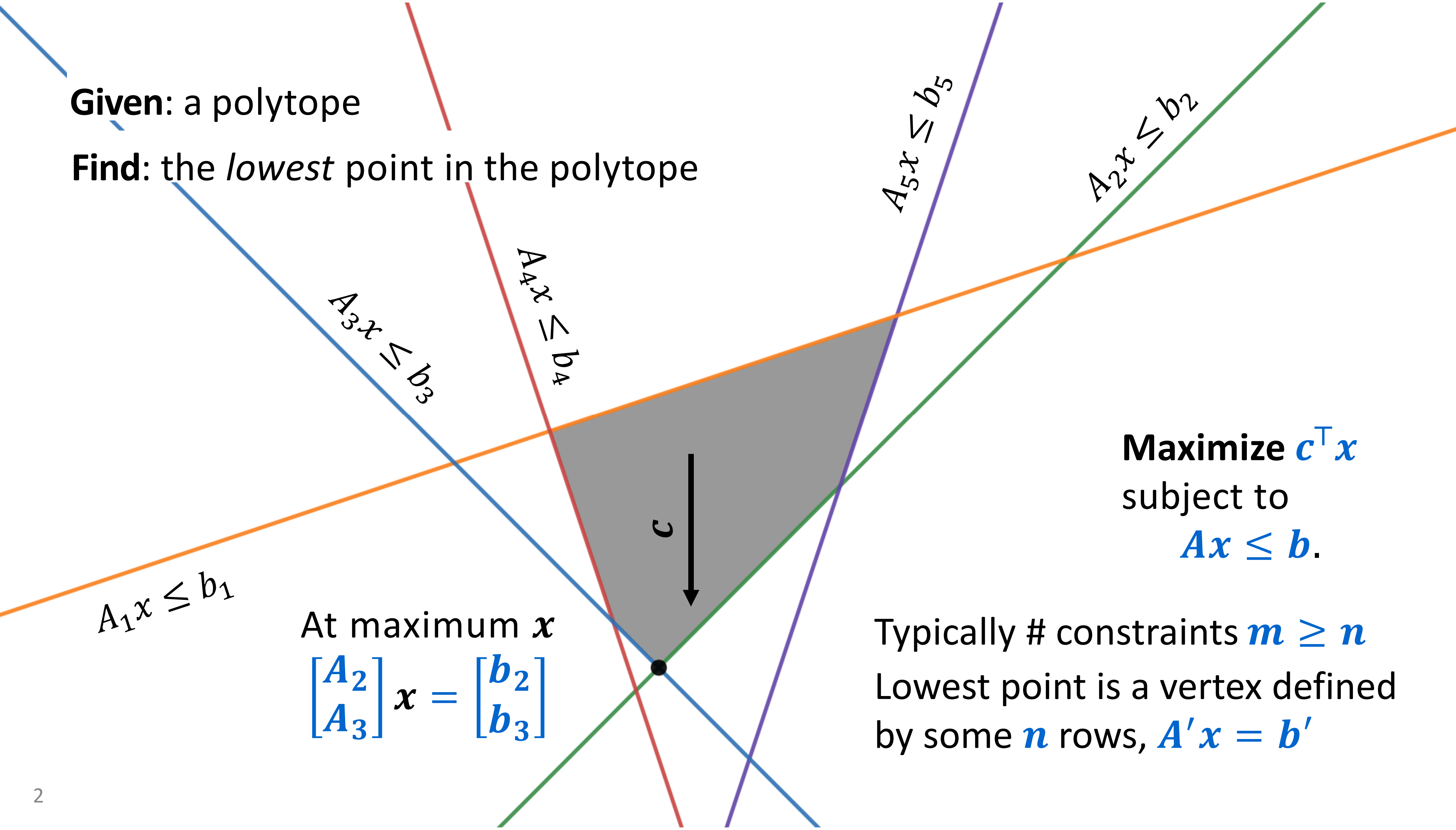
CSE 421
Introduction to Algorithms

Lecture 21: Linear Programming Duality

Given: a polytope

Find: the *lowest* point in the polytope



Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$\boldsymbol{Ax} \le \boldsymbol{b}$.

At maximum $\boldsymbol{x}$

$$\begin{bmatrix} \boldsymbol{A_2} \\ \boldsymbol{A_3} \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{b_2} \\ \boldsymbol{b_3} \end{bmatrix}$$

Typically # constraints $\boldsymbol{m} \ge \boldsymbol{n}$

Lowest point is a vertex defined by some $\boldsymbol{n}$ rows, $\boldsymbol{A'x} = \boldsymbol{b'}$

Max Flow in Standard Form LP

Maximize $\sum_{e \text{ out of } s} x_e$

subject to

$0 \le x_e \le c(e)$ for every $e \in E$

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

→

Maximize $c^\top x$

subject to

$$Ax \le b$$
$$x \ge 0$$

This is for the c above. Nothing to do with capacities!

1. $c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise} \end{cases}$
2. $x_e \le c(e)$
3. $\sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e \le 0$
4. $\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e \le 0$
5. $x \ge 0$

Replace equality constraints by a pair of inequalities

Minimization converted to Maximization

Minimize $\boldsymbol{c}^{\top}\boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

⟷

Maximize $(-\boldsymbol{c})^{\top}\boldsymbol{x}$
subject to
$$(-\boldsymbol{A})\boldsymbol{x} \leq (-\boldsymbol{b})$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Shortest Paths

Given: Directed graph $G = (V, E)$
vertices s, t in V

Find: (length of) shortest path from s to t

Claim: *Length* ℓ of the shortest path is the solution (minimum value) for this program.

Proof sketch: A shortest path yields a solution of cost ℓ. Optimal solution must be a combination of flows on shortest paths also cost ℓ; otherwise there is a part of the 1 unit of flow that gets counted on more than ℓ edges.

Minimize $\sum_{e} x_e$ Sum of flow on all edges

subject to

$$x \geq 0$$

$$\sum_{e \text{ out of } s} x_e = 1$$ Flow out of s is 1

$$\sum_{e \text{ into } t} x_e = 1$$ Flow into t is 1

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

Flow conservation

Shortest Paths

Given: Directed graph $G = (V, E)$
vertices s, t in V

Find: shortest path from s to t

Claim: *Length* ℓ of the shortest path is solution (minimum value) for this program.

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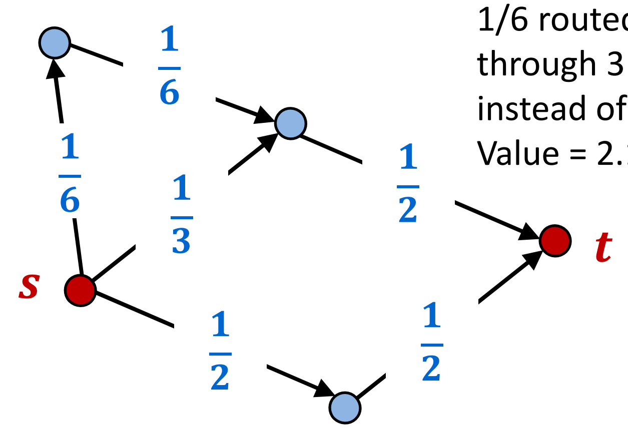
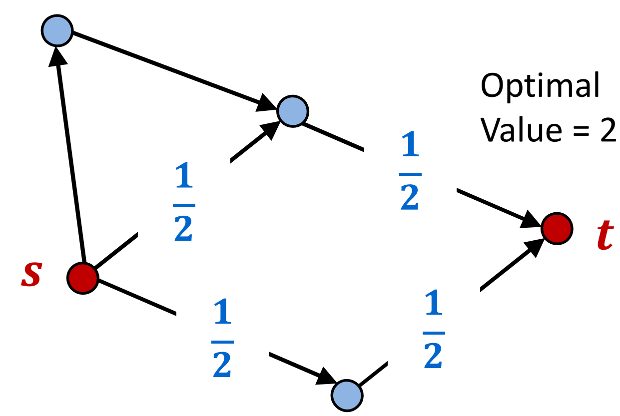


Not optimal
1/6 routed through 3 edges instead of 2.
Value = 2.166666



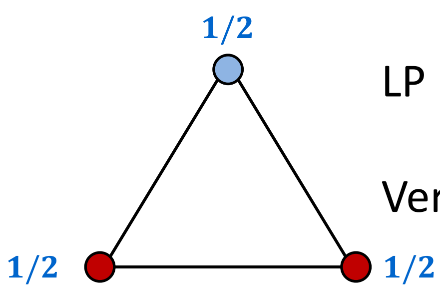
Optimal
Value = 2

Vertex Cover

Given: Undirected graph $G = (V, E)$

Find: smallest set of vertices touching all edges of G.

Doesn't work: To define a set we need $x_v = 0$ or $x_v = 1$



LP minimum $= \ 3/2$

Vertex Cover minimum $= \ 2$

Natural Variables for LP:

x_v for each $v \in V$

Minimize $\sum_v x_v$

subject to

$0 \le x_v \le 1$ for each node $v \in V$

$x_u + x_v \ge 1$ for each edge $\{u, v\} \in E$

This LP optimizes for a different problem: "**fractional vertex cover**".
x_v indicates the fraction of vertex v that is chosen in the cover.

What makes Max Flow different?

For Vertex Cover we only got a fractional optimum but for Max Flow can get integers.

- Why?
  - Ford-Fulkerson analysis tells us this for Max Flow.
  - Is there a reason we can tell just from the LP view?

Recall: Optimum is at some vertex $\boldsymbol{x}$ satisfying $\boldsymbol{A'x = b'}$ for some subset of exactly $\boldsymbol{n}$ constraints.

This means that $\boldsymbol{x = (A')^{-1}b'}$.

Entries of the matrix inverse are quotients of determinants of sub-matrices of $\boldsymbol{A'}$ so, for integer inputs, optimum is always rational.

Fact: Every full rank submatrix of MaxFlow matrix $\boldsymbol{A}$ has determinant $\boldsymbol{\pm 1}$

⇒ all denominators are $\boldsymbol{\pm 1}$ ⇒ integers. $\boldsymbol{A}$ is "totally unimodular"

Next: How **MaxFlow=MinCut** is an example of a general "duality" property of LPs

Duality

Coefficients 1, 0, and 2

Maximize $x_1 + 2x_3$
subject to

$$
\begin{array}{lr}
a & 2x_1 - x_2 + 3x_3 \le 1 \\
b & -x_1 + x_2 - x_3 \le 5 \\
 & x \ge 0
\end{array}
$$

Want coefficients of weighted sum ≥ all coefficients above

Claim: Optimum ≤ 6

Proof: Add the two LHS:

$$
\begin{aligned}
& 2x_1 - x_2 + 3x_3 \\
+ & (-x_1 + x_2 - x_3) \\
= & \ x_1 \qquad + 2x_3.
\end{aligned}
$$

Must be $\le$ sum of RHS $= 6$.

We multiplied the 1st inequality by $a = 1$, the 2nd by $b = 1$ and added.

Claim: For all $a, b \ge 0$ if

$$
\begin{aligned}
2a - b &\ge 1 \\
-a + b &\ge 0 \\
3a - b &\ge 2
\end{aligned}
$$

then Optimum $\le a + 5b$

Proof:

$$
\begin{aligned}
& x_1 \qquad + 2x_3 \\
\le\ & a(2x_1 - x_2 + 3x_3) \\
& + b(-x_1 + x_2 - x_3) \\
\le\ & 1a + 5b.
\end{aligned}
$$

Duality

Maximize $x_1 + 2x_3$
subject to

$$\begin{array}{lrcl} a & 2x_1 - x_2 + 3x_3 & \leq & 1 \\ b & -x_1 + x_2 - \quad x_3 & \leq & 5 \\ & x \geq 0 & & \end{array}$$

primal

Minimize $a + 5b$
subject to

$$\begin{array}{rcl} 2a - b & \geq & 1 \\ -a + b & \geq & 0 \\ 3a - b & \geq & 2 \\ a, b \geq 0 & & \end{array}$$

dual

We multiplied the 1st inequality by $a = 1$, the 2nd by $b = 1$ and added.

Claim: For all $a, b \geq 0$ if

$$2a - b \geq 1$$
$$-a + b \geq 0$$
$$3a - b \geq 2$$

then Optimum $\leq a + 5b$

Proof:

$$x_1 \qquad + \; 2x_3$$
$$\leq a(2x_1 - x_2 + 3x_3)$$
$$+b\,(-x_1 + x_2 - \quad x_3)$$
$$\leq 1a + 5b.$$

Duality

Maximize $x_1 + 2x_3$
subject to

$a \quad 2x_1 - x_2 + 3x_3 \leq 1$

$b \quad -x_1 + x_2 - x_3 \leq 5$ primal

$x \geq 0$

Minimize $a + 5b$
subject to

$$2a - b \geq 1$$
$$-a + b \geq 0$$ dual
$$3a - b \geq 2$$
$$a, b \geq 0$$

We multiplied the 1st inequality by $a = 1$, the 2nd by $b = 1$ and added.

Claim: For all $a, b \geq 0$ if

$$2a - b \geq 1$$
$$-a + b \geq 0$$
$$3a - b \geq 2$$

then Optimum $\leq a + 5b$

Proof: $x_1 + 2x_3$

$$\leq a(2x_1 - x_2 + 3x_3)$$
$$+b(-x_1 + x_2 - x_3)$$
$$\leq 1a + 5b.$$

Duality

Maximize $x_1 + 2x_3$
subject to

$$\begin{array}{lrl} a & 2x_1 - x_2 + 3x_3 \le 1 & \\ b & -x_1 + x_2 - \quad x_3 \le 5 & \text{primal} \\ & x \ge 0 & \end{array}$$

Maximize $-a - 5b$
subject to

$$\begin{array}{rl} -2a + b \le -1 & \\ a - b \le \quad 0 & \text{dual} \\ -3a + b \le -2 & \\ a, b \ge 0 & \end{array}$$

We multiplied the 1st inequality by $a = 1$, the 2nd by $b = 1$ and added.

Claim: For all $a, b \ge 0$ if

$$2a - b \ge 1$$
$$-a + b \ge 0$$
$$3a - b \ge 2$$

then Optimum $\le a + 5b$

Proof:
$$\begin{aligned} & x_1 \qquad + \ 2x_3 \\ \le\ & a(2x_1 - x_2 + 3x_3) \\ & +b\,(-x_1 + x_2 - \quad x_3) \\ \le\ & 1a + 5b. \end{aligned}$$

Duality

Maximize $x_1 + 2x_3$
subject to

$$\begin{array}{ll} a & 2x_1 - x_2 + 3x_3 \le 1 \\ b & -x_1 + x_2 - x_3 \le 5 \\ & x \ge 0 \end{array}$$

primal

Maximize $-a - 5b$
subject to

$$\begin{array}{ll} y_1 & -2a + b \le -1 \\ y_2 & a - b \le 0 \\ y_3 & -3a + b \le -2 \\ & a, b \ge 0 \end{array}$$

dual

What is the dual of the dual?

Minimize $-1y_1 - 2y_3$
subject to

$$\begin{array}{l} -2y_1 + y_2 - 3y_3 \ge -1 \\ y_1 - y_2 + y_3 \ge -5 \\ y \ge 0 \end{array}$$

equivalent to

Maximize $y_1 + 2y_3$
subject to

$$\begin{array}{l} 2y_1 - y_2 + 3y_3 \le 1 \\ -y_1 + y_2 - y_3 \le 5 \\ y \ge 0 \end{array}$$

Duality

primal

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$$
$$\boldsymbol{x} \ge \boldsymbol{0}$$

dual

Minimize $\boldsymbol{b}^\top \boldsymbol{y}$
subject to
$$\boldsymbol{A}^\top \boldsymbol{y} \ge \boldsymbol{c}$$
$$\boldsymbol{y} \ge \boldsymbol{0}$$

dual

Maximize $(-\boldsymbol{b})^\top \boldsymbol{y}$
subject to
$$(-\boldsymbol{A})^\top \boldsymbol{y} \le -\boldsymbol{c}$$
$$\boldsymbol{y} \ge \boldsymbol{0}$$

Theorem: The dual of the dual is the primal.

Proof:

dual of dual

Minimize $(-\boldsymbol{c})^\top \boldsymbol{x}$
subject to
$$((-\boldsymbol{A})^\top)^\top \boldsymbol{x} \ge -\boldsymbol{b}$$
$$\boldsymbol{x} \ge \boldsymbol{0}$$

$\equiv$

dual of dual

Minimize $-\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$-\boldsymbol{A}\boldsymbol{x} \ge -\boldsymbol{b}$$
$$\boldsymbol{x} \ge \boldsymbol{0}$$

dual of dual

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$$
$$\boldsymbol{x} \ge \boldsymbol{0}$$

Duality

primal

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

dual

Minimize $\boldsymbol{b}^\top \boldsymbol{y}$
subject to
$$\boldsymbol{A}^\top \boldsymbol{y} \geq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

Theorem: The dual of the dual is the primal.

Theorem (Weak Duality): Every solution to primal has a value that is at most that of every solution to dual.

Proof: We constructed the dual to give upper bounds on the primal.

Duality

primal

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

dual

Minimize $\boldsymbol{b}^\top \boldsymbol{y}$
subject to
$$\boldsymbol{A}^\top \boldsymbol{y} \geq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

Theorem: The dual of the dual is the primal.

Theorem (Weak Duality): Every solution to primal has a value that is at most that of every solution to dual.

Theorem (Strong Duality): If primal has a solution of finite value, then that value is equal to optimal solution of dual.

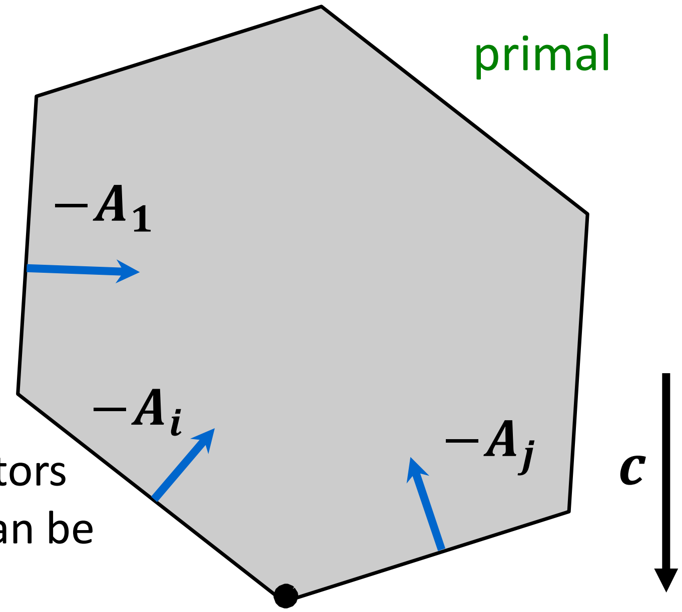
Duality

primal	dual
Maximize $\boldsymbol{c}^\top \boldsymbol{x}$	**Minimize** $\boldsymbol{b}^\top \boldsymbol{y}$
subject to	subject to
$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$	$\boldsymbol{A}^\top \boldsymbol{y} \geq \boldsymbol{c}$
$\boldsymbol{x} \geq \boldsymbol{0}$	$\boldsymbol{y} \geq \boldsymbol{0}$

Theorem (Strong Duality): If primal has a solution of finite value, then that value is equal to optimal solution of dual.

Fact: At vertex, $\boldsymbol{n}$ inequalities are tight

$$\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'.$$



Physics: Coefficient vectors $\boldsymbol{y}' \geq \boldsymbol{0}$ for tight rows can be combined to get $\boldsymbol{c}^\top$.

E.g. there are $\boldsymbol{y}_i, \boldsymbol{y}_j \geq \boldsymbol{0}$ s.t. $\boldsymbol{y}_i \boldsymbol{A}_i + \boldsymbol{y}_j \boldsymbol{A}_j = \boldsymbol{c}^\top$.

Set $\boldsymbol{y}_k$ for all other rows to $\boldsymbol{0}$, get $\boldsymbol{y}^\top \boldsymbol{A} = (\boldsymbol{y}')^\top \boldsymbol{A}' = \boldsymbol{c}^\top$

so $\boldsymbol{A}^\top \boldsymbol{y} = \boldsymbol{c}$.

Then

$$\boldsymbol{b}^\top \boldsymbol{y} = (\boldsymbol{b}')^\top \boldsymbol{y}' = (\boldsymbol{A}'\boldsymbol{x})^\top \boldsymbol{y}' = \boldsymbol{x}^\top (\boldsymbol{A}')^\top \boldsymbol{y}' = \boldsymbol{x}^\top \boldsymbol{A}^\top \boldsymbol{y}$$
$$= \boldsymbol{x}^\top \boldsymbol{c} = \boldsymbol{c}^\top \boldsymbol{x}$$

since $\boldsymbol{x}^\top \boldsymbol{c}$ and $\boldsymbol{c}^\top \boldsymbol{x}$ are just numbers.

Saving dual variables for equalities

Maximize $x_1 + 4x_2$
subject to

a' $\quad 3x_1 - 2x_2 \leq 5$

a'' $\quad -3x_1 + 2x_2 \leq -5$

$\ldots$

$x \geq 0$

Dual →

Minimize $5(a' - a'') + \ldots$
subject to

$3(a' - a'') + \ldots \geq 1$

$-2(a' - a'') + \ldots \geq 4$

$a', a'' \ldots \geq 0$

$a' - a''$ can take on any real value

Standard form conversion for equality ↑

Maximize $x_1 + 4x_2$
subject to

a $\quad 3x_1 - 2x_2 = 5$

$\ldots$

$x \geq 0$

Dual →

use direct conversion!

Minimize $5a + \ldots$
subject to

$3a + \ldots \geq 1$

$-2a + \ldots \geq 4$

$\ldots \geq 0$

No requirement that $a \geq 0$

Dual of Max Flow

Use a different names to avoid confusion with capacity vector

Maximize $g^\top x$
subject to
$$Ax \le h$$
$$x \ge 0$$

1. $g_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise} \end{cases}$

a_e 2. $x_e \le c(e)$

b_v 3. $\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e = 0$

4. $x \ge 0$

$v \in S - \{s, t\}$

Minimize $\sum_e c(e) a_e \equiv c^\top a$
subject to

$a_e + b_v \ge 1$ if $e = (s, v)$

$a_e - b_u \ge 0$ if $e = (u, t)$

$a_e - b_u + b_v \ge 0$ if $e = (u, v)$

$a \ge 0$ $u, v \in S - \{s, t\}$

More uniform way to write Max Flow Dual

Minimize $\sum_e c(e) a_e \equiv c^{\top} a$

subject to

$$a_e + b_v \geq 1 \text{ if } e = (s, v)$$

$$a_e - b_u \geq 0 \text{ if } e = (u, t)$$

$$a_e - b_u + b_v \geq 0 \text{ if } e = (u, v)$$

$$u, v \in S - \{s, t\}$$

$$a \geq 0$$

Define

$$b_s = 1$$
$$b_t = 0$$

Minimize $\sum_e c(e) a_e \equiv c^{\top} a$

subject to

$$b_s = 1$$
$$b_t = 0$$

$$a_e - b_u + b_v \geq 0$$
$$\text{for } e = (u, v)$$

$$a \geq 0$$

Simpler to read Max Flow Dual

Minimize $\sum_e c(e) a_e \equiv c^\top a$
subject to

$$b_s = 1$$
$$b_t = 0$$

$$a_e - b_u + b_v \geq 0$$
for $e = (u, v)$

$$a \geq 0$$

All the $c(e) \geq 0$, so we want the a_e as small as possible.

Minimize $\sum_e c(e) a_e \equiv c^\top a$
subject to

$$b_s = 1$$
$$b_t = 0$$

$$a_e = \max(b_u - b_v, 0)$$
for $e = (u, v)$

Minimize

$\sum_e c(e) a_e \equiv c^\top a$

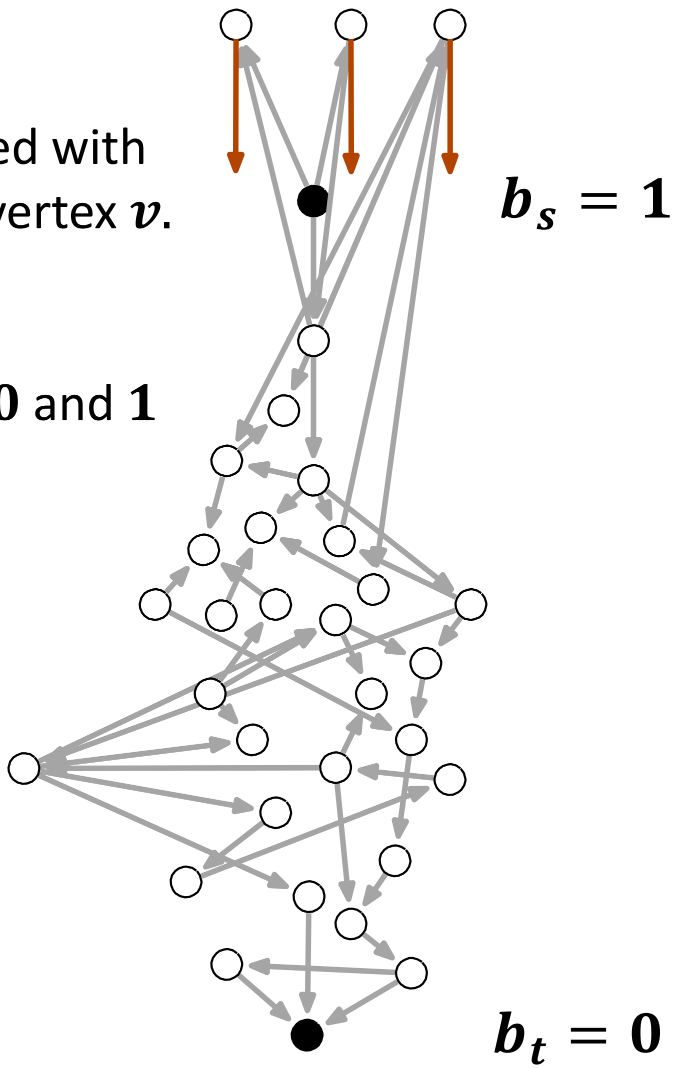
subject to

$$b_s = 1$$
$$b_t = 0$$

$$a_e = \max(b_u - b_v, 0)$$
for $e = (u, v)$

Claim: Optimum is achieved with
$0 \le b_v \le 1$ for every vertex v.

Proof:
Move b_v values between $\mathbf{0}$ and $\mathbf{1}$
Reduces:
$a_e =$ length if e is down
Doesn't change:
$a_e = \mathbf{0}$ if e is up
Overall solution improves.



Minimize

$\sum_e c(e) a_e \equiv c^\top a$

subject to

$$b_s = 1$$
$$b_t = 0$$
$$0 \le b_v \le 1$$
$$a_e = \max(b_u - b_v, 0)$$
$$\text{for } e = (u, v)$$

Claim: Optimum is achieved with

$0 \le b_v \le 1$ for every vertex v.

Proof:

Move b_v values between $\mathbf{0}$ and $\mathbf{1}$

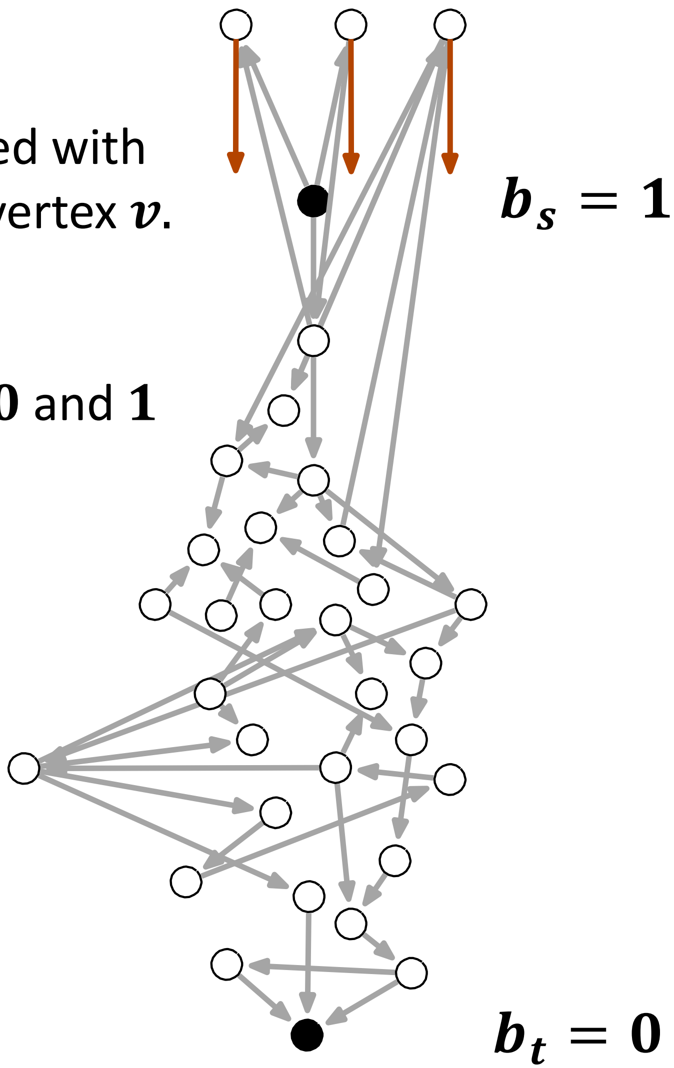
Reduces:

a_e = length if e is down

Doesn't change:

$a_e = 0$ if e is up

Overall solution improves.

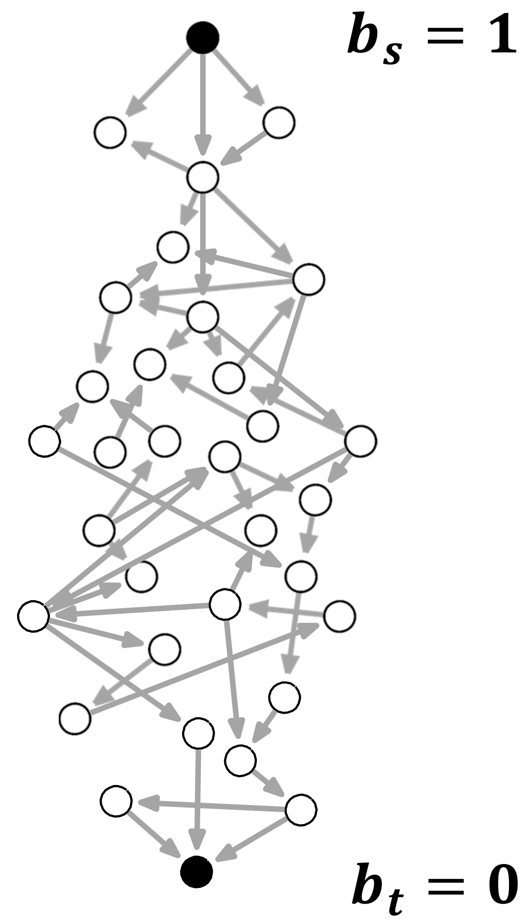


Minimize

$\sum_e c(e) a_e \equiv c^\top a$

subject to

$$b_s = 1$$
$$b_t = 0$$
$$0 \le b_v \le 1$$
$$a_e = \max(b_u - b_v, 0)$$
$$\text{for } e = (u, v)$$



Minimize

$\sum_e c(e) a_e \equiv c^\top a$

subject to

$$b_s = 1$$
$$b_t = 0$$
$$0 \le b_v \le 1$$
$$a_e = \max(b_u - b_v, 0)$$
for $e = (u, v)$

Claim: Optimum is achieved with $b_v = 0$ or $b_v = 1$ for every vertex v.

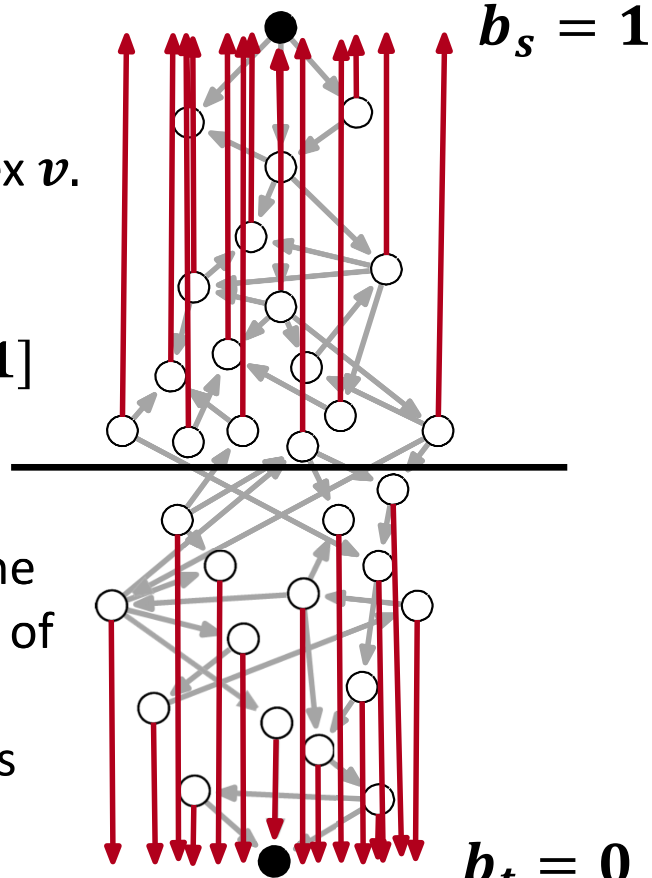
Proof:

Choose uniform random $r \in [0,1]$

Set $b_v = \begin{cases} 1 & \text{if } b_v \ge r \\ 0 & \text{if } b_v < r \end{cases}$

Expected value for random r is the same as the original since edge e of length a_e is cut w.p. a_e.

So... one of those random choices must be at least as good.



Minimize

$\sum_e \boldsymbol{c}(\boldsymbol{e})\boldsymbol{a}_e \equiv \boldsymbol{c}^\top \boldsymbol{a}$

subject to

$$\boldsymbol{b}_s = \mathbf{1}$$

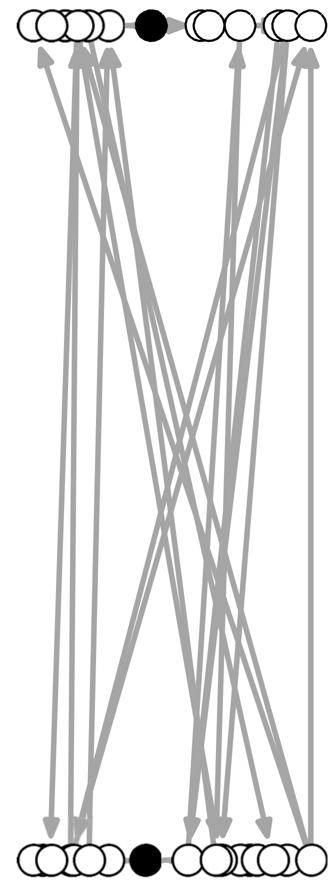
$$\boldsymbol{b}_t = \mathbf{0}$$

$$\boldsymbol{b}_v \in \{\mathbf{0}, \mathbf{1}\}$$

$$\boldsymbol{a}_e = \max(\boldsymbol{b}_u - \boldsymbol{b}_v, \mathbf{0})$$

for $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{v})$

MinCut!



$\boldsymbol{b}_s = \mathbf{1}$

$\boldsymbol{b}_t = \mathbf{0}$

Duality of Shortest Paths

Minimize $\sum_e x_e$

subject to

$\sum_{e \text{ out of } s} x_e = 1$

$\sum_{e \text{ into } t} x_e = 1$

$\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e = 0$
for all $v \in V - \{s, t\}$

$x \geq 0$

Duality of Shortest Paths

Minimize $\sum_e x_e$

subject to

a_s $\sum_{e \text{ into } s} x_e - \sum_{e \text{ out of } s} x_e = -1$

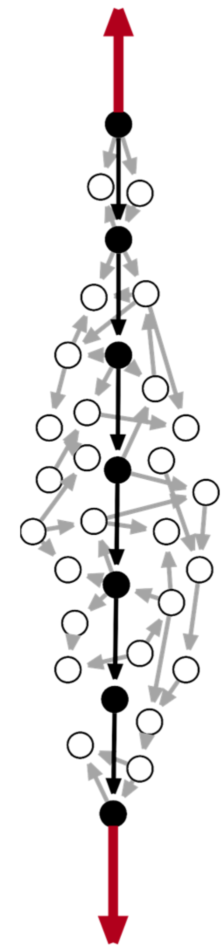
a_t $\sum_{e \text{ into } t} x_e - \sum_{e \text{ out of } t} x_e = 1$

a_v $\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e = 0$
for all $v \in V - \{s, t\}$

$x \geq 0$

Maximize $a_s - a_t$
subject to

$a_u - a_v \leq 1$
if $e = (u, v)$



Duality and Zero-Sum Games

Two player zero-sum game:

An $\boldsymbol{m} \times \boldsymbol{n}$ matrix $\boldsymbol{G}$

$\boldsymbol{G_{i,j}} =$ payoff to row player assuming:
row player uses strategy $\boldsymbol{i}$, and
column player uses strategy $\boldsymbol{j}$.

Column player's payoff for game $= -\boldsymbol{G_{i,j}}$

Example: Chess (idealized)

$\boldsymbol{i}$ specifies how white would move in every possible board configuration.

$\boldsymbol{j}$ specifies how black would move.

$$\boldsymbol{G_{i,j}} = \begin{cases} \boldsymbol{+1} & \text{White checkmates} \\ \boldsymbol{-1} & \text{Black checkmates} \\ \boldsymbol{0} & \text{Draw on board} \end{cases}$$

Randomized Strategy:

Probability distribution on row strategies:

- A column vector $\boldsymbol{x}$ with each $\boldsymbol{x_i} \geq \boldsymbol{0}$

$$\sum_i \boldsymbol{x_i} = \boldsymbol{1}$$

Probability distribution on column strategies:

- A column vector $\boldsymbol{y}$ with each $\boldsymbol{y_j} \geq \boldsymbol{0}$

$$\sum_j \boldsymbol{y_j} = \boldsymbol{1}$$

Expected payoff to row player:

$$\boldsymbol{x}^\top \boldsymbol{G}\, \boldsymbol{y}$$

Zero-Sum Game Example: Rock-Paper-Scissors

G	Rock	Paper	Scissors
Rock	**0**	**-1**	**1**
Paper	**1**	**0**	**-1**
Scissors	**-1**	**1**	**0**

The payoff matrix tells you how much the row player wins (and the column player loses).

Who decides on their strategy first

If row player commits to $\boldsymbol{x}$:

Row player will get payoff

$$\min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \min_{j} (\boldsymbol{x}^\top \boldsymbol{G})_j$$

So if row player plays first they can get payoff

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$$

If column player commits to $\boldsymbol{y}$:

Row player will get payoff

$$\max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \max_{i} (\boldsymbol{G}\, \boldsymbol{y})_i$$

So if column player plays first, row player can get payoff

$$\min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$$

Randomized Strategy:

Probability distribution on row strategies:

- A column vector $\boldsymbol{x}$ with each $\boldsymbol{x}_i \geq \mathbf{0}$

$$\sum_i \boldsymbol{x}_i = \mathbf{1}$$

Probability distribution on column strategies:

- A column vector $\boldsymbol{y}$ with each $\boldsymbol{y}_j \geq \mathbf{0}$

$$\sum_j \boldsymbol{y}_j = \mathbf{1}$$

Expected payoff to row player:

$$\boldsymbol{x}^\top \boldsymbol{G}\, \boldsymbol{y}$$

Zero-Sum Game Example: Rock-Paper-Scissors

G	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

The payoff matrix tells you how much the row player wins (and the column player loses).

		0	1	0
	G	Rock	Paper	Scissors
1/2	Rock	0	-1	1
1/4	Paper	1	0	-1
1/4	Scissors	-1	1	0

- Suppose that a player chose imbalanced probabilities:

If the row player chose with probabilities **1/2,1/4,1/4**
... then the column player could always choose Paper and get an expected payoff of **(-1)∗1/2+0∗1/4+1∗1/4 = -1/4**.

Von Neumann's MiniMax Theorem

If row player commits to $\boldsymbol{x}$:

Row player will get payoff

$$\min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G}\boldsymbol{y} = \min_{j}(\boldsymbol{x}^\top \boldsymbol{G})_{j}$$

So if row player plays first they can get payoff

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G}\boldsymbol{y}$$

If column player commits to $\boldsymbol{y}$:

Row player will get payoff

$$\max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G}\boldsymbol{y} = \max_{i}(\boldsymbol{G}\ \boldsymbol{y})_{i}$$

So if column player plays first, row player can get payoff

$$\min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G}\boldsymbol{y}$$

It doesn't matter who plays first!

Theorem:

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G}\boldsymbol{y} = \min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G}\boldsymbol{y}$$

Zero-Sum Game Example: Rock-Paper-Scissors

G	Rock	Paper	Scissors
Rock	0	-1	1
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Scissors	-1	1	0

The payoff matrix tells you how much the row player wins (and the column player loses).

		0	**1**	**0**
	G	Rock	Paper	Scissors
1/2	Rock	0	-1	1
1/4	Paper	1	0	-1
1/4	Scissors	-1	1	0

- Suppose that a player chose imbalanced probabilities:

If the row player chose with probabilities **1/2,1/4,1/4**
... then the column player could always choose Paper and get an expected payoff of **(-1)∗1/2+0∗1/4+1∗1/4 = -1/4**.

		1/3	**1/3**	**1/3**
	G	Rock	Paper	Scissors
1/3	Rock	0	-1	1
1/3	Paper	1	0	-1
1/3	Scissors	-1	1	0

- The Von Neumann Minimax Theorem is NOT about one player *playing* first (That would be bad for Rock-Paper-Scissors!)
- It is about the order in which players *reveal their probabilities*.
- For Rock-Paper-Scissors, if a player uses **1/3,1/3,1/3** then the payoff is **0** no matter what the other player chooses!
- If they choose anything else, the other player could do better.

Use Strong Duality to prove MiniMax Theorem

Theorem: $\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$

i.e., $\max_{\boldsymbol{x}} \min_{\boldsymbol{j}} (\boldsymbol{x}^\top \boldsymbol{G})_{\boldsymbol{j}} = \min_{\boldsymbol{y}} \max_{\boldsymbol{i}} (\boldsymbol{G}\,\boldsymbol{y})_{\boldsymbol{i}}$

Primal

Maximize $\boldsymbol{z}$

subject to

$\boldsymbol{w}$ $\quad \sum_i \boldsymbol{x}_i = \boldsymbol{1}$

$\boldsymbol{y}_j$ $\quad \boldsymbol{z} - (\boldsymbol{x}^\top \boldsymbol{G})_j \leq \boldsymbol{0}^*$

for all $\boldsymbol{j}$

$\boldsymbol{x} \geq \boldsymbol{0}$

*equivalent to $\boldsymbol{z} \leq \min_{\boldsymbol{j}} (\boldsymbol{x}^\top \boldsymbol{G})_{\boldsymbol{j}}$

Dual

Minimize $\boldsymbol{w}$

subject to

$\sum_j \boldsymbol{y}_j = \boldsymbol{1}$ — Coefficient of $\boldsymbol{z}$ must be $\boldsymbol{1}$

$\boldsymbol{w} - (\boldsymbol{G}\,\boldsymbol{y})_i \geq \boldsymbol{0}^*$ — Coefficient of $\boldsymbol{x}_i$ must be $\geq \boldsymbol{0}$

for all $\boldsymbol{i}$

$\boldsymbol{y} \geq \boldsymbol{0}$

*equivalent to $\boldsymbol{w} \geq \max_{\boldsymbol{i}} (\boldsymbol{G}\,\boldsymbol{y})_{\boldsymbol{i}}$