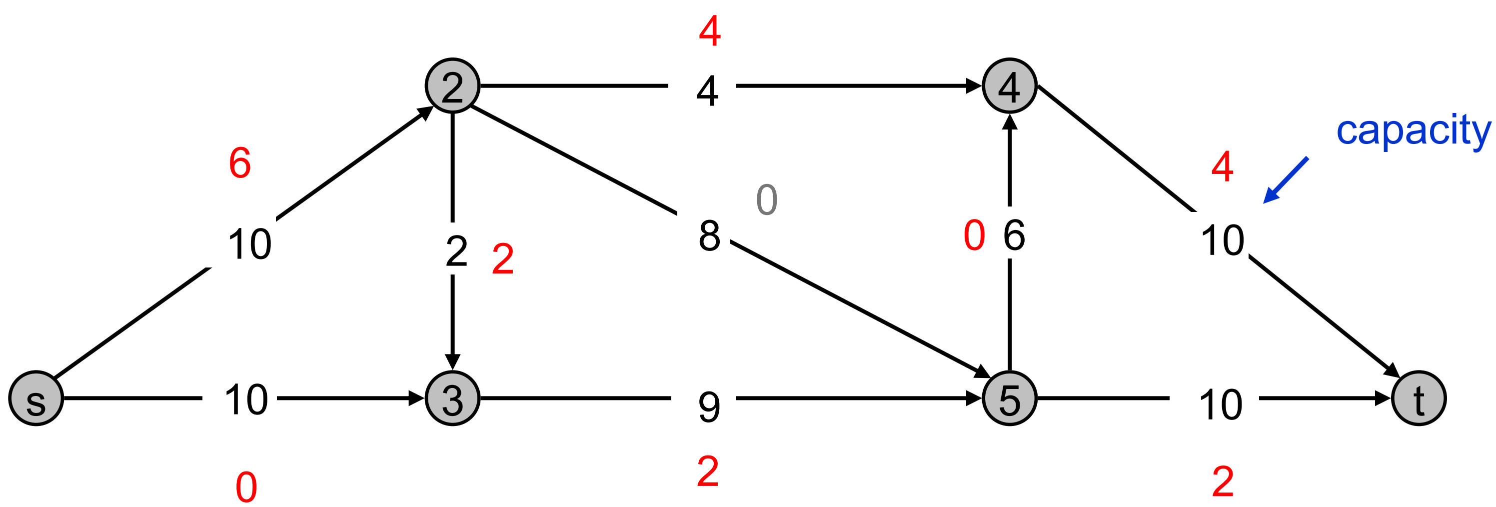

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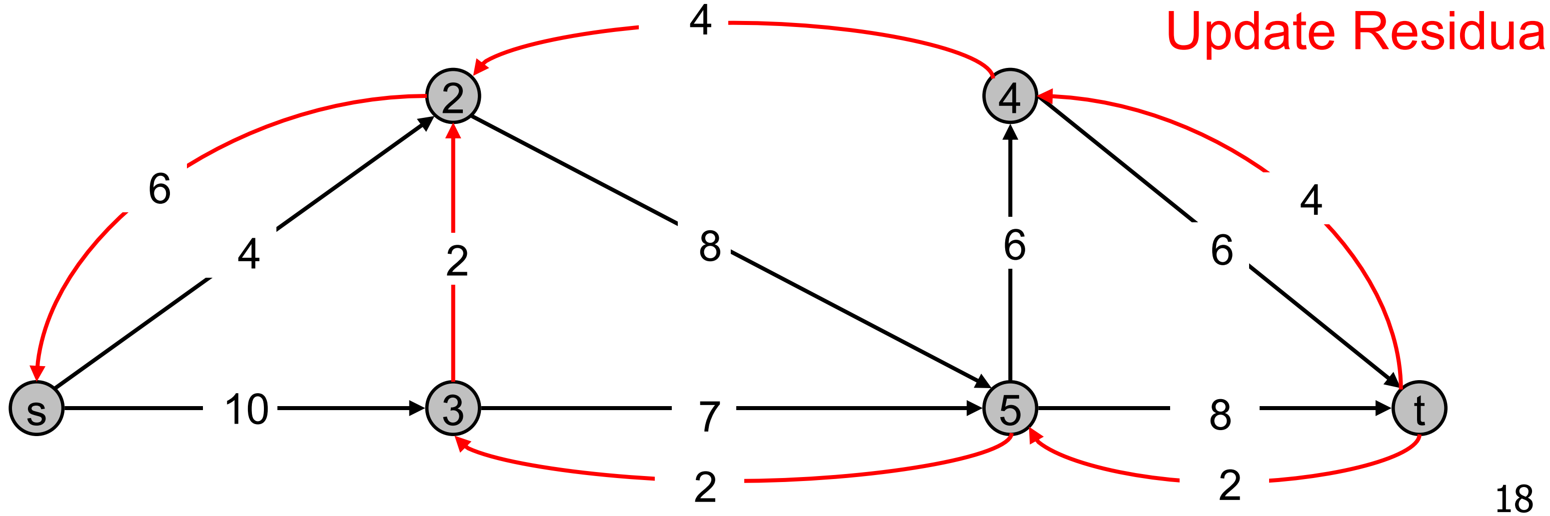

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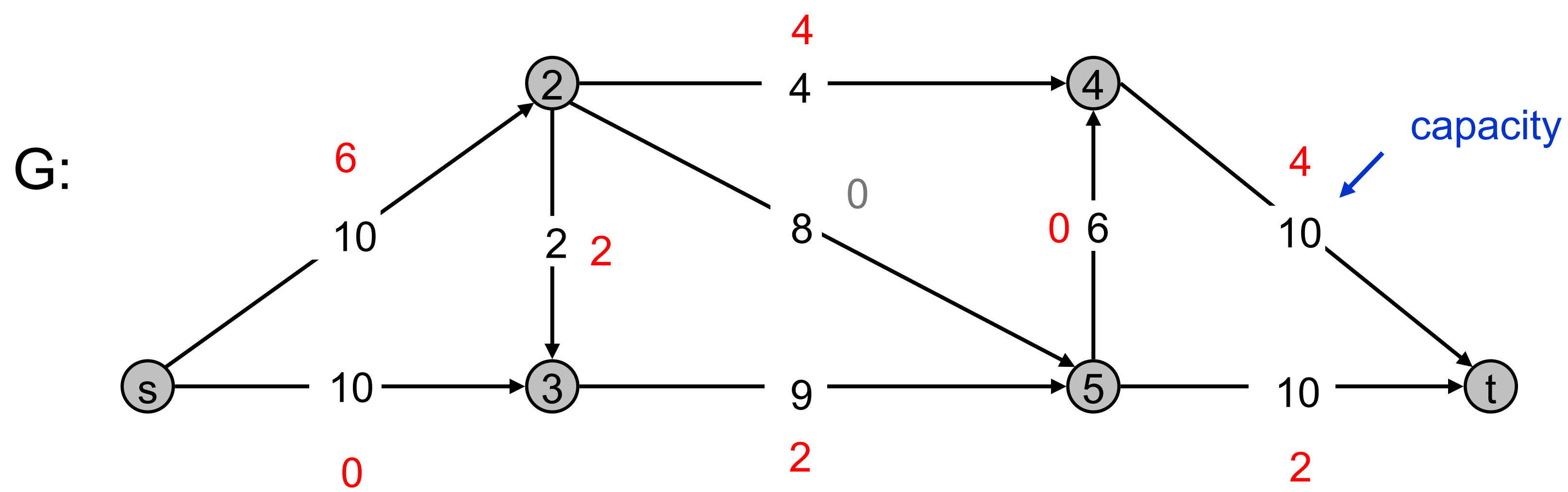

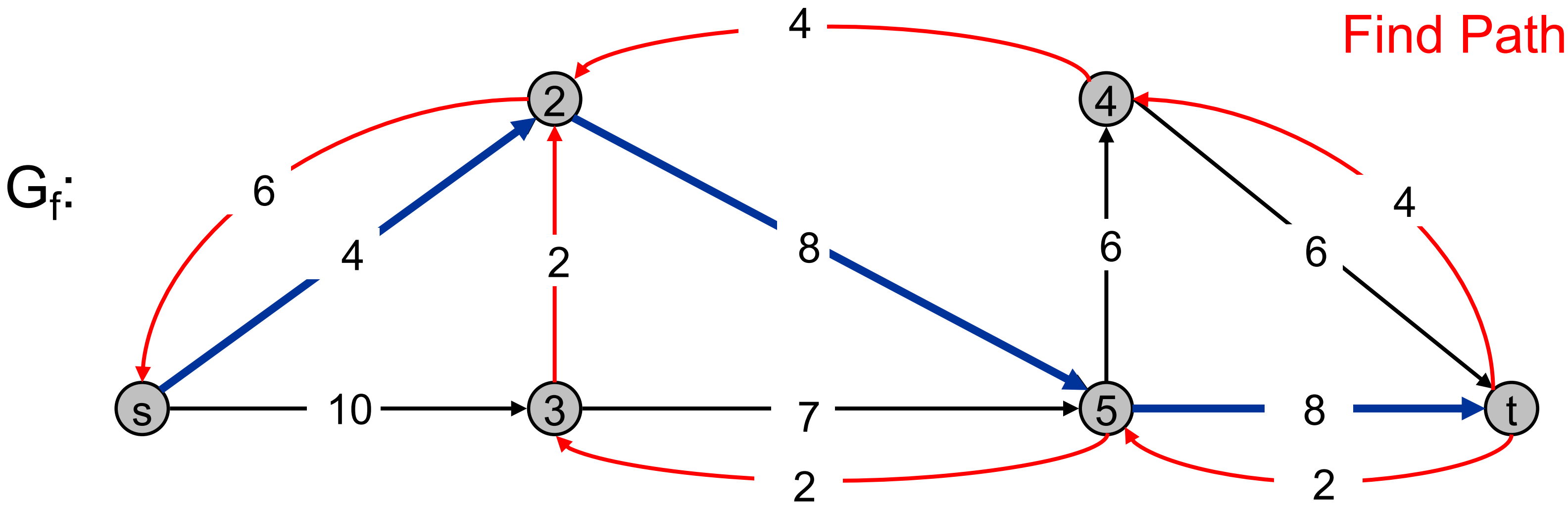

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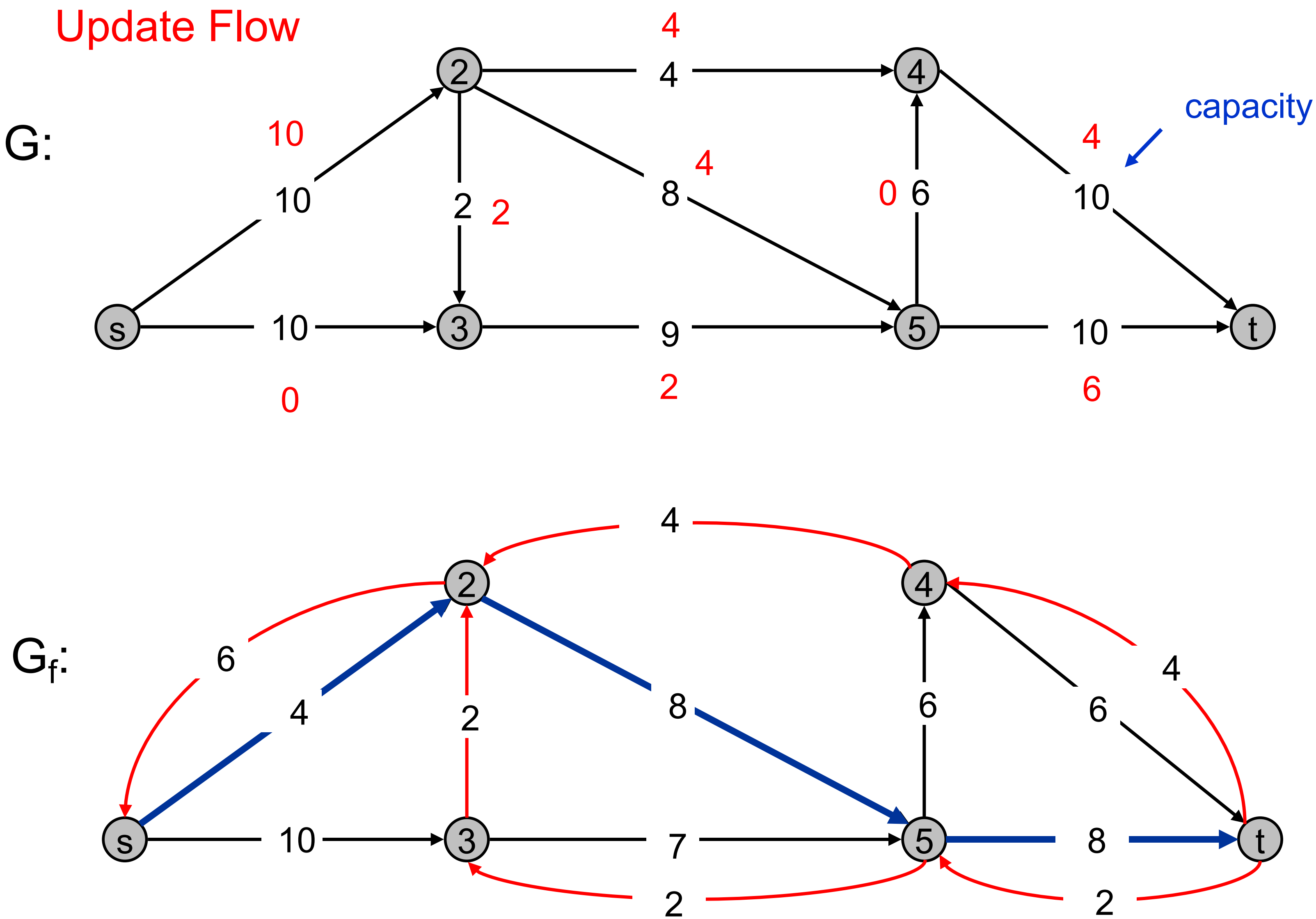

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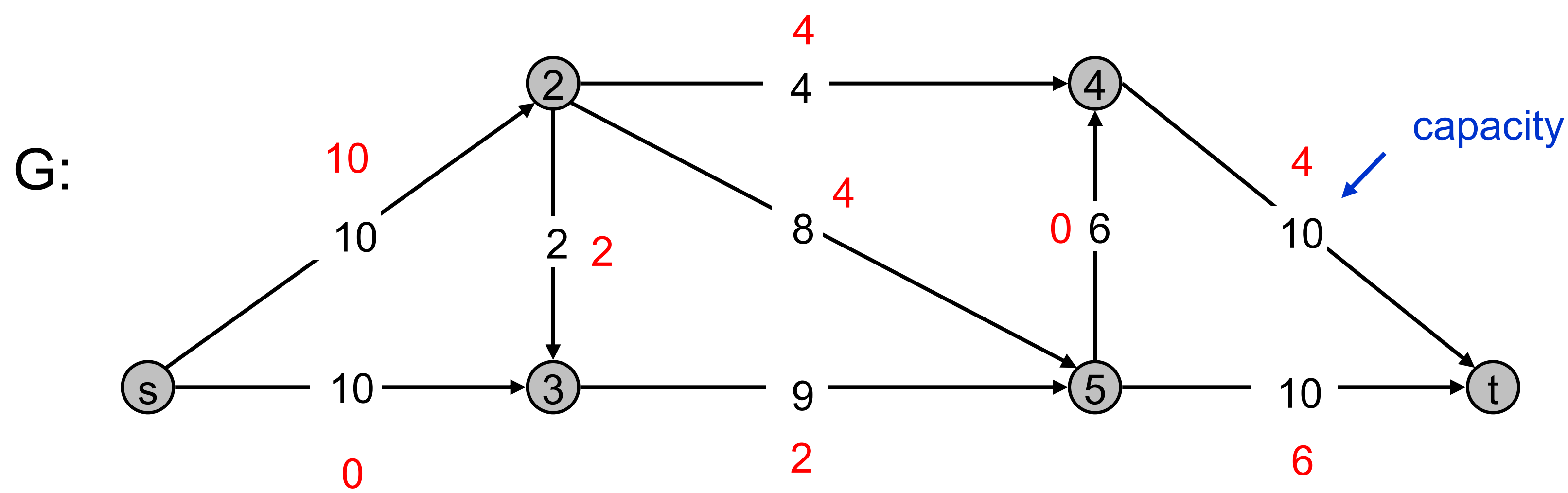

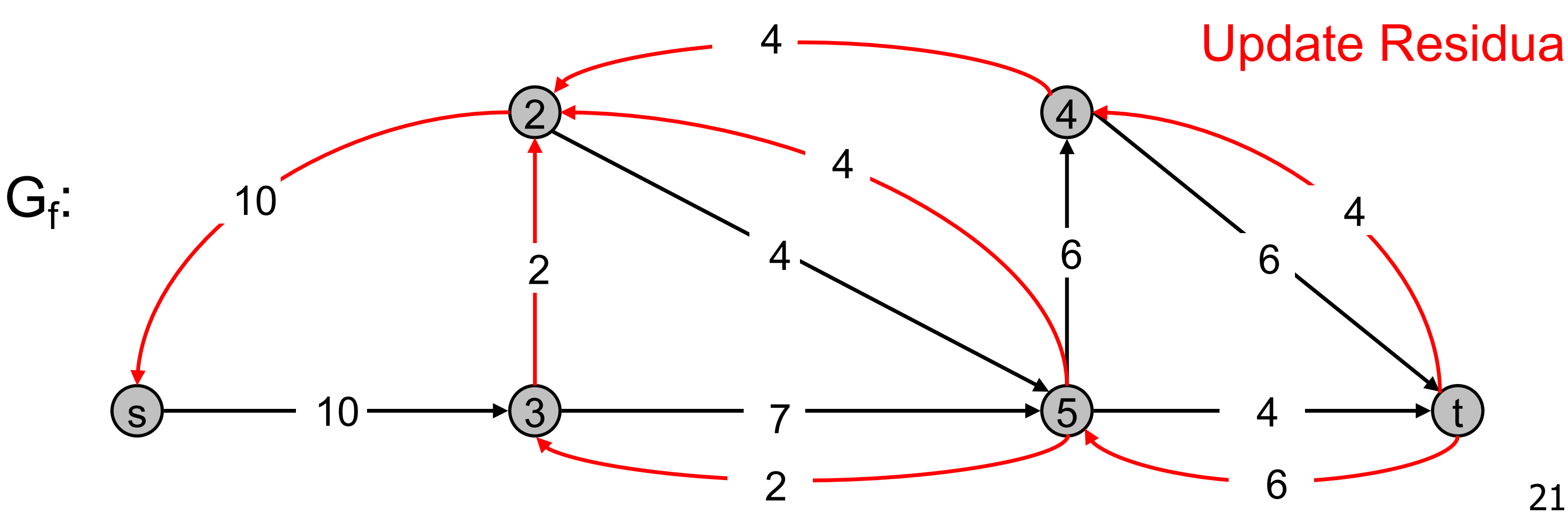

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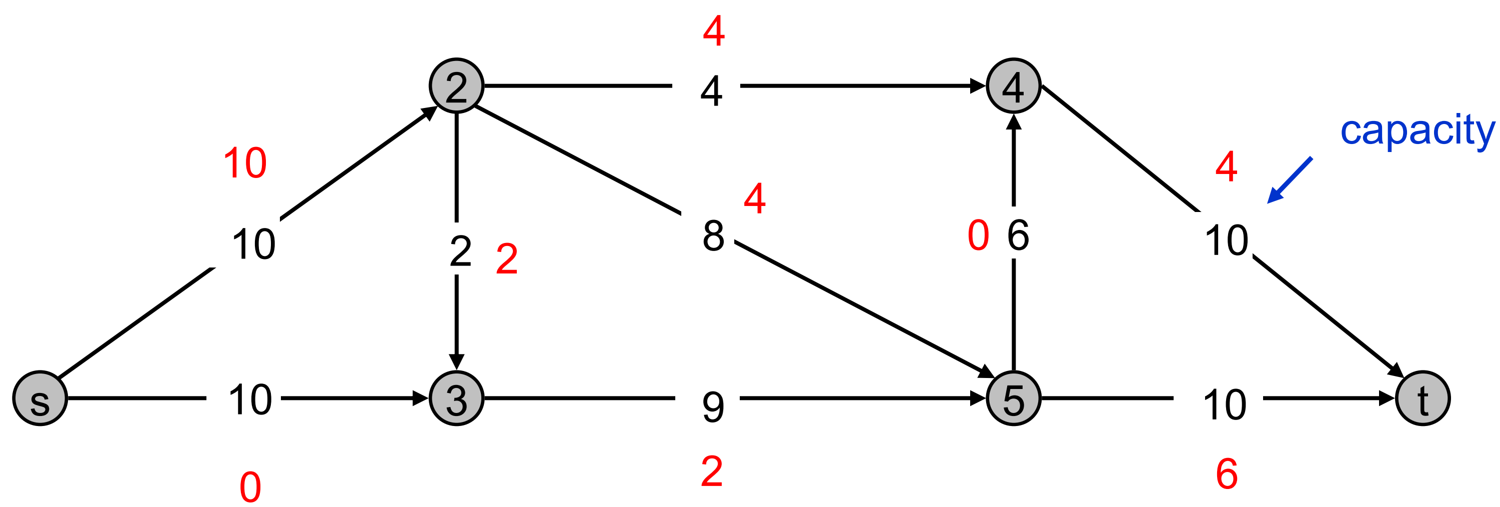

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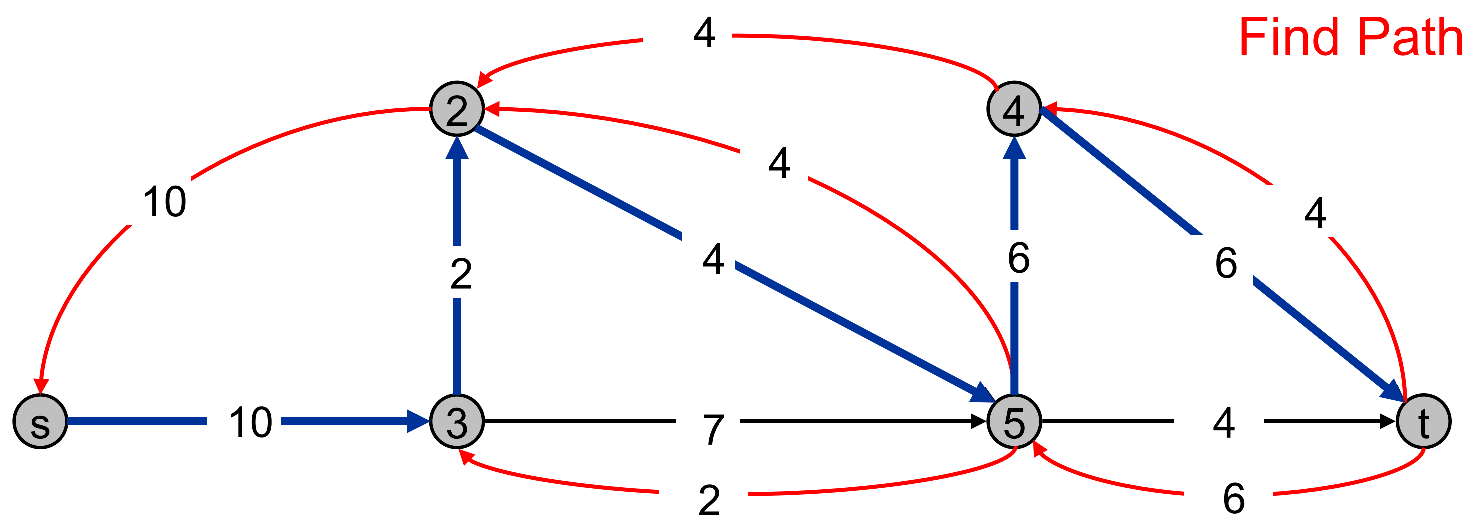
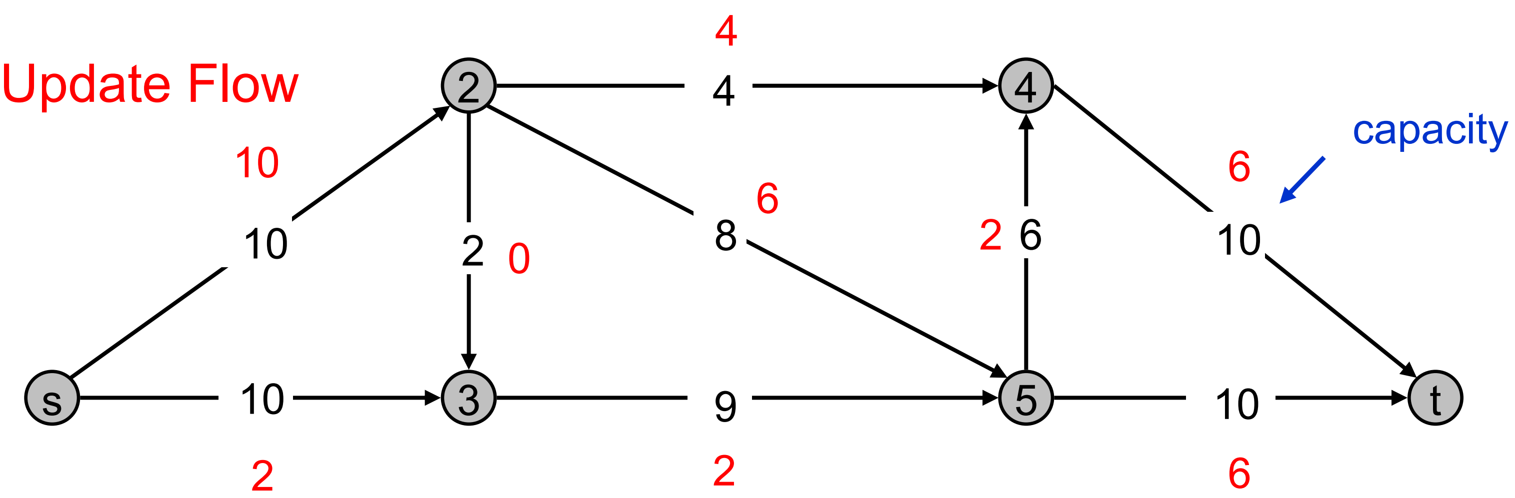


Ford-Fulkerson Alg: Greedy on G_f

Update Flow

G:



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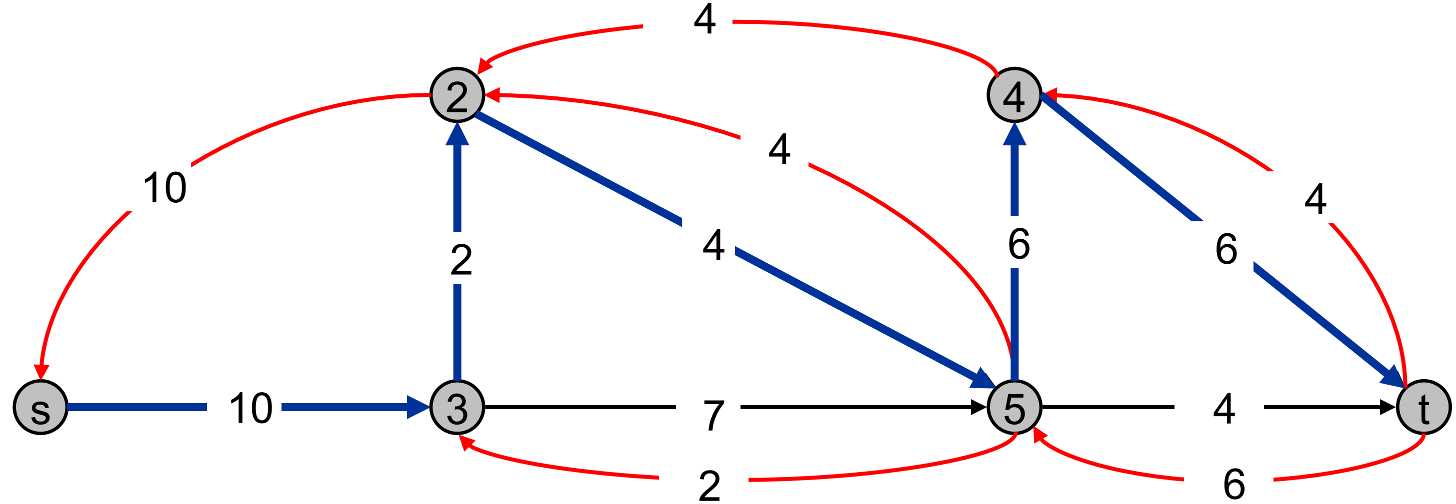

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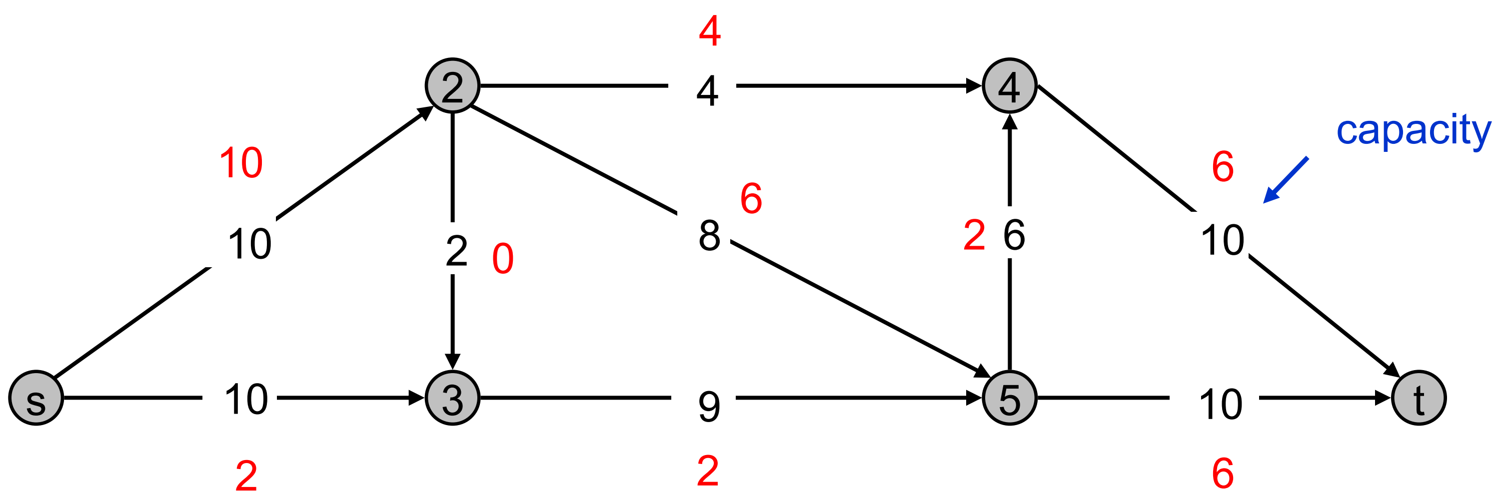

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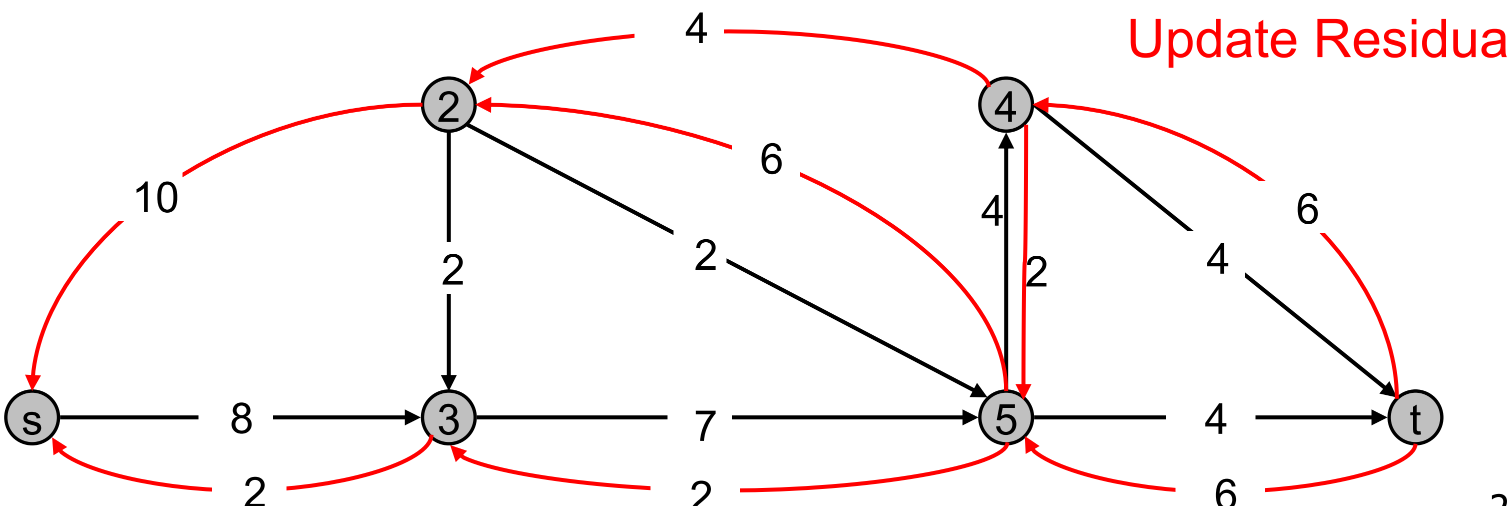
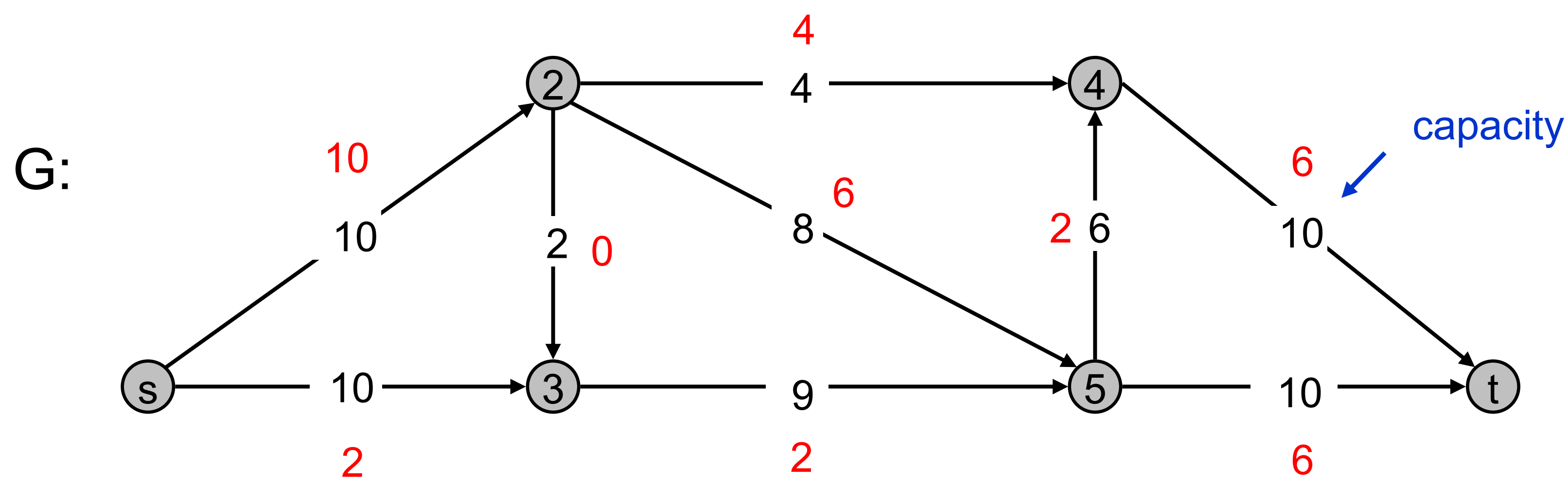
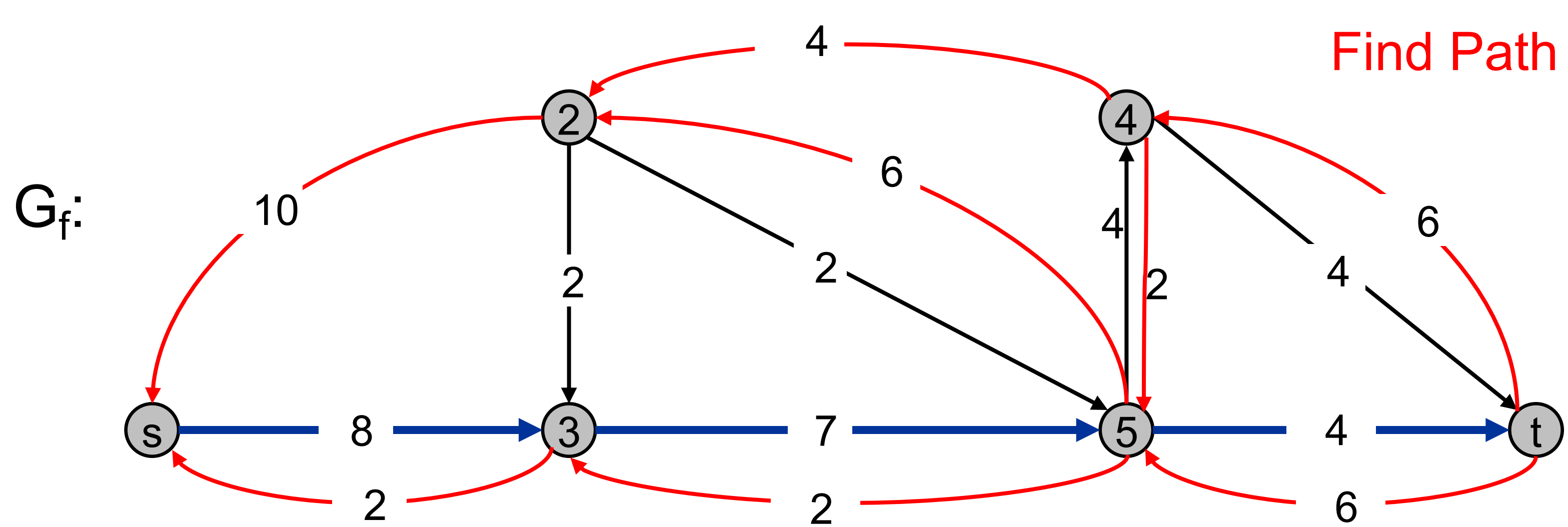


Ford-Fulkerson Alg: Greedy on G_f





Ford-Fulkerson Alg: Greedy on G_f

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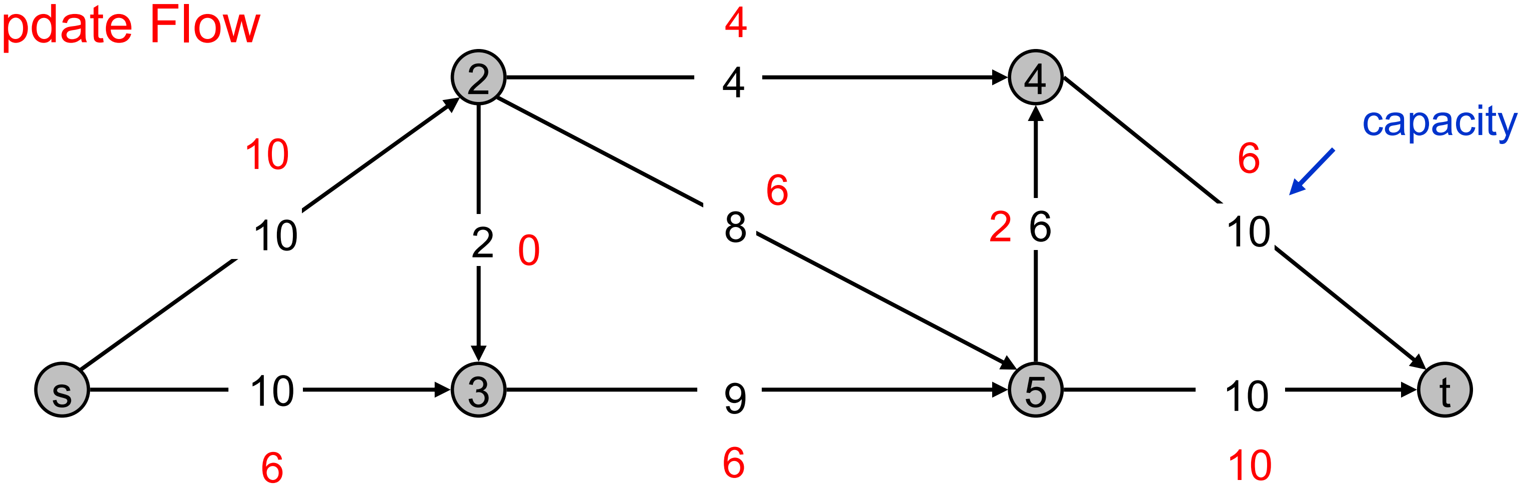

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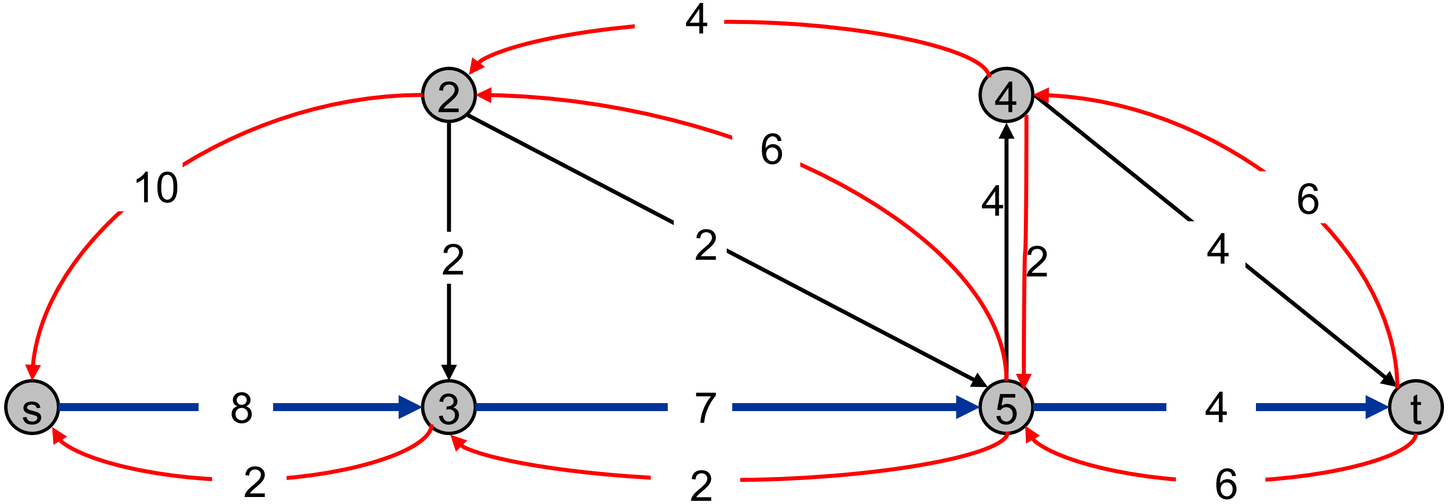

Ford-Fulkerson Alg: Greedy on G_f

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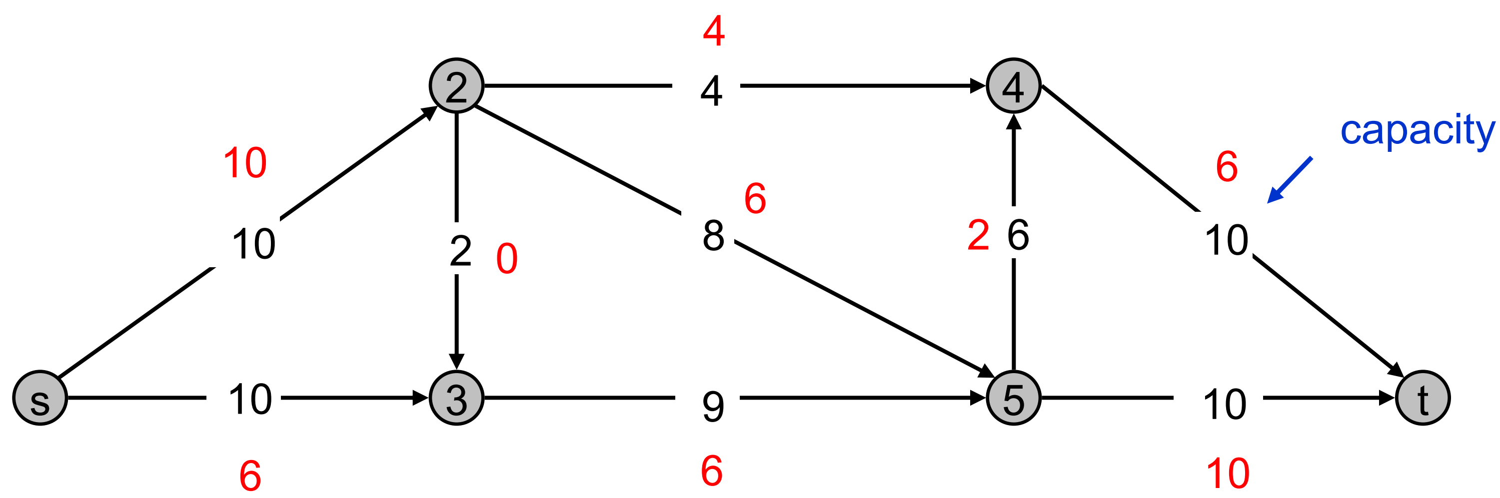

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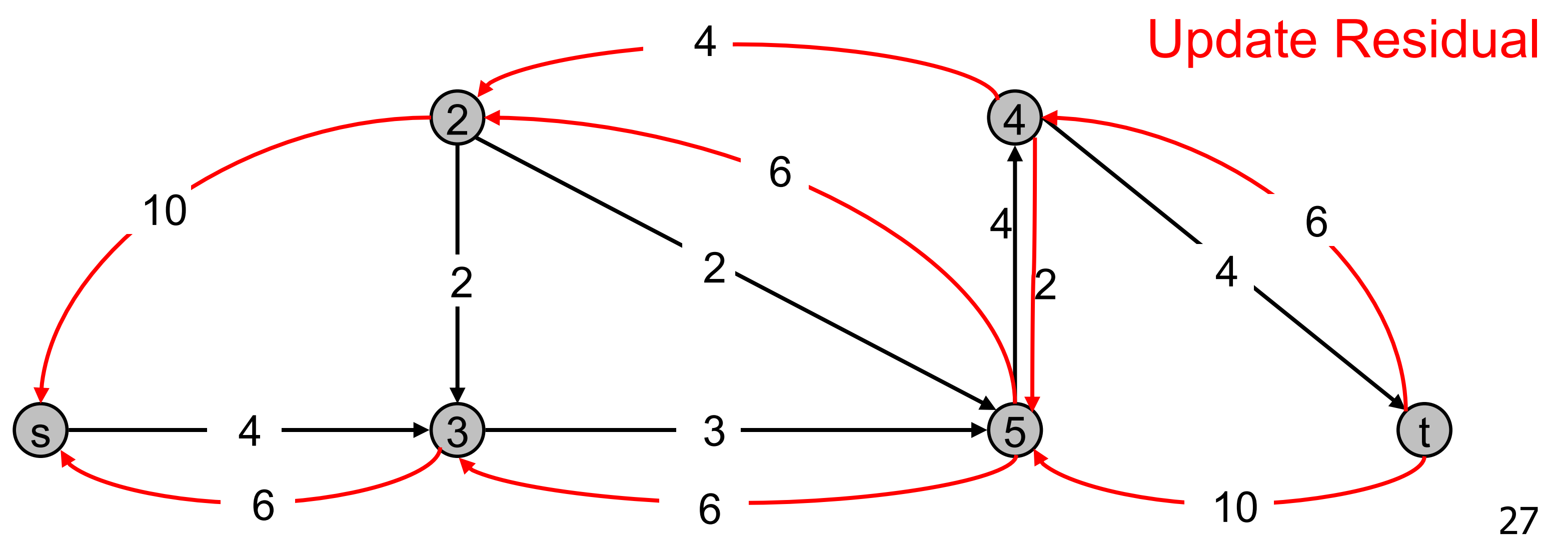

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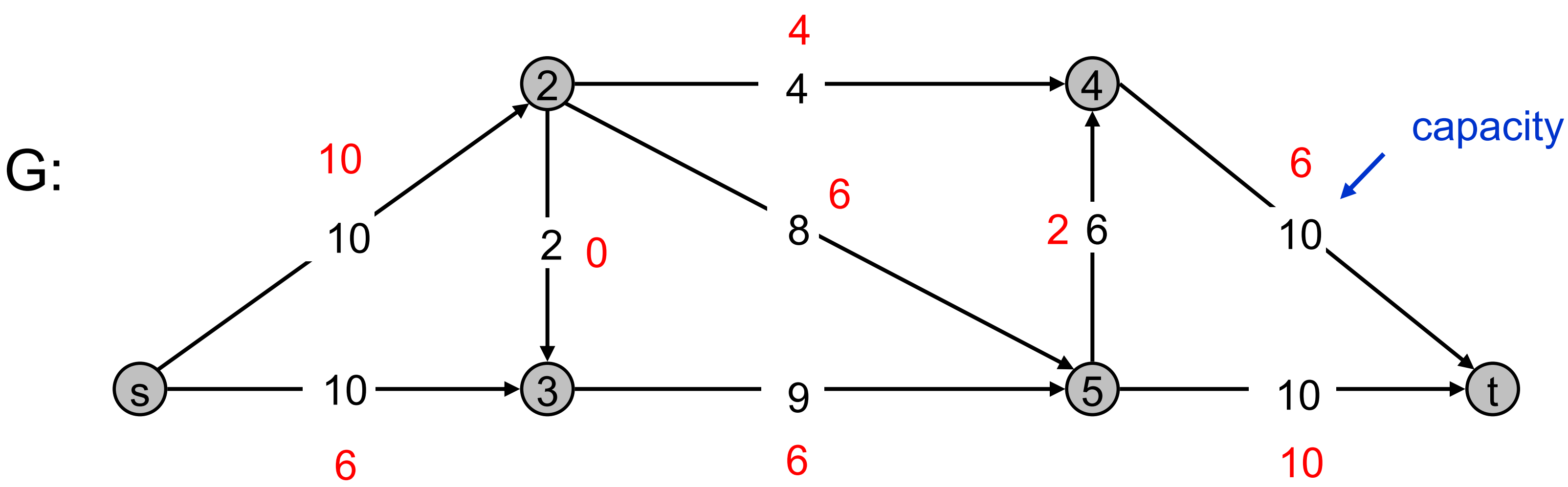
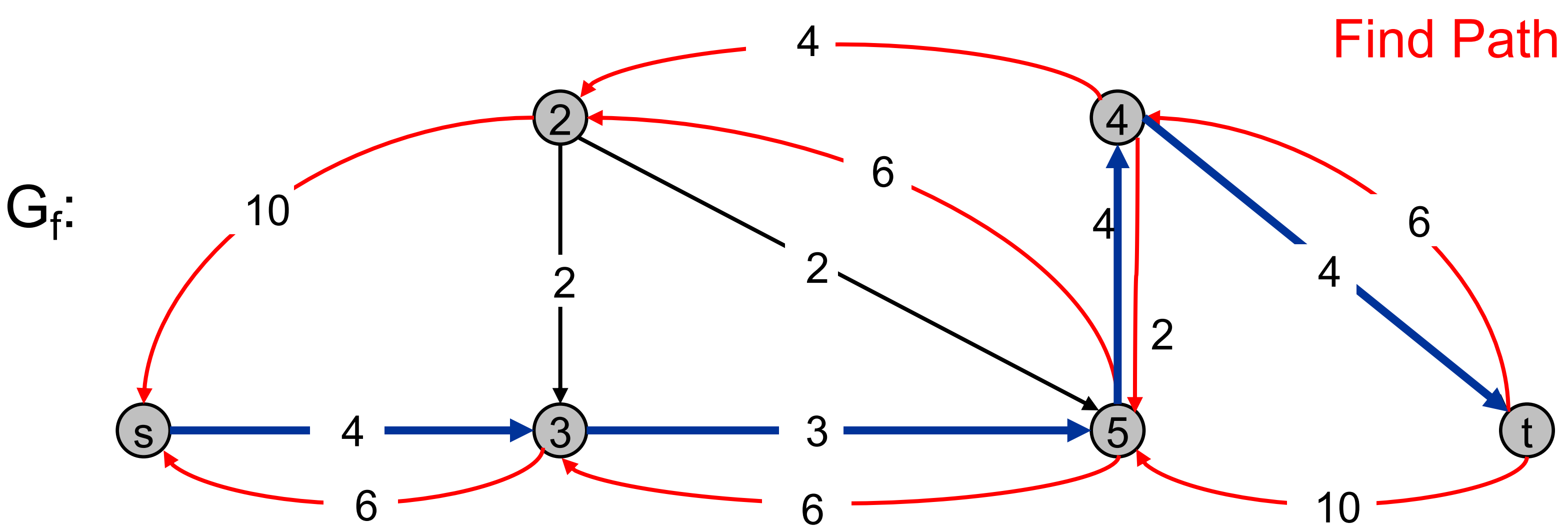
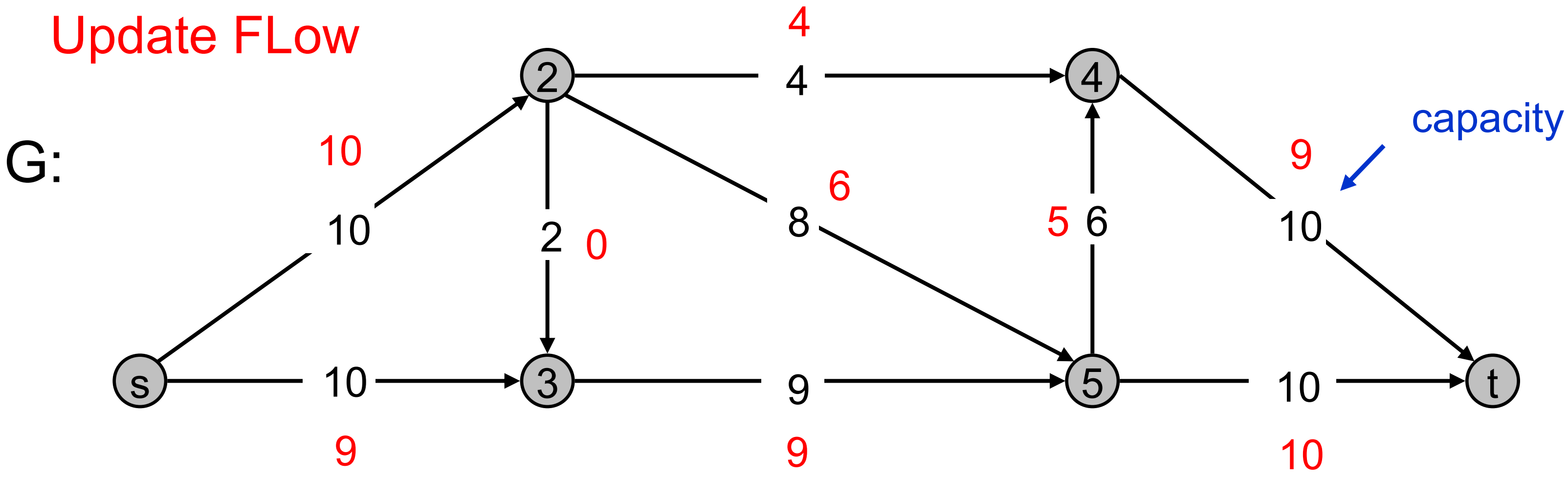
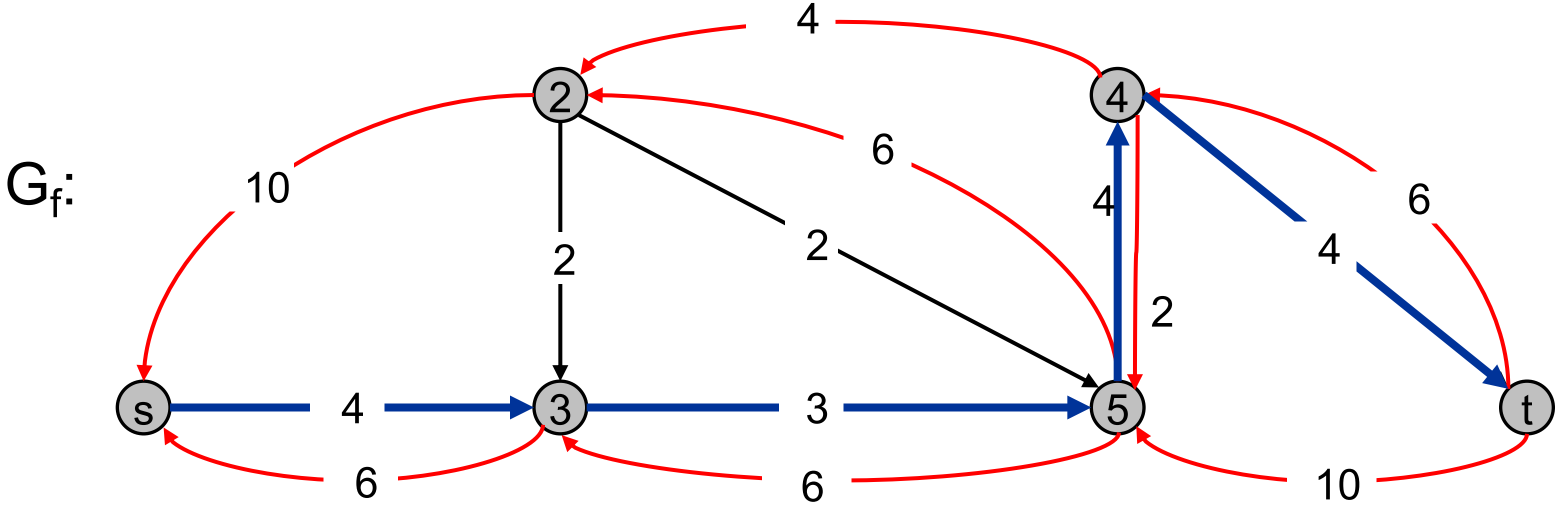




Ford-Fulkerson Alg: Greedy on G_f





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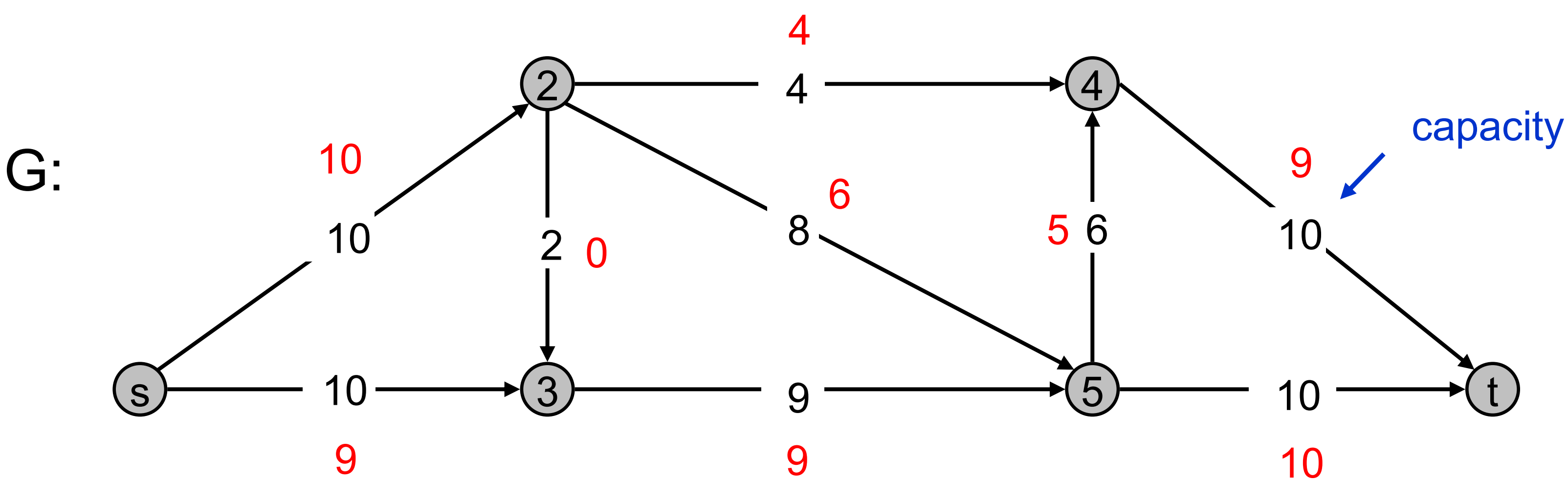
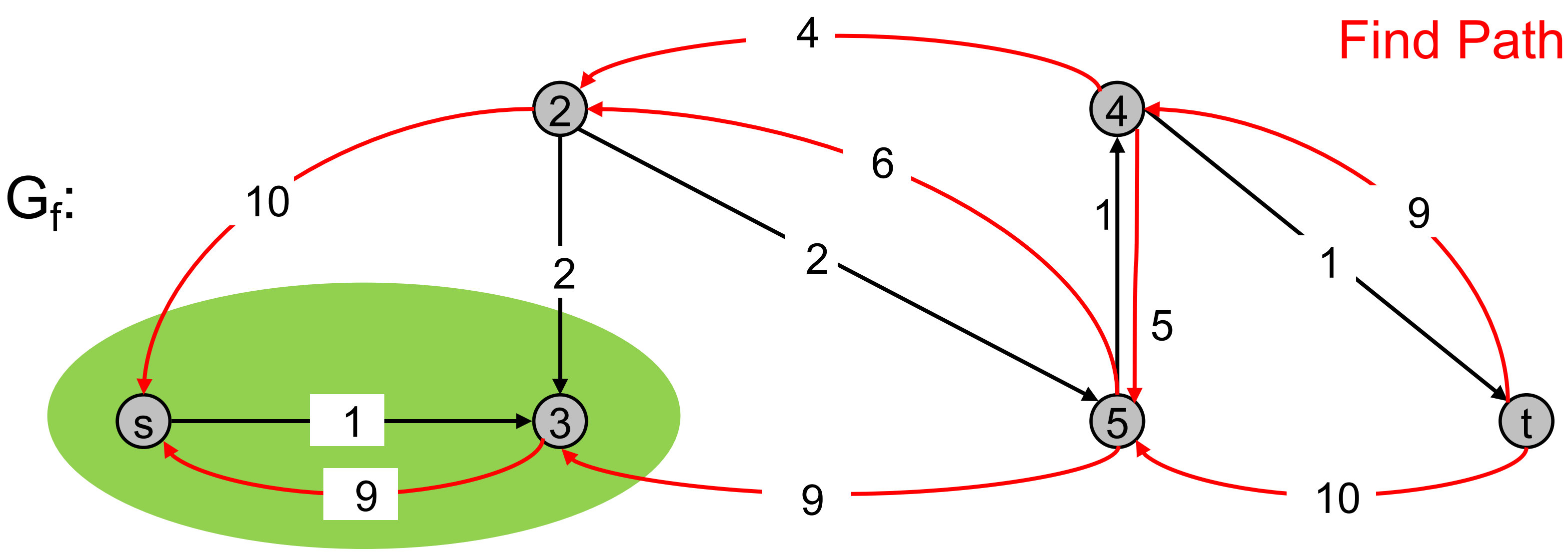




Augmenting Path Algorithm

```
Augment(f, c, P) {
   b ← bottleneck(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b
                 c(e) ← c(e) - b
                 c(eR) ← c(eR) + b

      else       f(e) ← f(e) - b
                 c(e) ← c(e) + b
                 c(eR) ← c(eR) - b
   }
   return f
}
```

Smallest capacity edge on P (annotation on `b ← bottleneck(P)`)

Forward edge (annotation on `if (e ∈ E)` branch)

Reverse edge (annotation on `else` branch)

$e^R \in P$ (annotation on `else`)

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0.  Gf is residual graph
   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
}
   return f
}
```

Max Flow Min Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Ford-Fulkerson 1956] The value of the max s-t flow is equal to the value of the min s-t cut.

Proof strategy. We prove both simultaneously by showing the TFAE:

(i) There exists a cut (A, B) such that v(f) = cap(A, B).

(ii) Flow f is a max flow.

(iii) There is no augmenting path relative to f.

(i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

(ii) $\Rightarrow$ (iii) We show contrapositive.

Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along that path.

Pf of Max Flow Min Cut Theorem

(iii) => (i)

No augmenting path for f => there is a cut (A,B): v(f)=cap(A,B)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A, s $\in$ A.
- By definition of f, t $\notin$ A.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&= \sum_{e \text{ out of } A} c(e) \\
&= cap(A, B)
\end{aligned}
$$

Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value $f(e)$ and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations, if f^* is optimal flow.

Pf. Each augmentation increase value by at least 1.

Corollary. If C = 1, Ford-Fulkerson runs in O(mn) time.

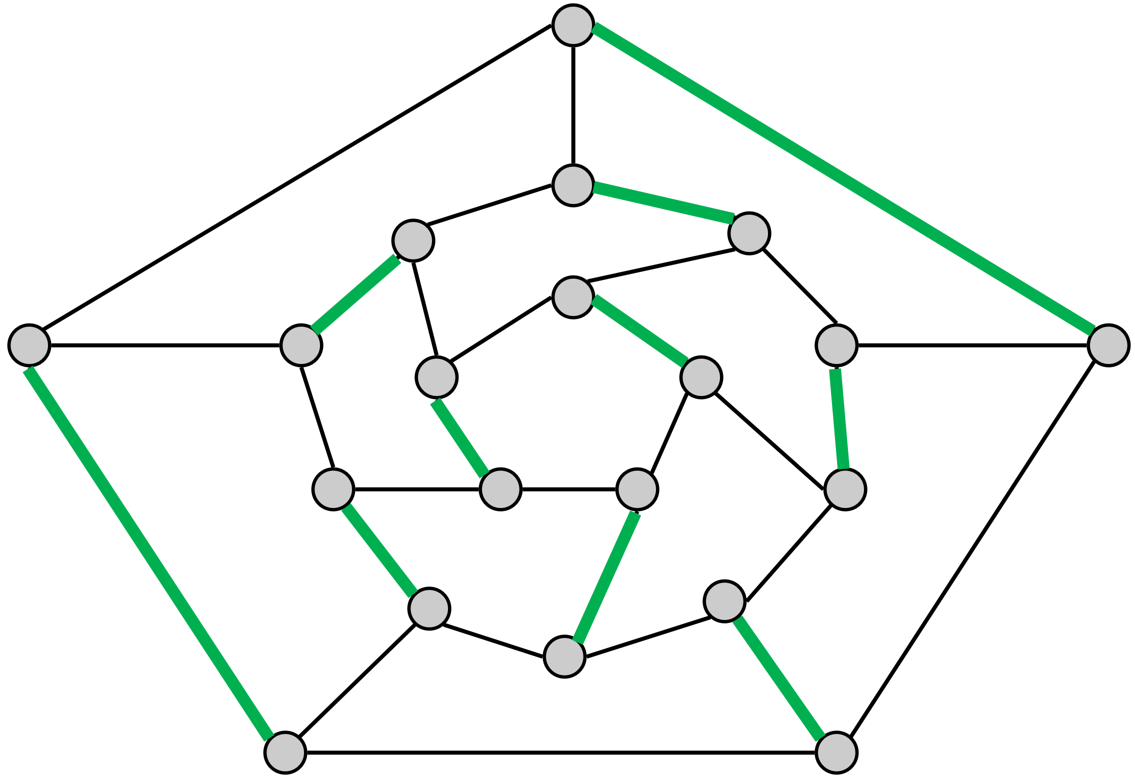
Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.

Pf. Since algorithm terminates, theorem follows from invariant.

Applications of Max Flow: Bipartite Matching

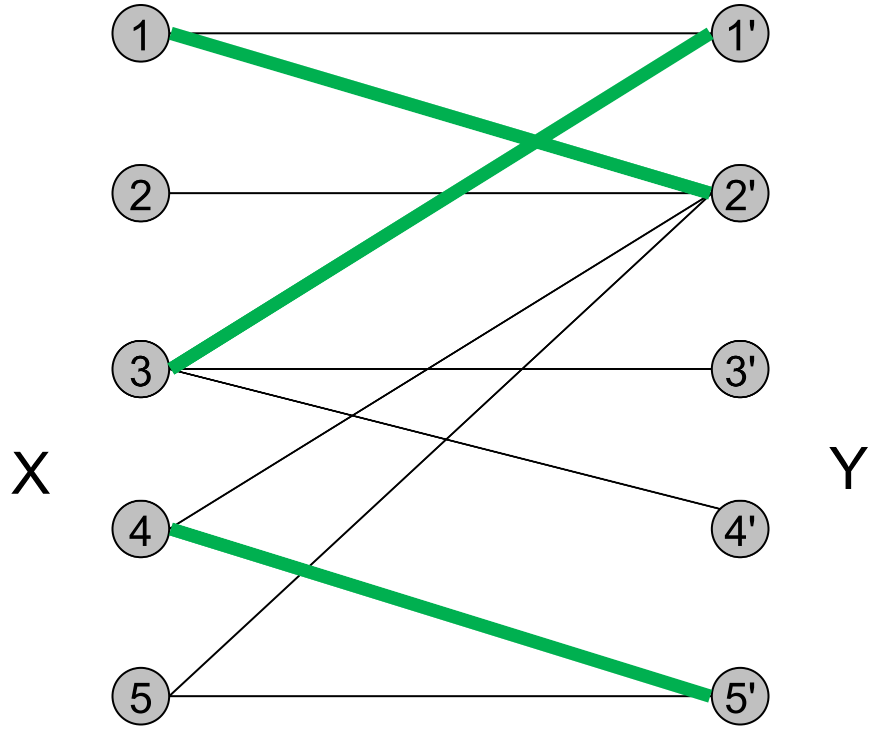
Maximum Matching Problem

Given an undirected graph G = (V, E).
A set $M \subseteq E$ is a matching if each node appears in at most one edge in M.
Goal: find a matching with largest cardinality.



Bipartite Matching Problem

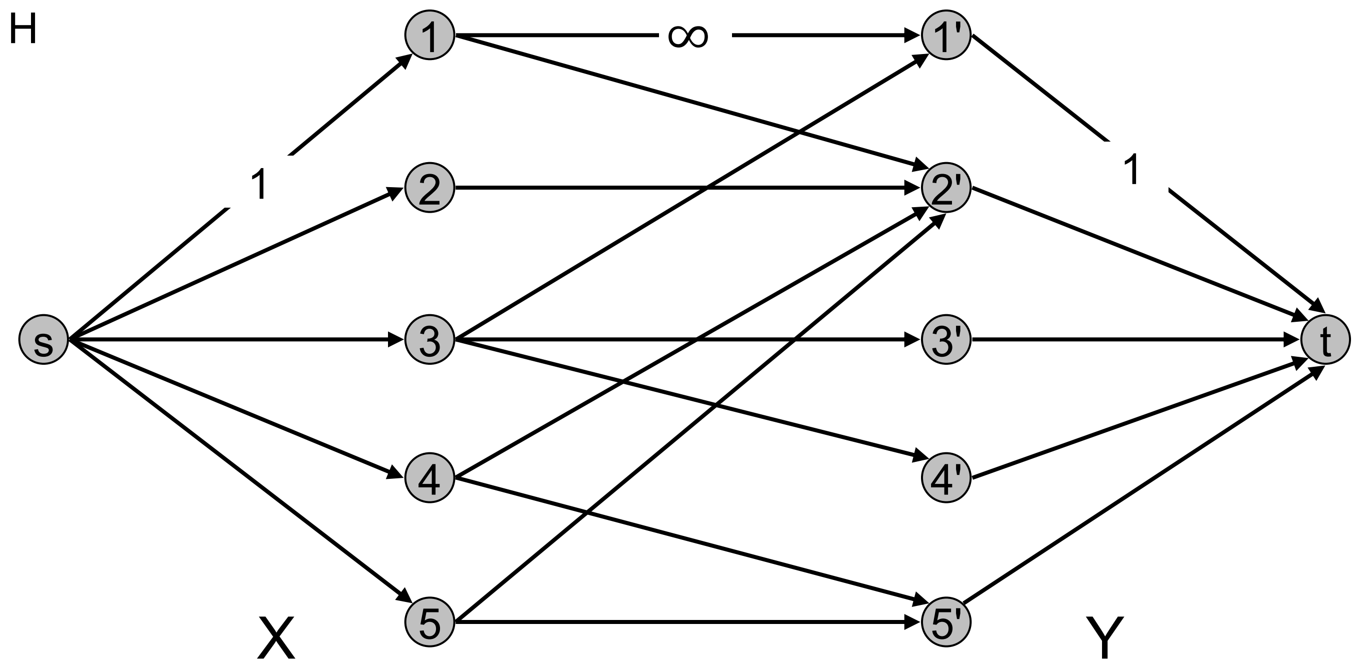
Given an undirected bibpartite graph $G = (X \cup Y, E)$
A set $M \subseteq E$ is a matching if each node appears in at most one edge in M.
Goal: find a matching with largest cardinality.



Bipartite Matching using Max Flow

Create digraph H as follows:

- Orient all edges from X to Y, and assign infinite (or unit) capacity.
- Add source s, and unit capacity edges from s to each node in L.
- Add sink t, and unit capacity edges from each node in R to t.



Bipartite Matching: Proof of Correctness

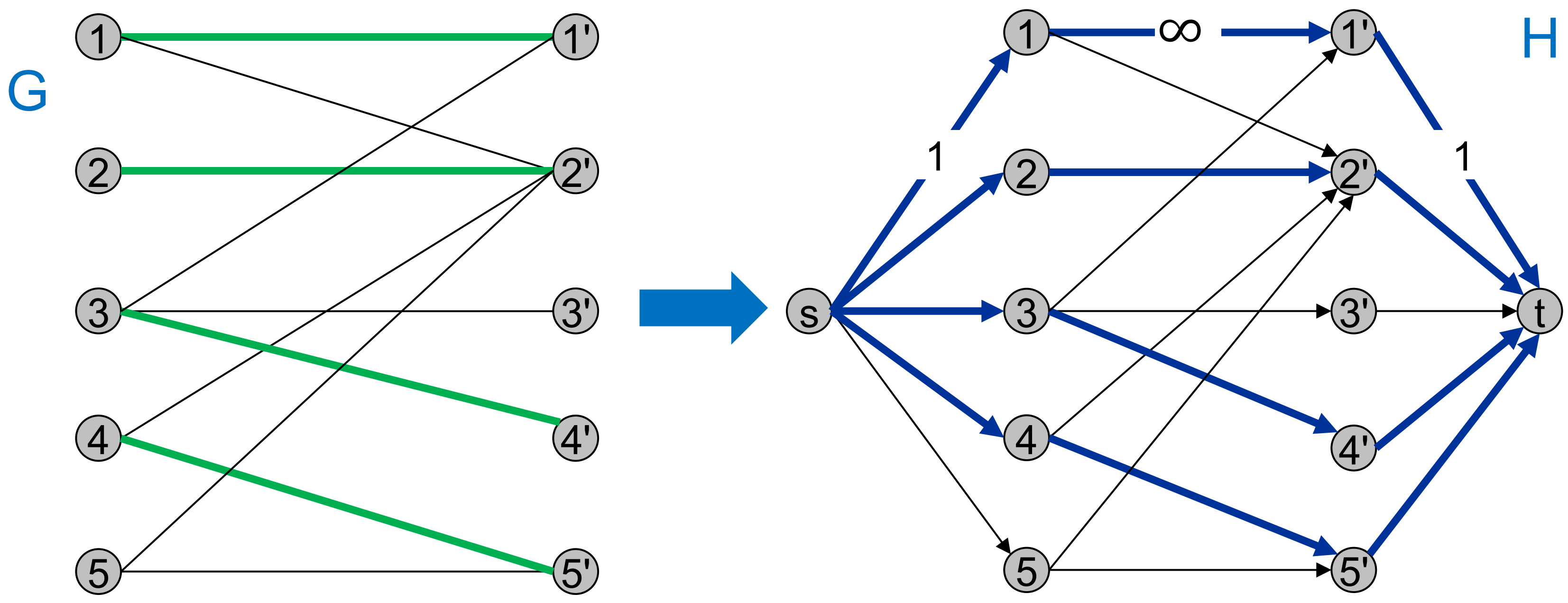
Thm. Max cardinality matching in G = value of max flow in H.

Pf. $\leq$

Given max matching M of cardinality k.

Consider flow f that sends 1 unit along each of k edges of M.

f is a flow, and has cardinality k. ▪



Bipartite Matching: Proof of Correctness

Thm. Max cardinality matching in G = value of max flow in H.

Pf. (of ≥) Let f be a max flow in H of value k.

Integrality theorem $\Rightarrow$ k is integral and we can assume f is 0-1.

Consider M = set of edges from X to Y with f(e) = 1.

- each node in X and Y participates in at most one edge in M
- |M| = k: consider s-t cut $(s \cup X, t \cup Y)$