**CSE421: Design and Analysis of Algorithms** May 26th, 2024

# Homework 8

*Shayan Oveis Gharan* Due: May 30th, 2024 at 23:59 PM

P1) For this problem no proof is needed. Translate the following LP into the standard form.

$$\begin{aligned} \min \quad & 3x_1 - x_2 \\ \text{s.t.,} \quad & x_1 + x_2 + x_3 = 1 \\ & 2x_1 - x_3 \geq x_2 - 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

P2) Suppose we are given a number of sets $S_1, \ldots, S_m$ where each $S_i$ is a subset of $\{1, \ldots, n\}$ and has a cost $c_i$. In the weighted set cover problem we want to choose a minimum cost family of these sets such that every element in $\{1, \ldots, n\}$ is in at least one set. Consider the following Linear program.

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} c_i x_i \\ \text{s.t.,} \quad & \sum_{i: j \in S_i} x_i \geq 1 \qquad \forall 1 \leq j \leq n \\ & x_i \geq 0 \qquad\qquad \forall 1 \leq i \leq m. \end{aligned}$$

Note that in the first constraint the sum is over all sets that contain element $j$.

a) Prove that the above LP is a relaxation for the weighted set cover problem. In other words, if OPT is the integer optimum of the weighted set cover problem and OPT-LP is the optimum of the LP, show that OPT-LP $\leq$ OPT.

b) Turn this LP into the standard form and write its dual in the standard form.

P3) Solve the following problem by writing a linear program. Given a directed graph $G = (V, E)$ and a pair of vertices $s, t$. Every edge $e$ has a capacity $c_e$ and a fee $p_e$. Every vertex $v \neq s, t$ has a capacity $c_v$ and a fee $p_v$. Given a demand $D$, we want to send $D$ units of flow from $s$ to $t$ such that the total flow going through each vertex $v \neq s, t$ is at most $c_v$ and the total payment is minimized: The total payment is $\sum_v f(v) p_v + \sum_e f(e) p_e$ where $f(v)$ is the total flow going through $v$.

You don't need to give proofs for this problem. Clearly explain what every variable of your LP represents and what every constraint of your LP represents.

P4) Directed Hamiltonian Path problem is defined as follows: Given a directed graph $G$ with two vertices $s, t \in V$, output "yes" if there is a directed Hamiltonian path from $s$ to $t$ and "no" otherwise.

Undirected Hamiltonian Path problem is defined as follows: Given an undirected graph $G$ with two vertices $a, b \in V$, output "yes" if there is an undirected Hamiltonian path from $a$ to $b$ and "no" otherwise.

Prove that Directed Hamiltonian Path $\leq_P$ Undirected Hamiltonian Path.

P5) **Extra Credit:** Prove that the Hamiltonian cycle problem in directed graphs is NP-Complete. You may use the fact that 3SAT is NP-Complete.