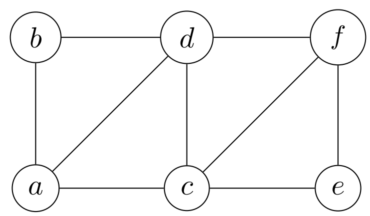
Homework 7

Shayan Oveis Gharan Due: May 22nd, 2024 at 23:59 PM

P1) Given an undirected graph $G = (V, E)$ and a set of vertices $S \subseteq V$ and a disjoint set of vertices $T \subseteq V$, i.e., $S \cap T = \emptyset$. Design a polynomial time algorithm that outputs the maximum number of vertex disjoint paths between vertices of S and T (note that every vertex in S and every vertex in T can be in **at most one** path).

For example in the following graph if $S = \{a, b, d\}$ and $T = \{e, f\}$ you should output 2 corresponding to the following two paths:

$$(d, f) \qquad (b, a, c, e)$$



P2) Given a directed unweighted graph $G = (V, E)$. Suppose that there are k edge disjoint paths from s to t and there are k edge disjoint paths from t to u for vertices $s, t, u \in V$. Prove that there are k edge disjoint paths from s to u.

P3) In this exercise we give a polynomial time algorithm to find the minimum vertex cover and maximum independent set in a bipartite graph $G = (X, Y, E)$. Following these steps:

a) **Optional:** Construct a flow network H from the given G just as in the maximum matching algorithm that we discuss in class. Simply, draw a graph similar to what we do in class and name different subsets of vertices accordingly.

b) Given a min cut $s - t$ cut (A, B) in H, construct a vertex cover $S \subseteq X \cup Y$ of G such that $|S| = cap(A, B)$

c) Conversely, given a min vertex cover $S \subseteq X \cup Y$ of G, construct a $s - t$ cut (A, B) in H such that $cap(A, B) = |S|$.

d) Write down the algorithm and use the above argument to prove that it correctly finds the min vertex cover of G.

e) **Optional:** Show that for any graph G the complement of the min vertex cover is the maximum independent set. In other words, if S is a minimum vertex cover then $V - S$ is a maximum independent set of G.

P4) Given an $n \times n$ chess board where some cells are removed. Design a polynomial time algorithm to find the maximum number of knights that can be placed on this board such that no two knights attack each other.

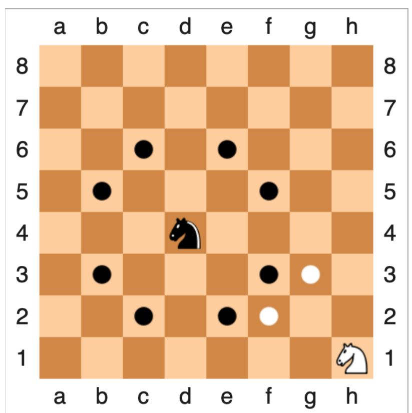
For example in the following 3×3 chessboard the removed cells are marked with X. You can put 4 knights such that no two can attack each other as we did in the right. The location of every knight is marked with a $\bullet$.

X		
		X
	X	

$\Rightarrow$

X		$\bullet$
	$\bullet$	X
$\bullet$	X	$\bullet$

Please see the following image for locations that a knight can attack. In general a knight can attack at most 8 cells if they exist. For example, the white knight can only attack two cells in the following picture.



P5) **Extra Credit:** You are given an $m \times n$ array of real numbers. Suppose that the numbers in each row add up to an integer and the numbers in each column add up to an integer. You want to substitute each number $A[i, j]$ with $\lfloor A[i, j] \rfloor$ or $\lceil A[i, j] \rceil$ such that the sum of the numbers in each row and each column remain invariant. Design a polynomial time algorithm that outputs the integer array.

For example, if the input is the left table you can output the right table. Note the sum of numbers in each row (and each column) of the left table is the same as the sum of the numbers of the same row (resp. the same column) in the right table.

0.4	0.1	1.5
0.6	1.9	0.5

$\Rightarrow$

1	0	1
0	2	1