CSE 421 Section 9

NP-completeness

Administrivia

Announcements & Reminders

- **HW6** regrade requests are open
- **HW7**
  - Due **tomorrow** 11/22 @ 11:59pm
  - Late submissions will be open until Sunday, 11/24 @ 11:59pm
- **HW8**
  - Will be released over the weekend
  - Due Wednesday, December 4th @ 11:59pm
- No section next week, happy Thanksgiving!

Definition review

Definition review

There were many new definitions in lecture recently that we'll review now.

To check your understanding, for each definition starting next slide, give an example of the definition!

Definition review

- **Problem**: a set of inputs and the correct outputs
- **Instance**: a single input to a problem
- **Decision problem**: a problem where the output is “yes” or “no”
- **Reduction**:

  $A \leq_p B$ “A reduces to B” “A is not harder than B” “Solve A using B”

  Formally, $A \leq_p B$ if there is an algorithm that solves A using polynomially many calls to a solver for B, running in polynomial time (excluding calls to B).

Definition review

- **P (“polynomial”)**: The set of **decision** problems A that can be solved in poly time.
- **NP (“nondeterministic polynomial”)**: The set of **decision** problems A for which YES-instances can be verified in poly time.

  Formally, there is a poly time algorithm $\textsc{VerifyA}$ such that for all inputs x,
    - If x is YES, there exists a poly length string y such that $\textsc{VerifyA}(x, y) =$ YES.
    - If x is NO, then for all poly length strings y, $\textsc{VerifyA}(x, y) =$ NO.
- **NP-hard**: A problem B is NP-hard if $A \leq_p B$ for all A in NP.
- **NP-complete**: A problem B is NP-complete if B is in NP and B is NP-hard.

Definition review

- **Boolean literal**: A Boolean variable x_i or its negation $\neg x_i$
- **Clause**: OR of zero or more literals
- **CNF formula**: AND of zero or more clauses
- **3SAT problem**:

  Input: A CNF formula with exactly 3 literals per clause
  Output: Is there an assignment to the variables that makes the formula true?

  3SAT is a fundamental **NP-complete** problem.

Practice with SAT

Problem 1 – SATisfy This

Determine whether each instance of 3-SAT is satisfiable. If it is, list a satisfying variable assignment.

a) $(\neg a \lor \neg b \lor c) \land (a \lor c \lor \neg d) \land (b \lor \neg c \lor \neg d) \land (\neg a \lor b \lor c) \land (\neg b \lor c \lor \neg d)$

b) $(\neg a \lor b \lor d) \land (\neg b \lor c \lor d) \land (a \lor \neg c \lor d) \land (a \lor \neg b \lor \neg d)$
$\land (b \lor \neg c \lor \neg d) \land (\neg a \lor c \lor \neg d) \land (a \lor b \lor c) \land (\neg a \lor \neg b \lor \neg c)$

Think about it with the people around you, then we'll discuss!

Solution

Problem 1 – SATisfy This

a) $(\neg a \lor \neg b \lor c) \land (a \lor c \lor \neg d) \land (b \lor \neg c \lor \neg d) \land (\neg a \lor b \lor c) \land (\neg b \lor c \lor \neg d)$

Satisfiable. Many possible solutions (students only need to list one of these):

- $a = 0, b = 0, c = 0, d = 0$
- $a = 0, b = 0, c = 1, d = 0$
- $a = 0, b = 1, c = 0, d = 0$
- $a = 0, b = 1, c = 1, d = 0$
- $a = 0, b = 1, c = 1, d = 1$
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Solution

Problem 1 – SATisfy This

b) $(\neg a \vee b \vee d) \wedge (\neg b \vee c \vee d) \wedge (a \vee \neg c \vee d) \wedge (a \vee \neg b \vee \neg d)$
$\wedge (b \vee \neg c \vee \neg d) \wedge (\neg a \vee c \vee \neg d) \wedge (a \vee b \vee c) \wedge (\neg a \vee \neg b \vee \neg c)$

Not satisfiable. Although you might be able to try some ad hoc arguments for why, there is generally no explanation significantly faster than “try everything”.

The next 16 slides show what that looks like.

Solution

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$$\wedge (\neg 1 \vee 1 \vee \neg 0) \wedge (1 \vee 1 \vee 1) \wedge (\neg 1 \vee \neg 1 \vee \neg 1)$$

$$(0 \vee 1 \vee 0) \wedge (0 \vee 1 \vee 0) \wedge (1 \vee 0 \vee 0) \wedge (1 \vee 0 \vee 1) \wedge (1 \vee 0 \vee 1)$$
$$\wedge (0 \vee 1 \vee 1) \wedge (1 \vee 1 \vee 1) \wedge (0 \vee 0 \vee 0)$$

Solution

Problem 1 – SATisfy This

b) $(\neg a \lor b \lor d) \land (\neg b \lor c \lor d) \land (a \lor \neg c \lor d) \land (a \lor \neg b \lor \neg d)$
$\land (b \lor \neg c \lor \neg d) \land (\neg a \lor c \lor \neg d) \land (a \lor b \lor c) \land (\neg a \lor \neg b \lor \neg c)$

Checking assignment: $a = 1, b = 1, c = 1, d = 1$

$$(\neg a \lor b \lor d) \land (\neg b \lor c \lor d) \land (a \lor \neg c \lor d) \land (a \lor \neg b \lor \neg d) \land (b \lor \neg c \lor \neg d)$$
$$\land (\neg a \lor c \lor \neg d) \land (a \lor b \lor c) \land (\neg a \lor \neg b \lor \neg c)$$

$$(\neg 1 \lor 1 \lor 1) \land (\neg 1 \lor 1 \lor 1) \land (1 \lor \neg 1 \lor 1) \land (1 \lor \neg 1 \lor \neg 1) \land (1 \lor \neg 1 \lor \neg 1)$$
$$\land (\neg 1 \lor 1 \lor \neg 1) \land (1 \lor 1 \lor 1) \land (\neg 1 \lor \neg 1 \lor \neg 1)$$

$$(0 \lor 1 \lor 1) \land (0 \lor 1 \lor 1) \land (1 \lor 0 \lor 1) \land (1 \lor 0 \lor 0) \land (1 \lor 0 \lor 0)$$
$$\land (0 \lor 1 \lor 0) \land (1 \lor 1 \lor 1) \land (0 \lor 0 \lor 0)$$

Reductions

How to prove NP-hardness

In previous weeks of this class, you've seen reductions of the following form:

$$A \le_p B$$

your problem

an algorithm to be used as a library,
i.e. stable matching, network flow

"**My problem is easy** to solve, because I can just use B."

How to prove NP-hardness

Now, for NP-hardness, we need to do the opposite.

$$A \le_p B$$

a problem that is known to be hard

your problem

“**My problem is hard**, because if it were easy, then A would be easy, but A is hard.”

In other words, we convert **from instances of the hard problem to your problem**.

NOT solving your problem!

How to prove NP-completeness

Show B is in NP:

1. State what the certificate is.
2. Say why the certificate can be checked in polynomial time.

Show B is NP-hard:

3. Identify an NP-hard problem A and say, “We will reduce from A to B”.
4. Define a reduction function f, which converts instances of A into instances of B.
5. Say why f is computable in polynomial time.
6. Show that “x is a YES-instance for A” $\Rightarrow$ “$f(x)$ is a YES-instance for B”.
    - **Convert a certificate** for x into a certificate for $f(x)$.
7. Show that “$f(x)$ is a YES-instance for B” $\Rightarrow$ “x is a YES-instance for A”.
    - **Convert a certificate** for $f(x)$ into a certificate for x.

Problem 2 – 5SAT

Define the problem **5SAT** to be:

Input: A CNF formula with exactly 5 literals per clause

Output: Is there an assignment to the variables that makes the formula true?

We will show that 5SAT is NP-complete.

First, we will show that 5SAT is in NP.

a) State what the certificate is.
b) Say why the certificate can be checked in polynomial time.

Think about this briefly!

Solution

Problem 2 – 5SAT

a) State what the certificate is.

Certificate: An assignment to the variables that makes the formula true.

b) Say why the certificate can be checked in polynomial time.

Verifier: Takes as input the original 5SAT input and the certificate (the assignment).
Just apply the assignment to every clause, and return whether all clauses are satisfied.
Runs in linear time.

Problem 2 – 5SAT

Recall **3SAT**:

Input: A CNF formula with exactly 3 literals per clause

Output: Is there an assignment to the variables that makes the formula true?

We will now prove that 5SAT is NP-hard with a reduction involving 3SAT.

c) Fill in the blank: "We will reduce from __ to __". Which is A, and which is B?

Think about this briefly!

Solution

Problem 2 – 5SAT

Recall **3SAT**:

Input: A CNF formula with exactly 3 literals per clause

Output: Is there an assignment to the variables that makes the formula true?

We will now prove that 5SAT is NP-hard with a reduction involving 3SAT.

c) Fill in the blank: “We will reduce from __ to __”. Which is A, and which is B?

We will reduce from A = 3SAT to B = 5SAT.

In other words, convert instances of 3SAT into instances of 5SAT.

Problem 2 – 5SAT

c) Fill in the blank: “We will reduce from __ to __”. Which is A, and which is B?

d) Define a reduction function f, which converts instances of A into B.

Previous Solution

Problem 2 – 5SAT

c) Fill in the blank: "We will reduce from __ to __". Which is A, and which is B?

We will reduce from A = 3SAT to B = 5SAT.

In other words, convert instances of 3SAT into instances of 5SAT.

d) Define a reduction function f, which converts instances of A into B.

Think about it with the people around you, then we'll discuss!

Solution

Problem 2 – 5SAT

c) Fill in the blank: "We will reduce from __ to __". Which is A, and which is B?

We will reduce from A = 3SAT to B = 5SAT.

In other words, convert instances of 3SAT into instances of 5SAT.

d) Define a reduction function f, which converts instances of A into B.

Let $C_1, C_2, \ldots, C_m$ be the clauses of the 3SAT instance.

Create two dummy variables y and z. For each clause C_i, create four clauses:

$$(C_i \vee y \vee z) \qquad (C_i \vee \neg y \vee z) \qquad (C_i \vee y \vee \neg z) \qquad (C_i \vee \neg y \vee \neg z)$$

Our 5SAT instance is the AND of all $4m$ clauses described above.

Problem 2 – 5SAT

e) Say why f is computable in polynomial time.

Think about this briefly!

Solution

Problem 2 – 5SAT

e) Say why f is computable in polynomial time.

We loop through the clauses and create a constant number of new clauses for each, thus linear time.

Problem 2 – 5SAT

To prove the correctness:

f) Show that "x is a YES-instance for A" $\Rightarrow$ "$f(x)$ is a YES-instance for B".
(Remember: convert a certificate for x into a certificate for $f(x)$!)

Think about it with the people around you, then we'll discuss!

Solution

Problem 2 – 5SAT

To prove the correctness:

f) Show that "x is a YES-instance for A" ⇒ "$f(x)$ is a YES-instance for B".
(Remember: convert a certificate for x into a certificate for $f(x)$!)

- There is an assignment α that makes the original 3SAT YES-instance true.
- Let us define an assignment β that satisfies our constructed 5SAT instance.

$\beta(x_i) = \alpha(x_i)$ for all x_i in the original instance

$\beta(y) = 0$ (or 1, doesn't matter)

$\beta(z) = 0$ (or 1, doesn't matter)

- Satisfies our constructed 5SAT instance because every clause contains one of the original 3SAT clauses.

Problem 2 – 5SAT

To prove the correctness:

g) Show that "$f(x)$ is a YES-instance for B" $\Rightarrow$ "x is a YES-instance for A".
(Remember: convert a certificate for $f(x)$ into a certificate for x!)

Think about it with the people around you, then we'll discuss!

Solution

Problem 2 – 5SAT

To prove the correctness:

g) Show that "$f(x)$ is a YES-instance for B" ⇒ "x is a YES-instance for A".
(Remember: convert a certificate for $f(x)$ into a certificate for x!)

- There is an assignment β that makes the formula we constructed true.
- But our formula has clauses

$$(C_i \vee y \vee z) \qquad (C_i \vee \neg y \vee z) \qquad (C_i \vee y \vee \neg z) \qquad (C_i \vee \neg y \vee \neg z)$$

  - In one of these clauses, the literals involving y and z will both be false in β.
  - Because β satisfies every clause, it must satisfy C_i alone.
- Thus, if we define $\alpha(x_i) = \beta(x_i)$ for all x_i, then α satisfies the original instance.

Problem 3 – Reduction with different types

The Integer Linear Programming problem (**ILP**) is:

Input: An integer matrix A and integer vector b

Output: Is there an integer vector x such that $Ax \leq b$?

Decision version!
Nothing to optimize for.

In lecture, you saw that **3SAT** $\leq_P$ **ILP** via a long series of reductions.
Prove this directly by a single reduction. (We will skip showing ILP is in NP today.)

c) Fill in the blank: "We will reduce from __ to __". Which is A, and which is B?

Think about this briefly!

Solution

Problem 3 – Reduction with different types

The Integer Linear Programming problem (**ILP**) is:

Input: An integer matrix A and integer vector b

Output: Is there an integer vector x such that $Ax \leq b$?

Decision version!
Nothing to optimize for.

In lecture, you saw that **3SAT** $\leq_P$ **ILP** via a long series of reductions.
Prove this directly by a single reduction.

c) Fill in the blank: "We will reduce from __ to __". Which is A, and which is B?

We will reduce from A = 3SAT to B = ILP.
In other words, convert instances of 3SAT into instances of ILP.

Solution

Problem 3 – Reduction with different types

This is tricky, so let's think about solving this example with ILP:

$$(\neg w \lor \neg x \lor y) \land (w \lor y \lor \neg z)$$

1. What variables should we use for the ILP?
   Just w, x, y, and z
2. What constraints can we write to say that our variables must be Boolean?
   $0 \leq w, w \leq 1$, and similarly for x, y, and z
3. How do we encode negations, like $\neg w$?
   $1 - w$
4. What constraints can we write to say that every clause is satisfied?
   Sum of all literals ≥ 1, for example $(1 - w) + (1 - x) + y \geq 1$

These are fairly generic steps for ANY reduction from 3SAT!

Problem 3 – Reduction with different types

d) Define a reduction function f, which converts instances of A into B.

(We did an example on the previous slide, so the question is just: how to write this generally?)

Think about it with the people around you, then we'll discuss!

Solution

Problem 3 – Reduction with different types

d) Define a reduction function f, which converts instances of A into B.

To avoid confusion, denote $y_1, \ldots, y_n$ the variables in the 3SAT instance and $C_1, C_2, \ldots, C_m$ the clauses, and we will use x_i as the ILP variable corresponding to y_i.

Our constraints are:

- $0 \leq x_i$ and $x_i \leq 1$ for all $i = 1, \ldots, n$
- $\sum_{y_i \in C_j}(x_i) + \sum_{\neg y_i \in C_j}(1 - x_i) \geq 1$ for all $j = 1, \ldots, m$

We would convert these to standard form.

Problem 3 – Reduction with different types

e) Say why f is computable in polynomial time.

Think about this briefly!

Solution

Problem 3 – Reduction with different types

e) Say why f is computable in polynomial time.

We create 2 constraints for every variable and 1 constraint for every clause. Every constraint is a row in the ILP input matrix A, so the reduction takes $(m+n)n$ time, which is polynomial.

Problem 3 – Reduction with different types

To prove the correctness:

f) Show that "x is a YES-instance for A" $\Rightarrow$ "$f(x)$ is a YES-instance for B".
(Remember: convert a certificate for x into a certificate for $f(x)$!)

Think about it with the people around you, then we'll discuss!

Solution

Problem 3 – Reduction with different types

To prove the correctness:

f) Show that "x is a YES-instance for A" $\Rightarrow$ "$f(x)$ is a YES-instance for B".
(Remember: convert a certificate for x into a certificate for $f(x)$!)

- There is an assignment α that makes the original 3SAT YES-instance true.
- Let x be the vector whose ith entry is $\alpha(y_i)$. We claim that x satisfies the ILP.
  - Certainly $0 \leq x_i$ and $x_i \leq 1$ for all i.
  - Note that $\sum_{y_i \in C_j}(x_i) + \sum_{\neg y_i \in C_j}(1 - x_i)$ is a sum of non-negative terms.
  - Because α satisfies C_j, there is $y_i \in C_j$ for which $\alpha(y_i) = 1$, so $x_i = \alpha(y_i) = 1$, or $\neg y_i \in C_j$ for which $\alpha(y_i) = 0$, in which case $1 - x_i = 1 - \alpha(y_i) = 1$.
  - Either way, we have found a term that evaluates to 1, so the expression is ≥ 1.

Problem 3 – Reduction with different types

To prove the correctness:

g) Show that "$f(x)$ is a YES-instance for B" $\Rightarrow$ "x is a YES-instance for A".
(Remember: convert a certificate for $f(x)$ into a certificate for x!)

Think about it with the people around you, then we'll discuss!

Solution

Problem 3 – Reduction with different types

To prove the correctness:

g) Show that "$f(x)$ is a YES-instance for B" ⇒ "x is a YES-instance for A".
(Remember: convert a certificate for $f(x)$ into a certificate for x!)

- There is a vector x that satisfies all ILP constraints.
- Let $\alpha(y_i) = x_i$. We claim that α satisfies the original 3SAT instance.
  - α is a valid assignment because x_i is an integer with $0 \le x_i$ and $x_i \le 1$.
  - Because $\sum_{y_i \in C_j}(x_i) + \sum_{\neg y_i \in C_j}(1 - x_i) \ge 1$, at least one of the terms must be positive, in which case the relevant literal is true in C_j, similar to the other direction.

Summary

- When you want **to show B is NP-complete, do NOT solve B**!
  - Convert instances of another NP-hard problem A into instances of B.
- For the reduction proof, **convert certificates** of each problem to certificates of the other problem.

Thanks for coming to section this week!