CSE 421

Divide and Conquer: Median Approximation Algorithms

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Integer Multiplication

Integer Arithmetic

Add: Given two n-bit integers a and b, compute a + b.

Add

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

O(n) bit operations.

Multiply: Given two n-bit integers a and b, compute a × b. The "grade school" method:

Multiply

								1	1	0	1	0	1	0	1
							*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

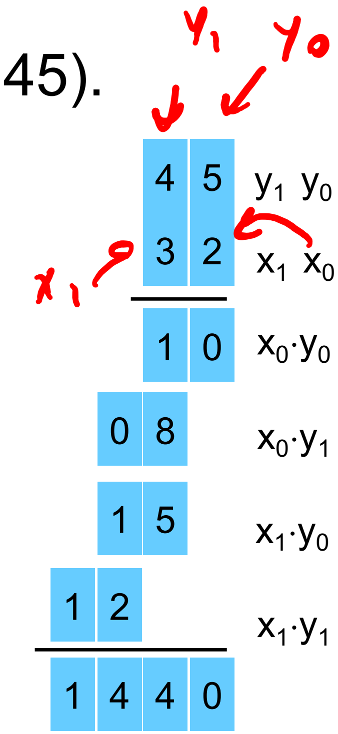
$O(n^2)$ bit operations.

How to use Divide and Conquer?

Suppose we want to multiply two 2-digit integers (32,45).

We can do this by multiplying four 1-digit integers

Then, use add/shift to obtain the result:



$$
\begin{aligned}
x &= 10x_1 + x_0 \\
y &= 10y_1 + y_0 \\
xy &= (10x_1 + x_0)(10y_1 + y_0) \\
&= 100\, x_1 y_1 + 10(x_1 y_0 + x_0 y_1) + x_0 y_0
\end{aligned}
$$

Same idea works when multiplying n-digit integers:

- Divide into 4 n/2-digit integers.
- Recursively multiply
- Then merge solutions

A Divide and Conquer for Integer Mult

Let x, y be two n-bit integers

Write $x = 2^{n/2}x_1 + x_0$ and $y = 2^{n/2}y_1 + y_0$

where x_0, x_1, y_0, y_1 are all n/2-bit integers.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0
\end{aligned}
$$

Therefore,

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n)$$

$T(n) = n^2$

We only need 3 values
$x_1y_1, x_0y_0, x_1y_0 + x_0y_1$
Can we find all 3 by only
3 multiplication?

So,

$$T(n) = \Theta(n^2).$$

Key Trick: 4 multiplies at the price of 3

$$x = 2^{n/2} \cdot x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0$$

$$\alpha = x_1 + x_0$$
$$\beta = y_1 + y_0$$
$$\alpha\beta = (x_1 + x_0)(y_1 + y_0)$$
$$= x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0$$
$$(x_1 y_0 + x_0 y_1) = \alpha\beta - x_1 y_1 - x_0 y_0$$

Key Trick: 4 multiplies at the price of 3

Theorem [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in O(n$^{1.585\ldots}$) bit operations.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \Rightarrow \alpha = x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \Rightarrow \beta = y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^n \cdot \underset{\text{A}}{x_1 y_1} + 2^{n/2} \cdot \underset{\alpha\beta - A - B}{(x_1 y_0 + x_0 y_1)} + \underset{\text{B}}{x_0 y_0}
\end{aligned}
$$

To multiply two n-bit integers:

- Add two n/2 bit integers.
- Multiply three n/2-bit integers.
- Add, subtract, and shift n/2-bit integers to obtain result.

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n) \Rightarrow T(n) = O\left(n^{\log_2 3}\right) = O(n^{1.585\ldots})$$

Integer Multiplication (Summary)

- Naïve: $\Theta(n^2)$
- Karatsuba: $\Theta(n^{1.585\ldots})$
- Amusing exercise: generalize Karatsuba to do 5 size n/3 subproblems

  This gives $\Theta(n^{1.46\ldots})$ time algorithm

- Best known algorithm runs in $\Theta(n \log n)$ using fast Fourier transform

  but mostly unused in practice (unless you need really big numbers - a billion digits of π, say)

- Best lower bound $O(n)$: A fundamental open problem

Median

Selecting k-th smallest

Problem: Given numbers $x_1, \dots, x_n$ and an integer $1 \le k \le n$ output the k-th smallest number

$$\mathrm{Sel}(\{x_1, \dots, x_n\}, k)$$

A simple algorithm: Sort the numbers in time O(n log n) then return the k-th smallest in the array.

Can we do better?

Yes, in time $O(n)$ if $k = 1$ or $k = 2$.

Can we do $O(n)$ for all possible values of k?

Assume all numbers are distinct for simplicity.

An Idea

Choose a number w from $x_1, \dots, x_n$

Define

- $S_<(w) = \{x_i : x_i < w\}$
- $S_=(w) = \{x_i : x_i = w\}$
- $S_>(w) = \{x_i : x_i > w\}$

Can be computed in linear time

Solve the problem recursively as follows:

- If $k \leq |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \leq |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

Ideally want $|S_<(w)|, |S_>(w)| \leq n/2$. In this case ALG runs in $O(n) + O\left(\frac{n}{2}\right) + O\left(\frac{n}{4}\right) + \cdots + O(1) = O(n)$.

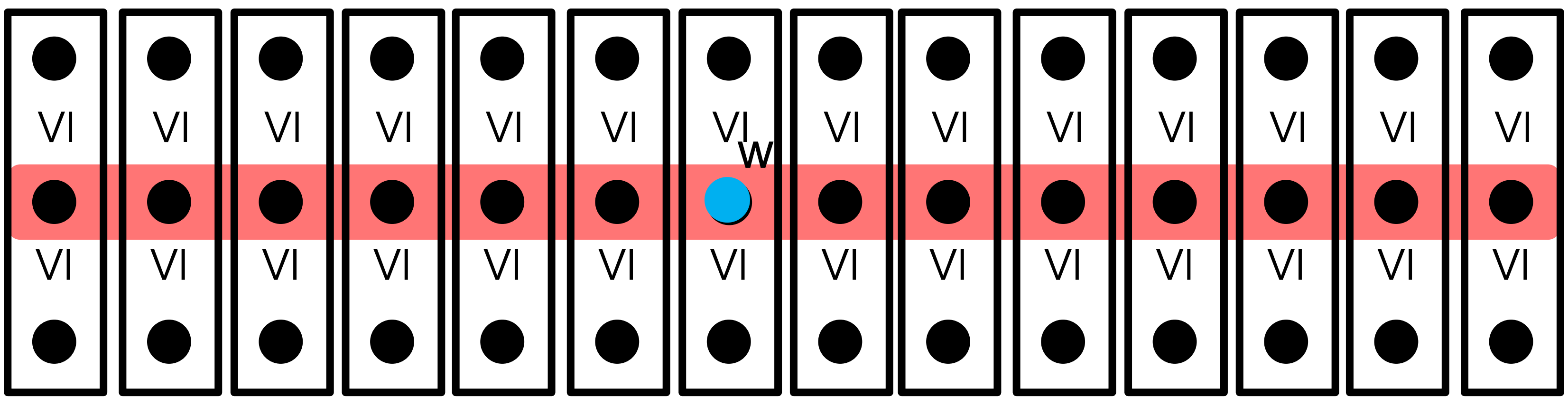
How to choose w?

Suppose we choose w uniformly at random similar to the pivot in quicksort.

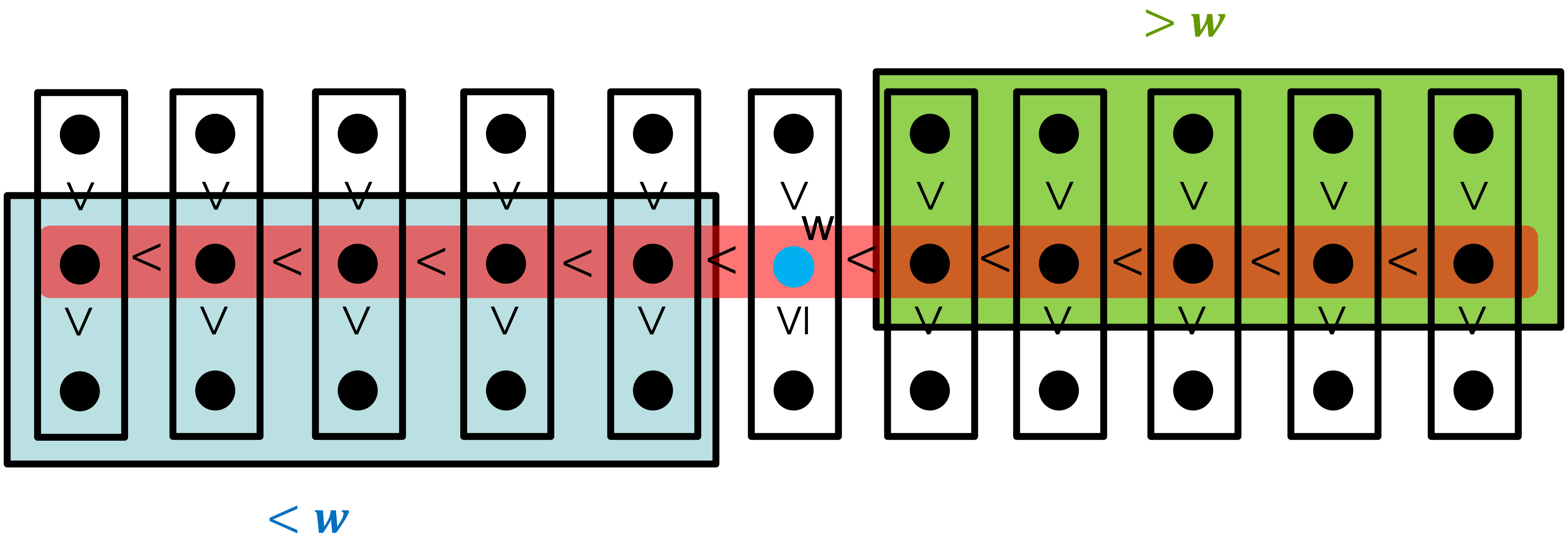
Then, $\mathbb{E}[|S_<(w)|] = \mathbb{E}[|S_>(w)|] = n/2$. Algorithm runs in $O(n)$ in expectation.

Can we get $O(n)$ running time deterministically?

- Partition numbers into sets of size 3.
- Sort each set (takes O(n))
- $w = Sel(midpoints, n/6)$



How to lower bound $|S_<(w)|, |S_>(w)|$?

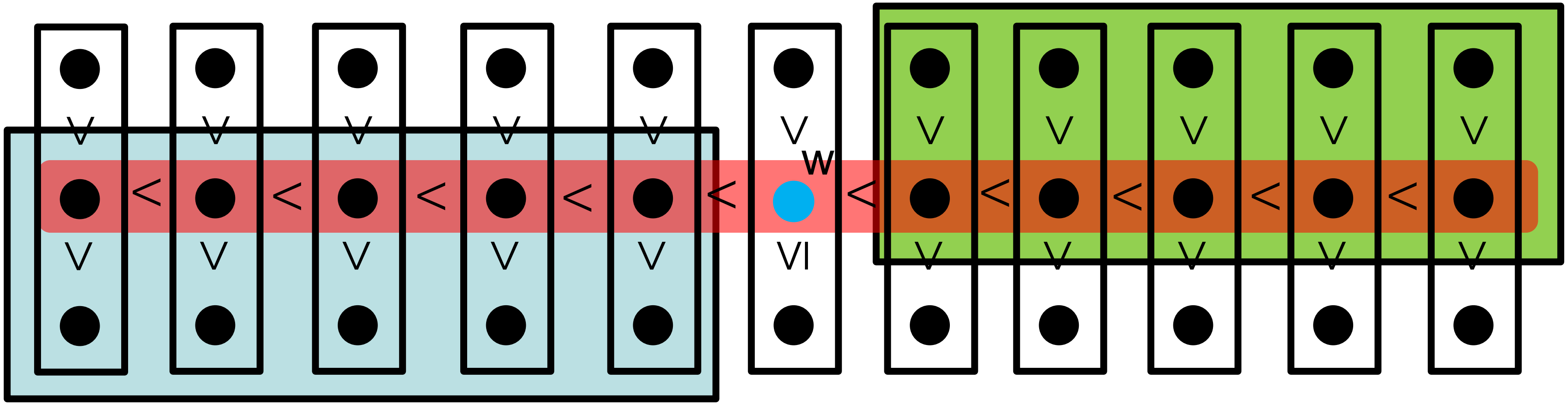


- $|S_<(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$
- $|S_>(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$.

$$\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$$

So, what is the running time?

Asymptotic Running Time?



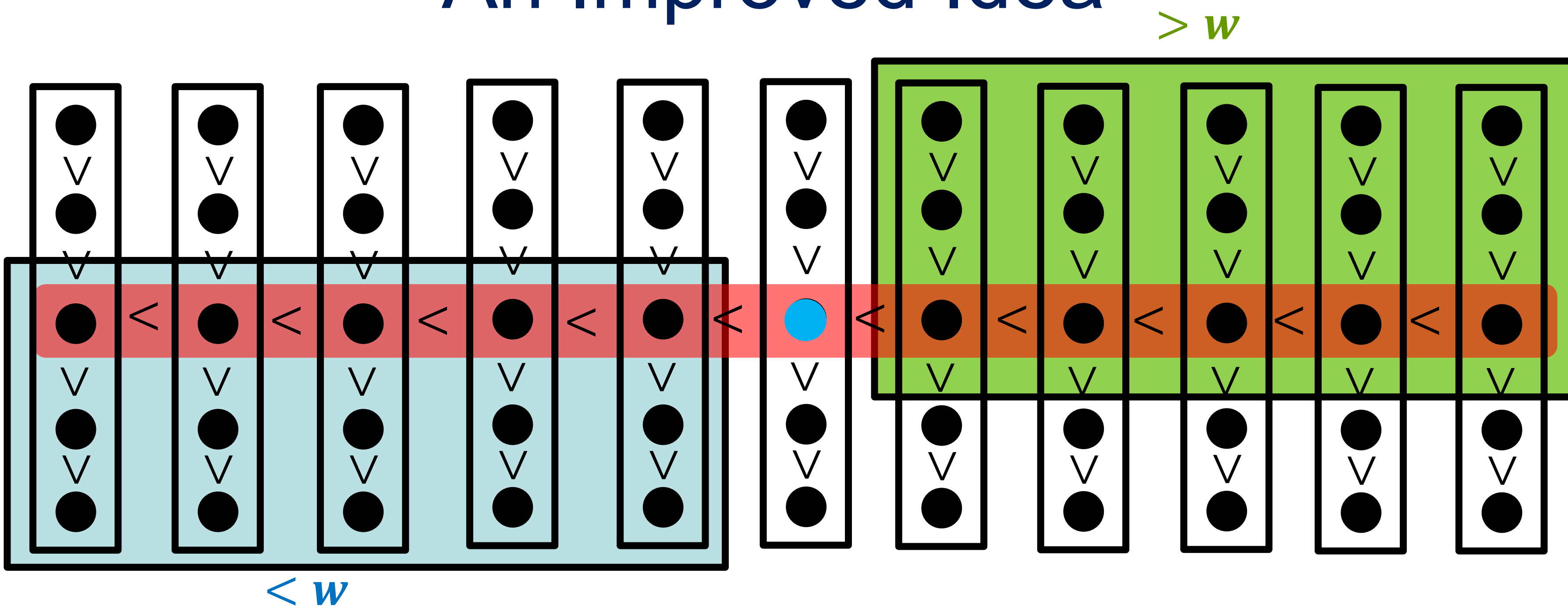
- If $k \leq |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \leq |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - S_<(w) - S_=(w))$

O(nlog n) again?
So, what is the point?

Where $\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n) \Rightarrow T(n) = O(n \log n)$$

An Improved Idea



Partition into n/5 sets. Sort each set and set $w = Sel(midpoints, n/10)$

- $|S_<(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$
- $|S_>(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$

$$\Rightarrow \quad \frac{3n}{10} \leq |S_<(w)|, |S_>(w)| \leq \frac{7n}{10}$$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n) \Rightarrow T(n) = O(n)$$

An Improved Idea

```
Sel(S, k) {
   n ← |S|
   If (n < ??) return ??
   Partition S into n/5 sets of size 5
   Sort each set of size 5 and let M be the set of medians, so
|M|=n/5
   Let w=Sel(M,n/10)
   For i=1 to n{
     If x_i < w add x to S_<(w)
     If x_i > w add x to S_>(w)
     If x_i = w add x to S_=(w)
   }
   If (k ≤ |S_<(w)|)
     return Sel(S_<(w), k)
   else if (k ≤ |S_<(w)| + |S_=(w)|)
     return w;
   else
     return Sel(S_>(w), k − |S_<(w)| − |S_=(w)|)
}
```

We can maintain each set in an array

D&C Summary

Idea:

“Two halves are better than a whole”

- if the base algorithm has super-linear complexity.

“If a little's good, then more's better”

- repeat above, recursively

- Applications: Many.
  - Binary Search, Merge Sort, (Quicksort),
  - Root of a Function
  - Closest points,
  - Integer multiplication
  - Median

Approximation Algorithms

How to deal with NP-complete Problem

Many of the important problems in real world are NP-complete.

SAT, Set Cover, Graph Coloring, TSP, Max IND Set, Vertex Cover, …

So, we cannot find optimum solutions in polynomial time.

What to do instead?

- Find optimum solution of special cases (e.g., random inputs)

- Find near optimum solution in the worst case

Approximation Algorithm

Polynomial-time Algorithms with a guaranteed approximation ratio.

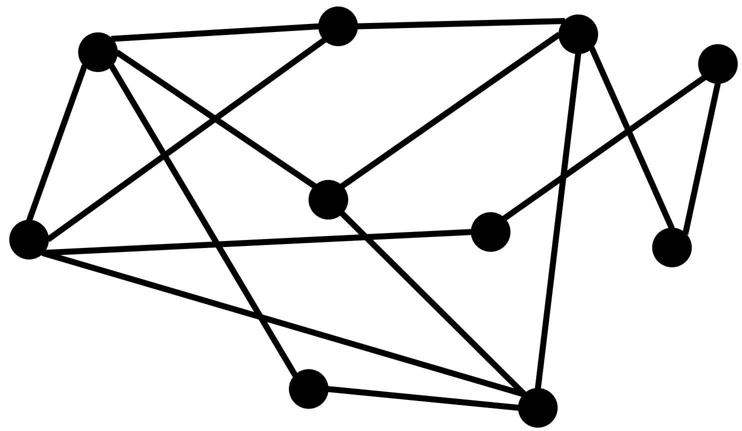
$$\alpha = \frac{\text{Cost of computed solution}}{\text{Cost of the optimum}}$$

worst case over all instances.

Goal: For each NP-hard problem find an approximation algorithm with the best possible approximation ratio.

Vertex Cover

Given a graph G=(V,E), Find smallest set of vertices touching every edge



Greedy Algorithm?

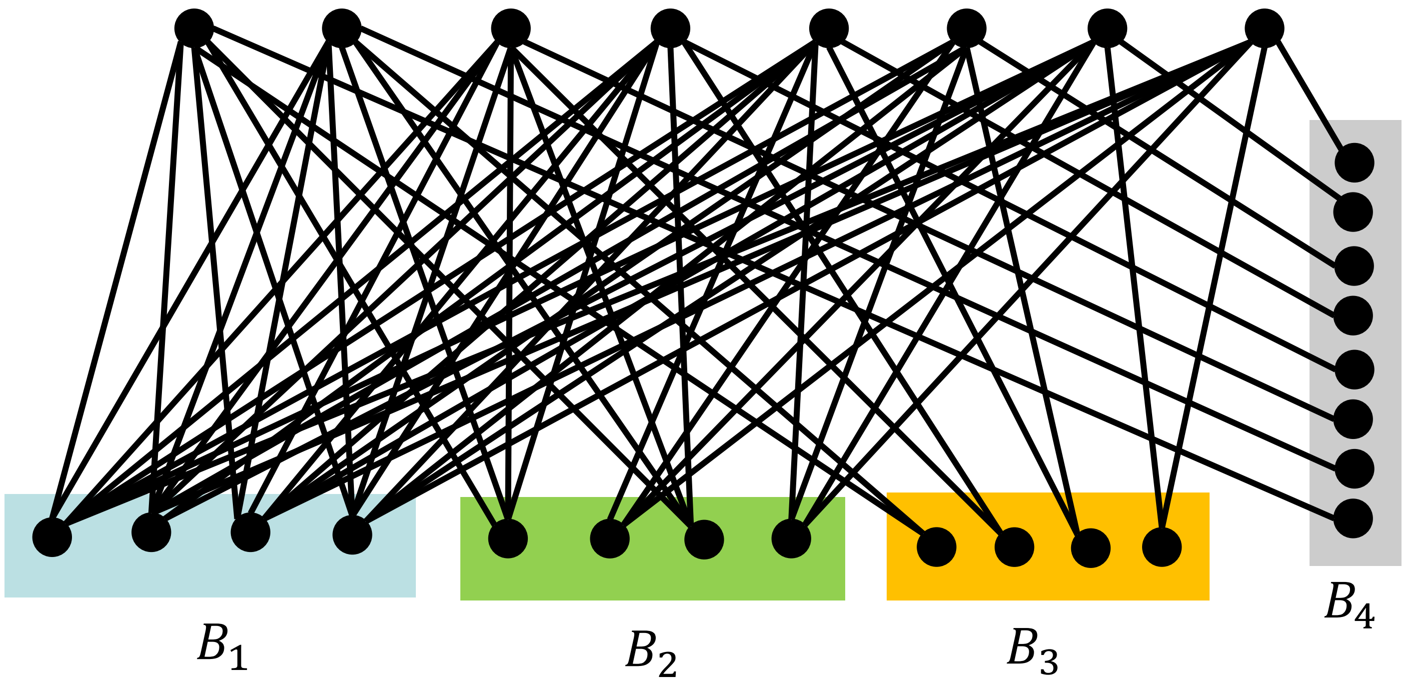
Greedy algorithms are typically used in practice to find a (good) solution to NP-hard problems

Strategy (1): Iteratively, include a vertex that covers most new edges

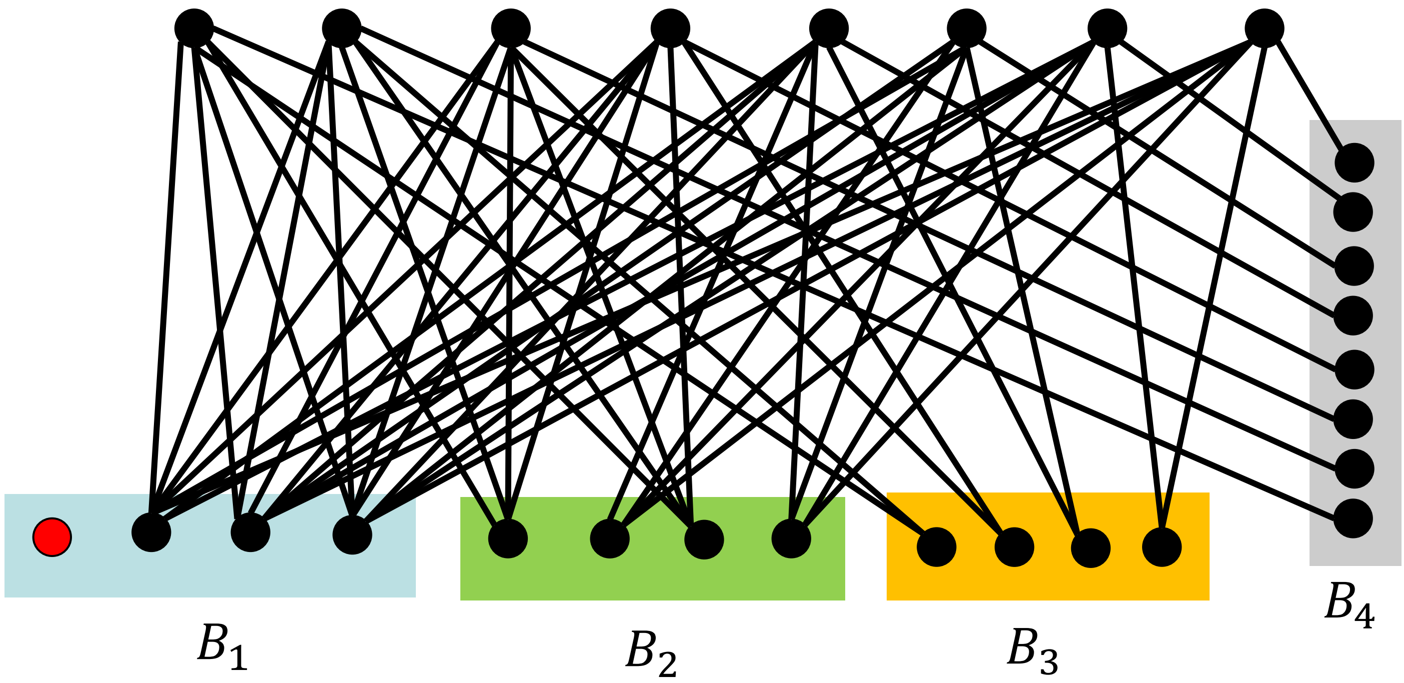
Q:Does this give an optimum solution?

A: No,

Greedy (1): Pick vertex that covers the most



Greedy (1): Pick vertex that covers the most



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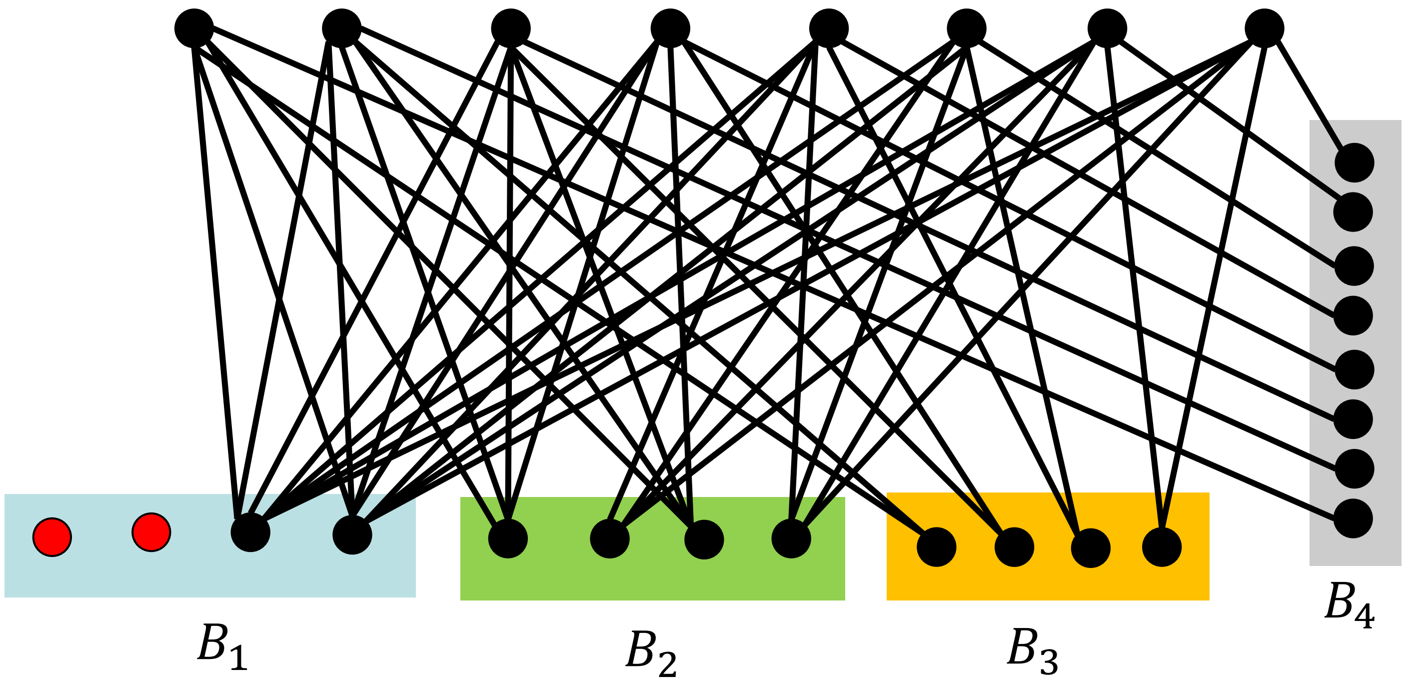

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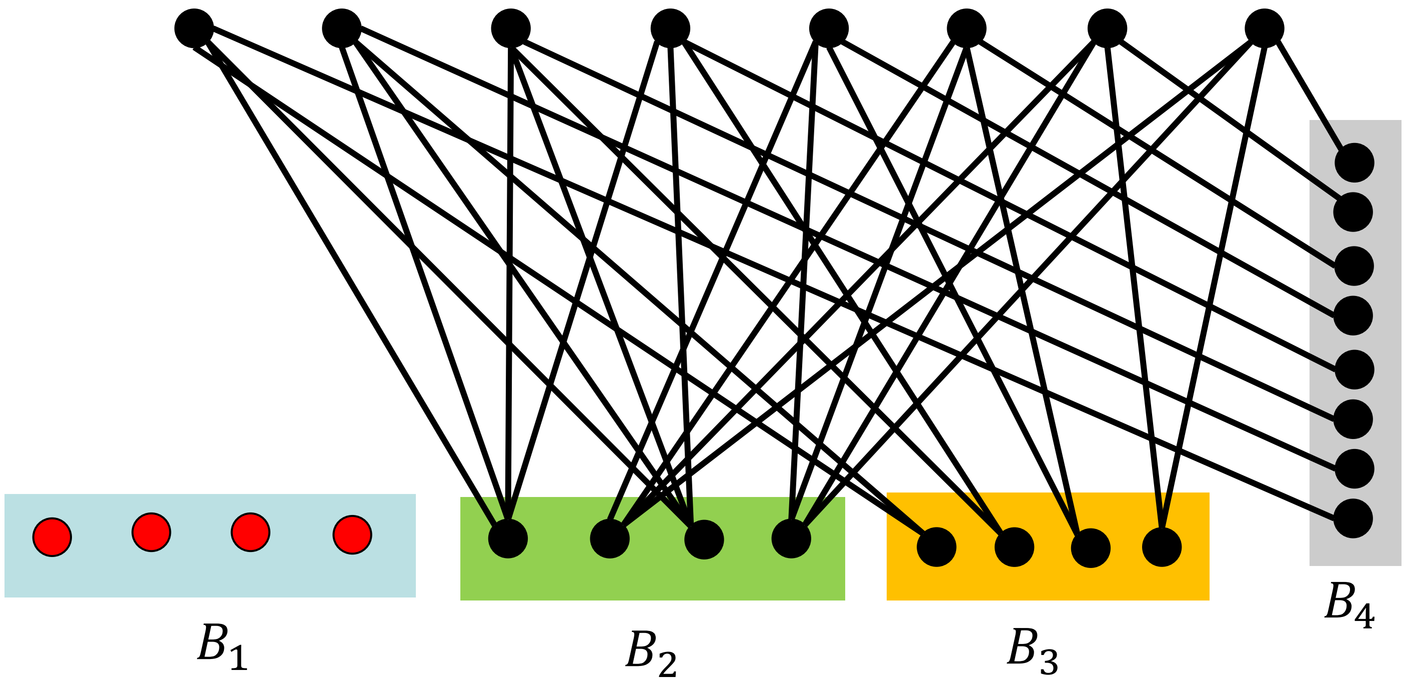

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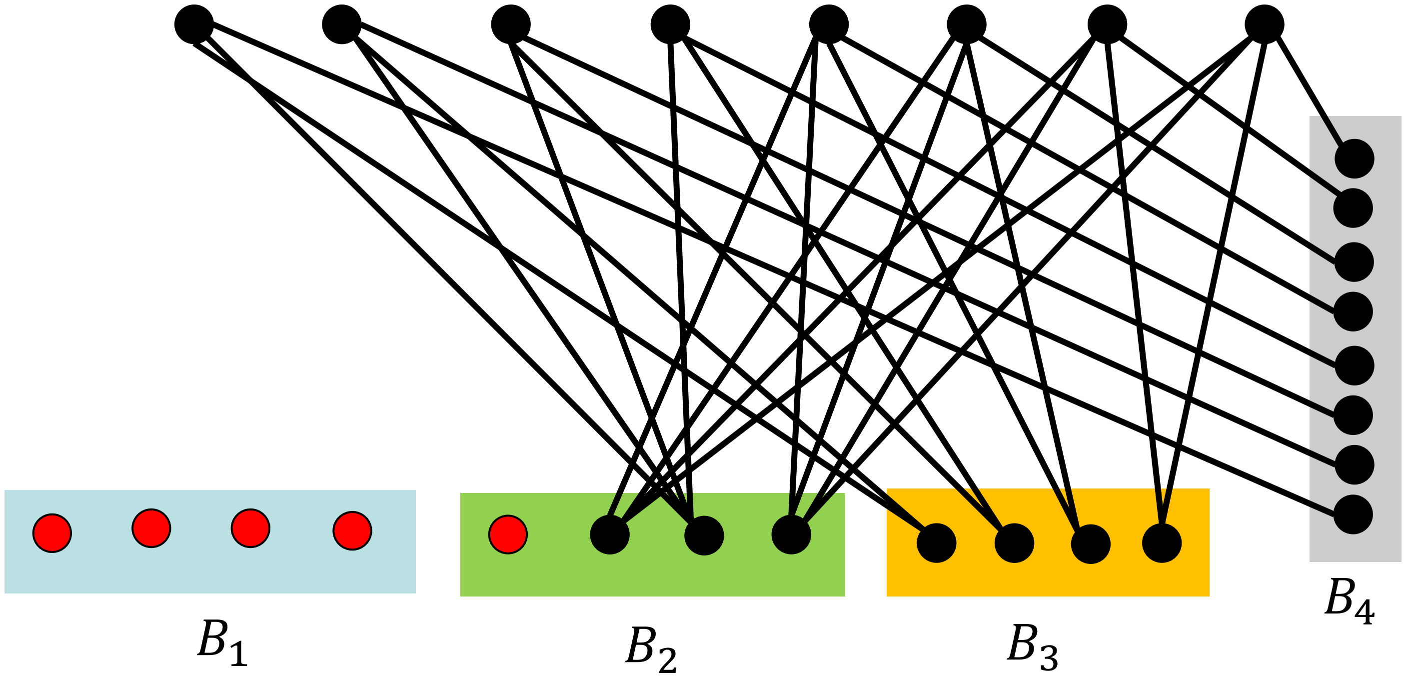

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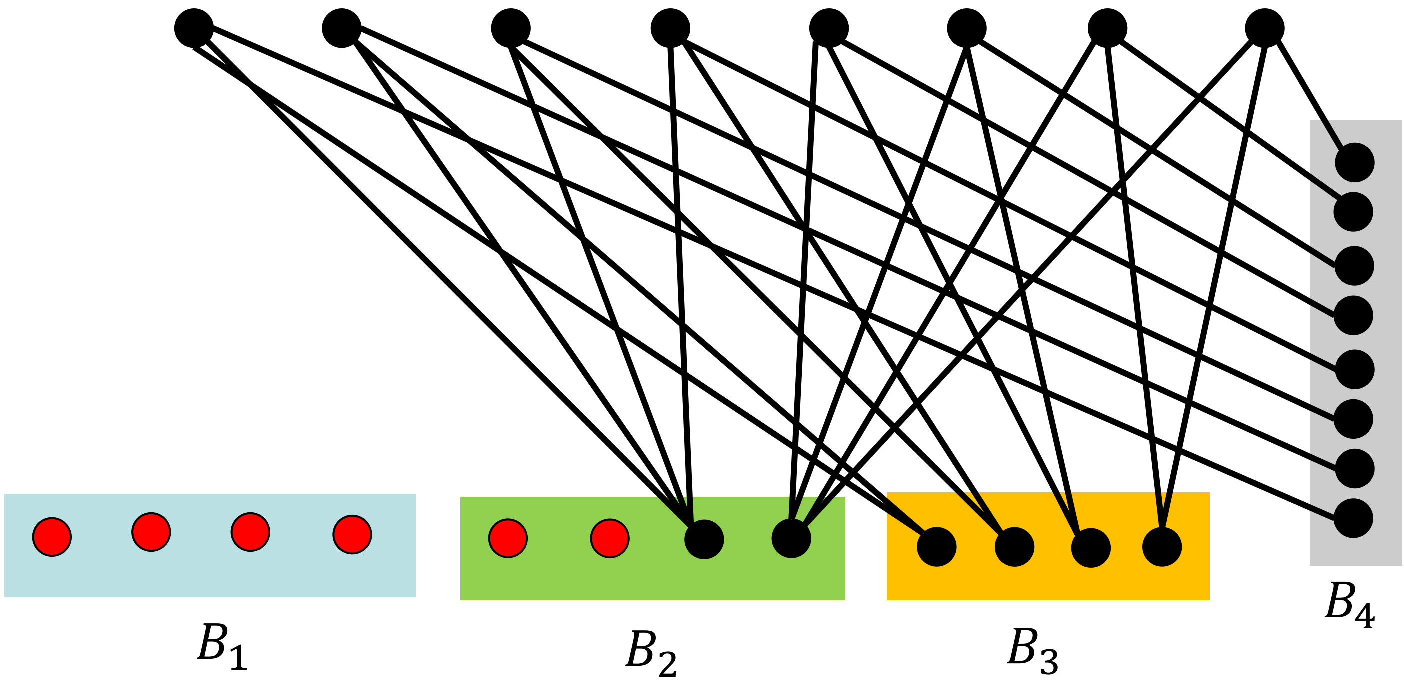

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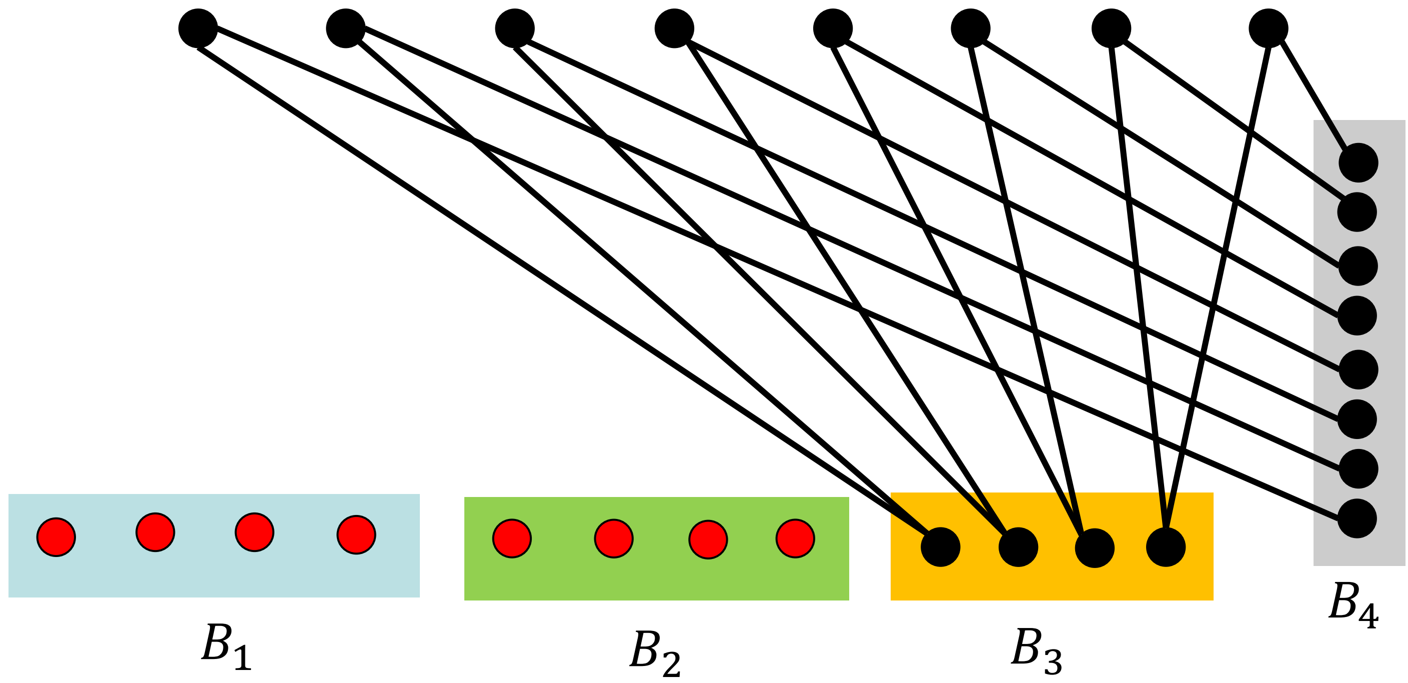

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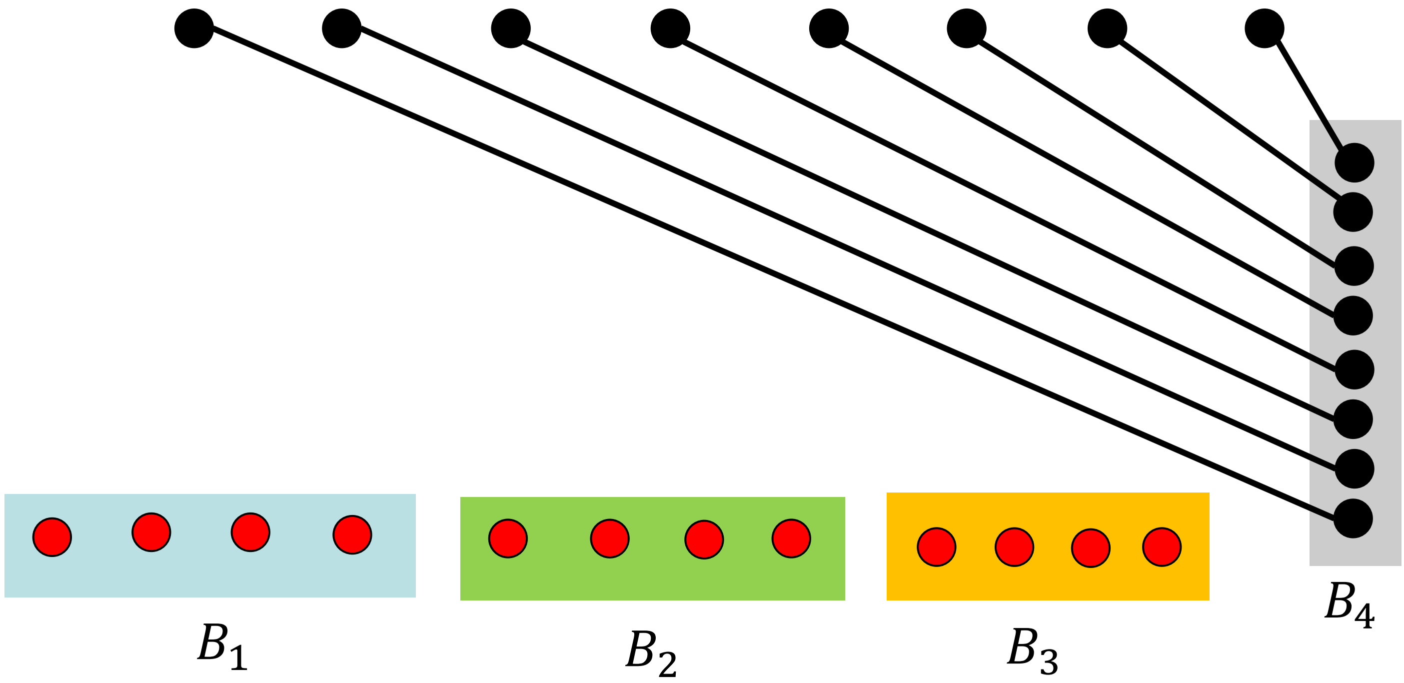
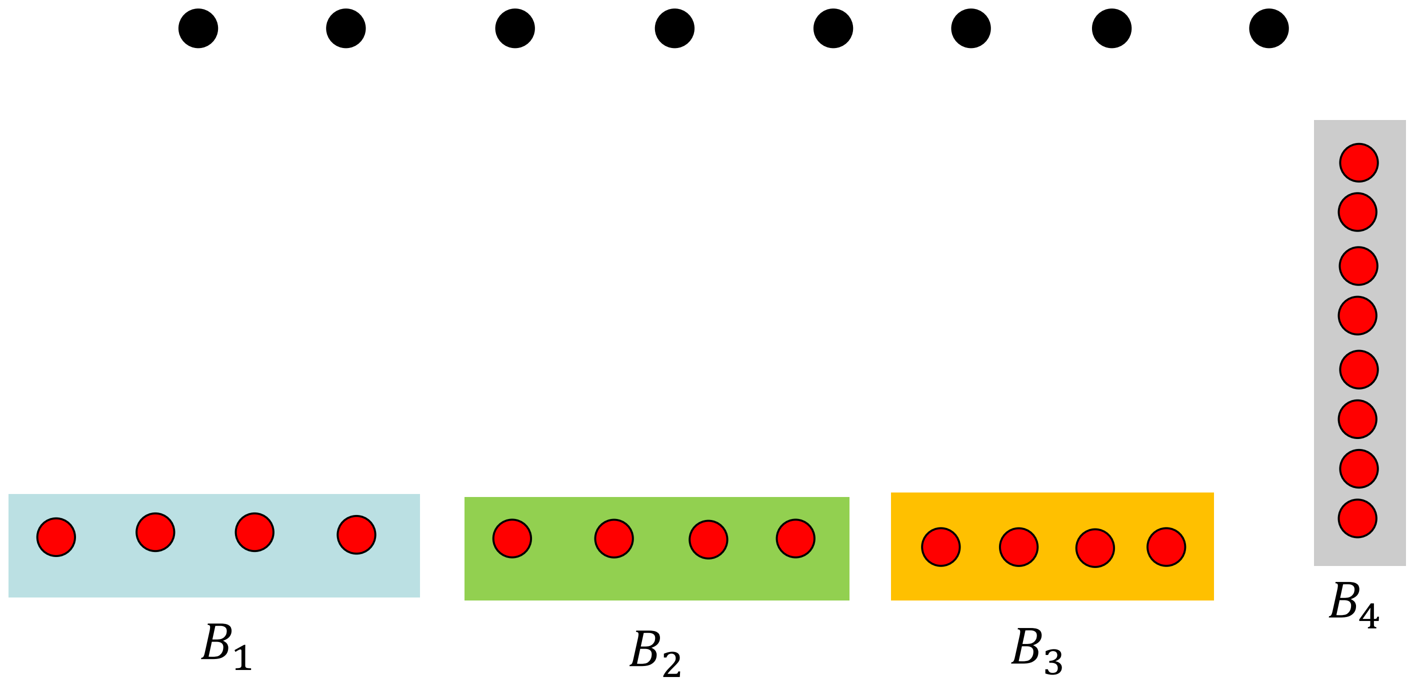


Greedy (1): Pick vertex that covers the most



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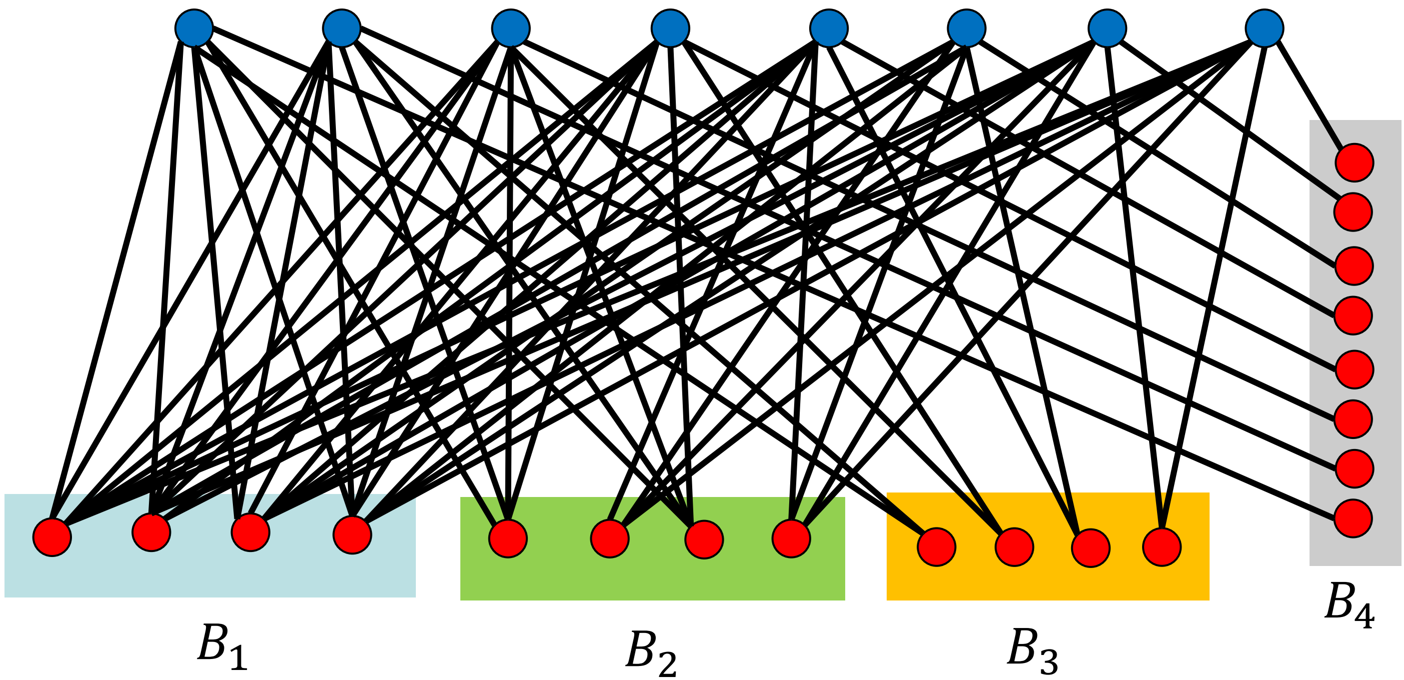
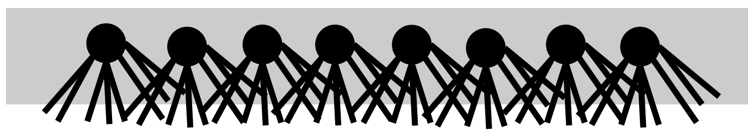


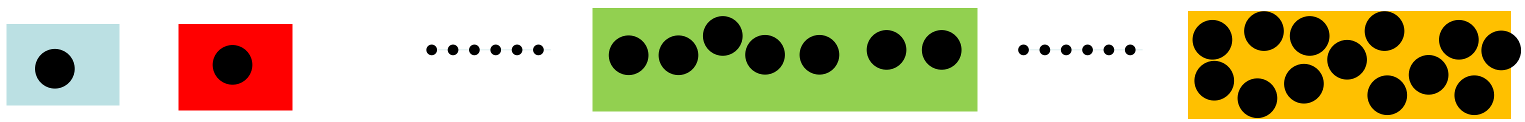
Greedy Vertex cover = 20

OPT Vertex cover = 8

Greedy (1): Pick vertex that covers the most

n vertices. Each vertex has one edge into each B_i





B_n B_{n-1} $|B_i| = n/i$ B_1

Each vertex in B_i has i edges to top

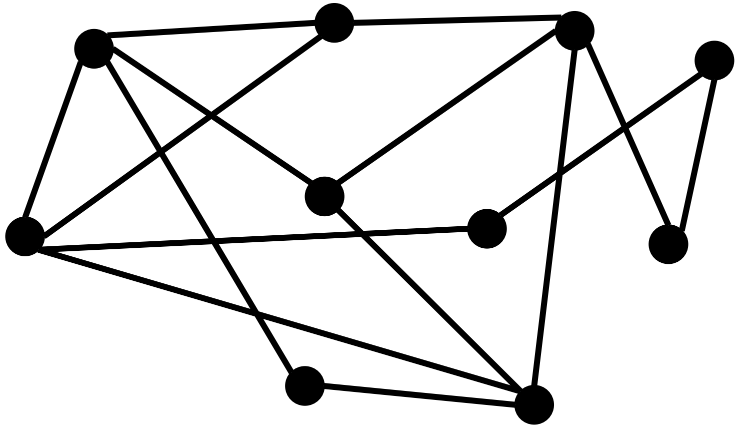
Greedy pick bottom vertices = $n + \frac{n}{2} + \frac{n}{3} + \cdots + 1 \approx n \ln n$

OPT pick top vertices = n

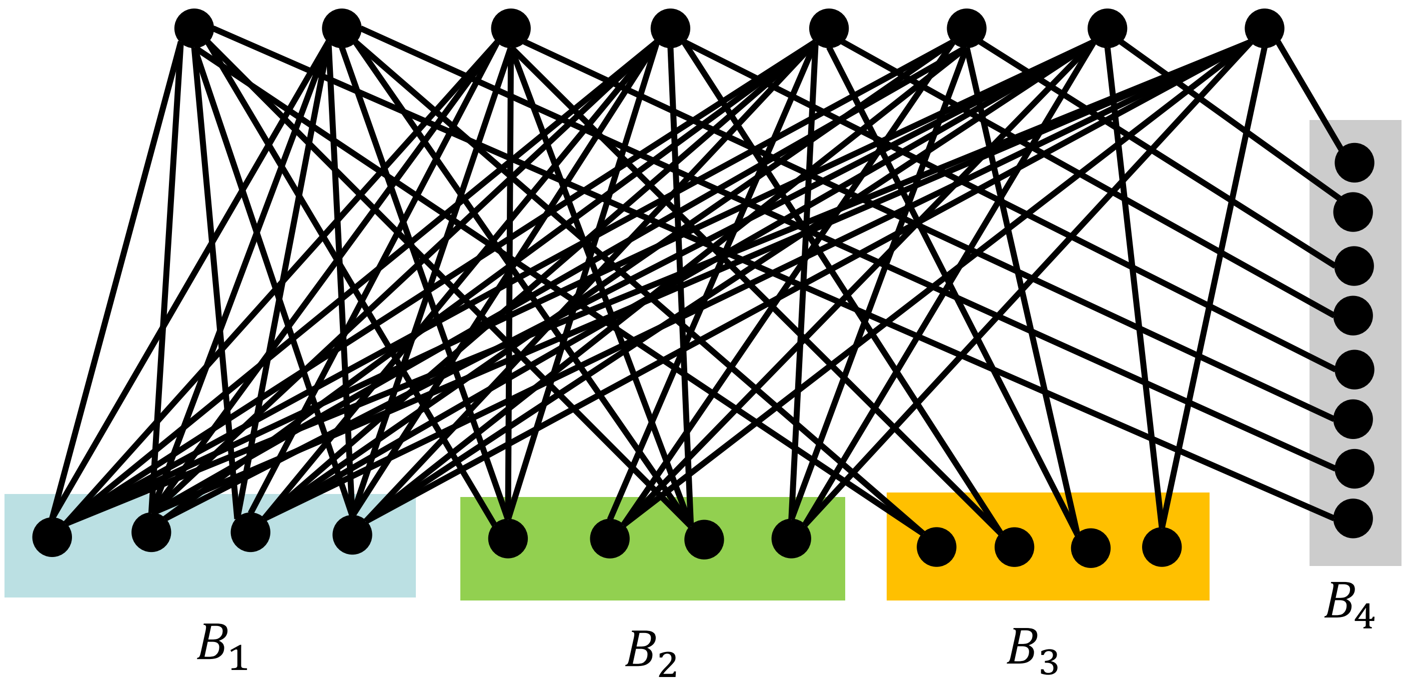
A Different Greedy Rule

Greedy 2: Iteratively, pick both endpoints of an uncovered edge.

Vertex cover = 6



Greedy 2: Pick Both endpoints of an uncovered edge



Greedy vertex cover = 16

OPT vertex cover = 8

Greedy (2) gives 2-approximation

Thm: Size of greedy (2) vertex cover is at most twice as big as size of optimal cover

Pf: Suppose Greedy (2) picks endpoints of edges $e_1, \ldots, e_k$. Since these edges do not touch, every valid cover must pick one vertex from each of these edges!

i.e., $OPT \geq k$.

But the size of greedy cover is 2k. So, Greedy is a 2-approximation.