CSE 421
Introduction to Algorithms

Lecture 28: Dealing with NP-completeness:

Fixed Parameter Tractability

SAT Solving

Reminder/Announcement

- The Final Exam is Monday December 9, 2:30-4:45 pm in this room since nobody had a conflict with the extra time.

- I sent an email over the weekend with information about the exam and a sample final
  - It will be comprehensive and similar in style to the midterm.

What to do if the problem you want to solve is NP-hard

Maybe you only need to solve it if the solution size is small...

- What if you only need to find cliques or vertex covers of constant size?
- For both **Clique** and **Vertex Cover**, the obvious brute force algorithm would have time $\Theta(n^k)$: try all subsets of size k.
- For **Clique** the best algorithms known are all $n^{\Omega(k)}$
- However, **Vertex Cover** has a much better algorithm…

The theory of **fixed parameter tractability** looks at $\mathbf{NP}$ problems using a second parameter k in addition to input size n and seeks algorithms with running times $f(k) \cdot n^{O(1)}$ where f might be exponential.

Fixed Parameter Algorithms

The theory of **fixed parameter tractability** looks at $\mathbf{NP}$ problems using a second parameter $\boldsymbol{k}$ in addition to input size $\boldsymbol{n}$ and seeks algorithms with running times $\boldsymbol{f(k) \cdot n^{O(1)}}$ where $\boldsymbol{f}$ might be exponential.

Clique: Extra parameter $\boldsymbol{k}$ for clique size target:

Brute force algorithm: try all subsets of size $\boldsymbol{k}$ and check: $\boldsymbol{\Theta(k^2 n^k)}$ time.

Vertex-Cover: Extra parameter $\boldsymbol{k}$ for clique size target:

Brute force algorithm: try all subsets of size $\boldsymbol{k}$ and check: $\boldsymbol{\Theta(mn^k)}$ time.

- Neither is a good fixed parameter algorithm

Vertex-Cover Fixed Parameter Algorithm

```
Vertex-Cover(C, b) {
    if there is an edge (u, v) not covered by C {
        if b > 0 {
            Vertex-Cover(C ∪ {u}, b − 1)
            Vertex-Cover(C ∪ {v}, b − 1)
        }
    }
    else
        Output YES (and set C) and halt
    }
}
```

Call **Vertex-Cover**($\emptyset$, k)
if no answer, output **NO**

Analysis:

- Time to identify possible edge (u, v) not covered (and modify C) is $O(m + n)$
- # of recursive calls $\leq 2^k$
- Total runtime $O(2^k(m + n))$

More on Fixed Parameter Algorithms

Many graph problems can be given a second parameter $\boldsymbol{k}$ called the treewidth of the input graph.

- Treewidth 1 graphs are trees (technically forests).
- Multiple natural definitions of treewidth (here's one):
  - Graph $\boldsymbol{G = (V, E)}$ is treewidth at most $\boldsymbol{k}$ iff there is a tree $\boldsymbol{T}$ such that
    - each node $\boldsymbol{u}$ of $\boldsymbol{T}$ is labelled by a subset $\boldsymbol{V_u}$ of $\leq \boldsymbol{k}$ vertices in $\boldsymbol{V}$
    - for every edge $\boldsymbol{(v, w) \in E}$ there is a node $\boldsymbol{u}$ of $\boldsymbol{T}$ such that both $\boldsymbol{v, w \in V_u}$.
    - for every $\boldsymbol{v \in V}$ the set of nodes $\boldsymbol{u}$ in $\boldsymbol{T}$ with $\boldsymbol{v \in V_u}$ is connected in $\boldsymbol{T}$
  - The tree with the sets are called the tree decomposition of $\boldsymbol{G}$.
    The minimum $\boldsymbol{k}$ and tree decomposition can be found in linear time.
    The tree defines a natural elimination ordering for recursive algorithms on the graph.
- **Fact:** Obstacle to treewidth $\boldsymbol{k - 1}$: the $\boldsymbol{k \times k}$ grid graph.

Many NP-hard problems are efficiently solvable on graphs of bounded treewidth.

Treewidth also comes up in route-finding in Google Maps: Can't run full-blown Dijkstra on the whole graph every time a user requests a route.

What to do if the problem you want to solve is NP-hard

Try to make an exponential-time solution as efficient as possible.

e.g. Try to search the space of possible hints/certificates in a more efficient way and hope that it is quick enough.

Backtracking search

e.g., for **SAT**, search through the 2^n possible truth assignments...

...but set the truth values one-by-one so we can able to figure out whole parts of the space to avoid,

e.g. Given $F = (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (x_4 \vee \neg x_3) \wedge (x_1 \vee x_4)$

after setting $x_1 = 1$ and $x_2 = 0$ we don't even need to set x_3 or x_4 to know that it won't satisfy F.

Today: More clever backtracking search for **SAT** solutions

SAT Solving

SAT is an extremely flexible problem:

- The fact that **SAT** is an **NP**-complete problem says that we can re-express a huge range of problems as **SAT** problems

This means that good algorithms for **SAT** solving would be useful for a huge range of tasks.

Since roughly 2001, there has been a massive improvement in our ability to solve **SAT** on a wide range of practical instances

- These algorithms aren't perfect. They fail on many worst-case instances.

Satisfiability Algorithms

Local search: Solve **SAT** as a special case of **MaxSAT**

(incomplete, may fail to find satisfying assignment)

GSAT – random local search [Selman,Levesque,Mitchell 92]

Walksat – Metropolis [Kautz,Selman 96]

Backtracking search (complete)

- DPLL [Davis,Putnam 60], [Davis,Logeman,Loveland 62]
- CDCL: Adds clause learning and restarts

  GRASP, SATO, zchaff, MiniSAT, Glucose, etc.

CNF Satisfiability

SAT: satisfiability problem for CNF formulas with any clause size

Write CNFs with the ∧ between clauses implicit:

$$F = (x_1 \vee \overline{x_2} \vee x_4)(\overline{x_1} \vee x_3)(\overline{x_3} \vee x_2)(\overline{x_4} \vee \overline{x_3})$$

Write assignment as literals assigned true: $x_1, x_2, x_3, \overline{x_4}$

Defn: Given partial assignment x_3 where

$$F = (x_1 \vee \overline{x_2} \vee x_4)(\overline{x_1} \vee x_3)(\overline{x_3} \vee x_2)(\overline{x_4} \vee \overline{x_3})$$

define **simplify**(F, x_3) by

simplify(F, x_3)$= (x_1 \vee \overline{x_2} \vee x_4)$ x_2 $\overline{x_4}$

That is: remove satisfied clauses and remove unsatisfied literals from clauses.

Note: F is satisfiable iff all clauses disappear under some assignment.

Backtracking search/DPLL

$t \leftarrow \varepsilon$

repeat

select a literal ℓ (some $\boldsymbol{x}$ or $\overline{\boldsymbol{x}}$)

$\boldsymbol{F} \leftarrow$ **simplify**$(\boldsymbol{F}, \ell)$; $t \leftarrow \text{append}(t, \ell)$ } free step

while $\boldsymbol{F}$ contains a $\mathbf{1}$-clause ℓ'

$\boldsymbol{F} \leftarrow$ **simplify**$(\boldsymbol{F}, \ell')$; $t \leftarrow \text{append}(t, \ell')$ } unit propagation

if $\boldsymbol{F}$ has no clauses **return** t as satisfying assignment

if $\boldsymbol{F}$ has an empty clause

backtrack to last free step and flip assignment (step no longer free)

Recursive view of DPLL (without unit propagation)

DPLL(F):

    if F is empty **report** satisfiable and **halt**

    if F contains the empty clause

        return

    else choose a literal x ← with unit propagation choose x to be the literal of a 1-clause if possible

        DPLL(**simplify**(F, x))

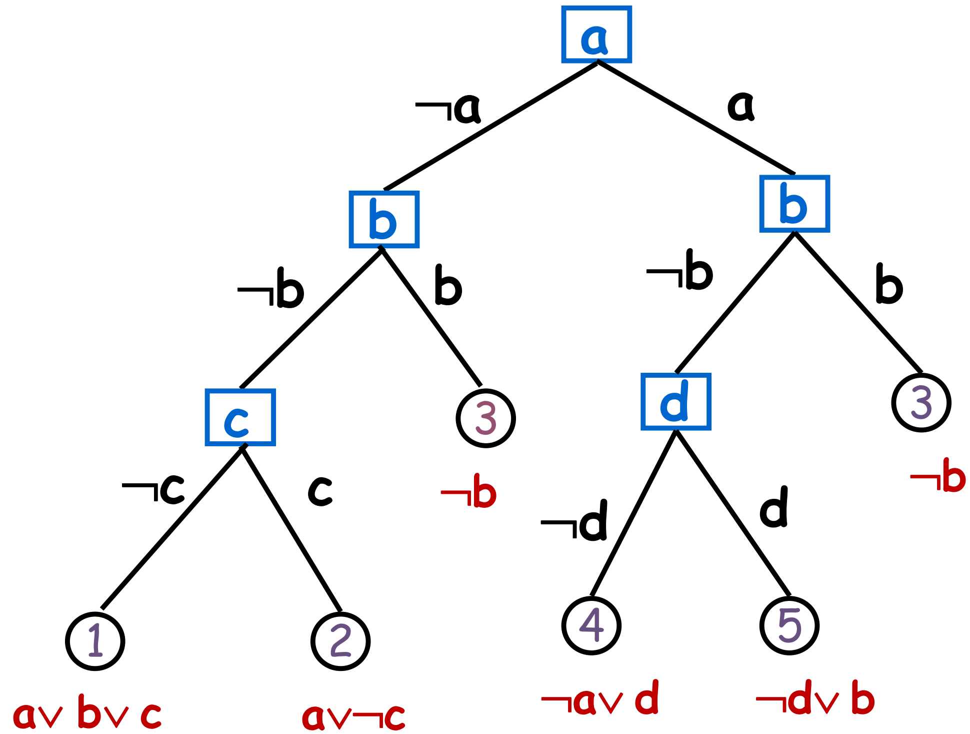
        DPLL(**simplify**$(F, \overline{x})$)

DPLL on UNSAT formula

Clauses

1. a∨ b∨ c
2. a∨¬c
3. ¬b
4. ¬a∨ d
5. ¬d∨ b

Residual Formula



Extending DPLL: Clause Learning

- When backtracking in DPLL, add new clauses corresponding to causes of failure of the search

- Added conflict clauses
  - Capture *reasons* of conflicts
  - Obtained via *unit propagations* from known ones
  - Reduce future search by producing conflicts sooner

Conflict Graph: Graph of Unit Propagations

At each conflict (derivation of them empty clause) the negations of the predecessor node labels across any cut form an implied clause.

- if clause is false then could derive ⊥

Known Clauses

$(p \vee q \vee a)$

$(\neg a \vee \neg b \vee \neg t)$

$(t \vee \neg x_1)$

$(t \vee \neg x_2)$

$(t \vee \neg x_3)$

$(x_1 \vee x_2 \vee x_3 \vee y)$

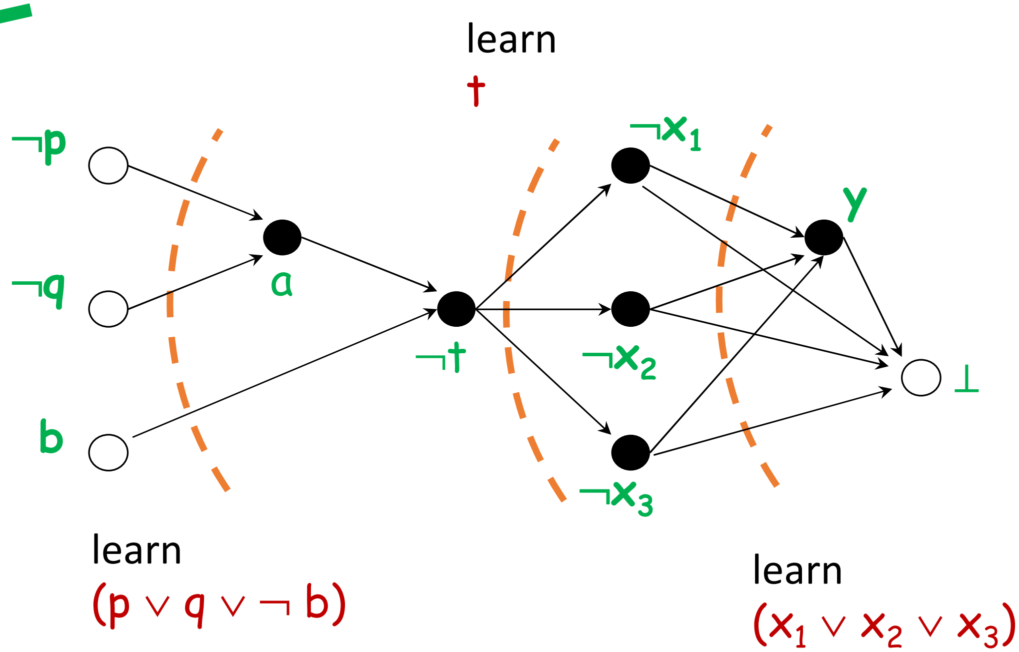
$(x_1 \vee x_2 \vee x_3 \vee \neg y)$

Decisions

p = false

q = false

b = true



Best Current SAT Solvers

Conflict-Directed Clause-Learning (CDCL) Algorithms
Minisat, Glucose, MapleSAT, CaDiCaL

They rely on many optimizations:

- No explicit computation of residual formulas, just fast calculation of the unit propagations that will happen. “watched literals”
- No explicit backtracking: New clauses always chosen to generate unit propagations higher in the tree. “asserting clauses”
- Heuristics based on learned clauses to decide what free choices to make. “VSIDS”
- Pruning of cache of learned clauses so only recently used ones are kept.
- Periodic restarting search with original formula plus learned clauses.
- etc...

Best Current SAT Solvers

Conflict-Directed Clause-Learning (CDCL) Algorithms

Minisat, Glucose, MapleSAT, CaDiCaL

They work well on many practical formulas even with hundreds of thousands of variables or more.

- Often used in proving properties of human-produced designs.
- They are incorporated in software verification tools and a variety of automated reasoning (SMT Solvers)
- We really don’t know why they work so well.
- Definitely worth a try!

However, they provably perform very badly even on some small formulas of a few hundred or thousand variables.   We have a pretty good idea why.

Other Exponential-Time Algorithms

Branch-and-bound search for optimization problems:

- **Branch**: Use backtracking search through a tree representing partial solutions
- **Bound**: In addition to keeping track of the best full solution found so far, at each step produce a bound on the quality of the best possible completion of the current partial solution
  - If that best possible completion is worse than the best full solution found so far, prune the search and backtrack instead.

Example: In backtracking search for **MetricTSP** one can use linear programming to provide lower bounds

Note: An excellent exact solver for **MetricTSP** called **Concorde** combines branch-and-bound and LP/ILP methods and will solve problems involving thousands of cities.

Other Heuristic Algorithms you might hear about

Genetic algorithms:

- View each solution as a **string** (analogy with DNA)
- Maintain a **population of good solutions**
- Allow **random mutations** of single characters of individual solutions
- **Combine two solutions** by taking part of one and part of another (analogy with crossover in sexual reproduction)
- Get rid of solutions that have the worst values and make multiple copies of solutions that have the best values (analogy with natural selection -- survival of the fittest).

*Usually very slow. In the rare cases when they produce answers with better objective function values than other methods they tend to produce very **brittle** solutions – that are very bad with respect to small changes to the requirements.*

Deep Neural Nets and NP-hardness?

- **Artificial neural networks**
  - based on very elementary model of human neurons
  - **Set up a circuit of artificial neurons**
    - each artificial neuron is an analog circuit gate whose computation depends on a set of **connection strengths**
  - **Train the circuit**
    - Adjust the connection strengths of the neurons by giving many positive & negative training examples and seeing if it behaves correctly
  - **The network is now ready to use**

Despite their wide array of applications, they have not been shown to be useful for NP-hard problems.

Quantum Computing and NP-hardness?

Use physical processes at the quantum level to implement "weird" kinds of circuit gates based on unitary transformations

- Quantum objects can be in a "superposition" of many pure states at once
  - Can have n objects together in a superposition of 2^n states
- Each quantum circuit gate operates on the whole superposition of states at once
  - Inherent parallelism but classical randomized algorithms have a similar parallelism: *not enough on its own*
  - Advantage over classical: copies interfere with each other.
- Exciting direction - theoretically able to factor efficiently.

  Major practical problems wrt errors, decoherence to be overcome.
- *Small brute force improvement but unlikely to produce exponential advantage for NP.*