# CSE 421
# Introduction to Algorithms

## Lecture 20: Linear Programming:
## A really very extremely big hammer

# Midterm

Midterm grades will be released at the end of this class.

- Breathe!
- These grades don't count for that much.
  - In past 421 I've had a student with a midterm grade in the mid-60's end with a 4.0 in the course.

**Histogram**

| | |
|---|---|
| 90s | **23** |
| 80s | **10** |
| 70s | **26** |
| 60s | **23** |
| 50s | **15** |
| 40s | **18** |
| <40 | **15** |

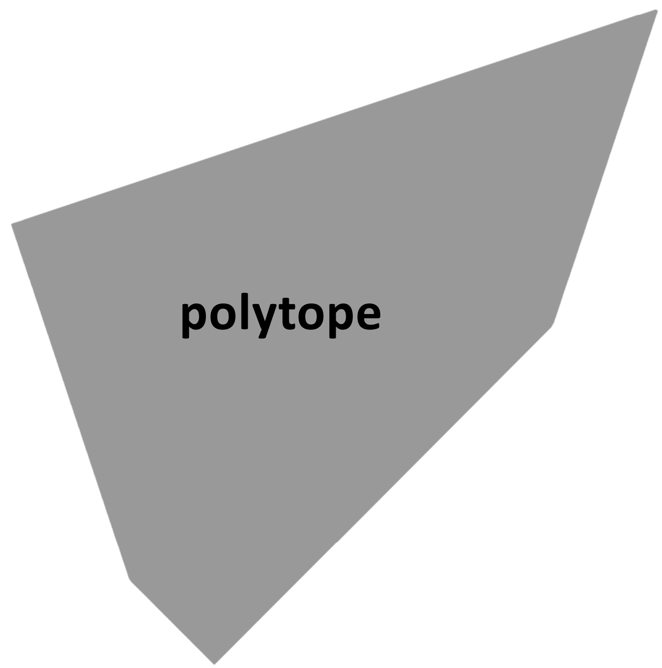
Median **68.5**

Average **65.28**

Standard Deviation **20.74**

**Given**: a polytope

**Find**: the *lowest* point in the polytope



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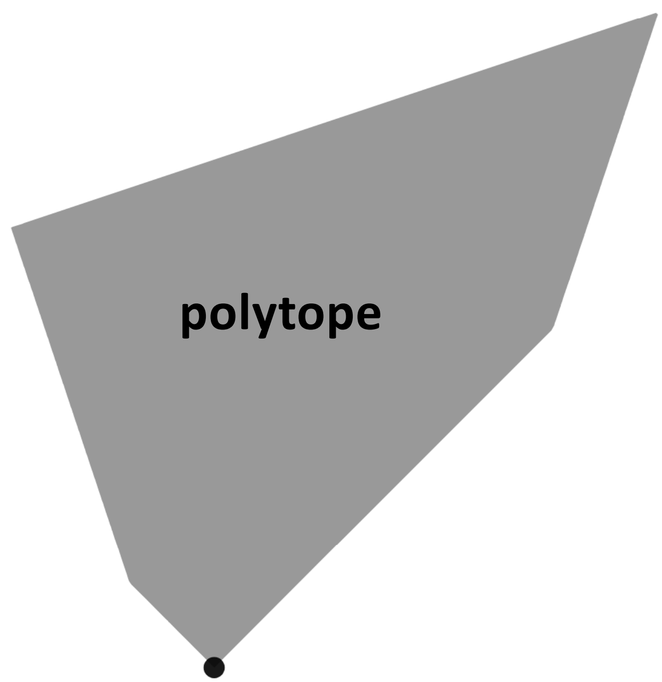
**Given**: a polytope

**Find**: the *lowest* point in the polytope




**We have fast algorithms for this!**

**Maximize** $z_1 + 2z_3$

subject to:

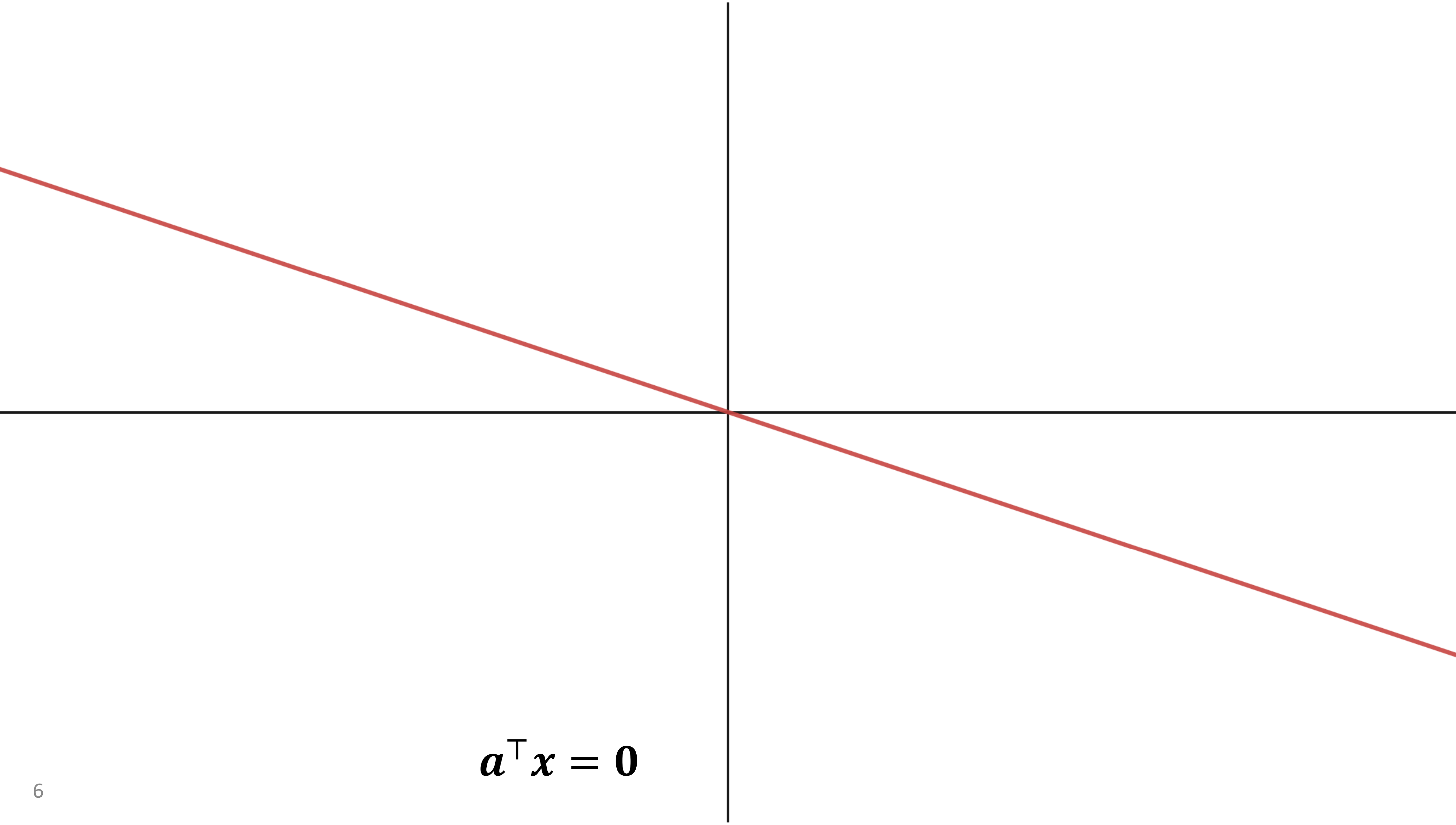
$$2z_1 - z_2 + 3z_3 \leq 1$$
$$-z_1 + z_2 - z_3 \leq 5$$

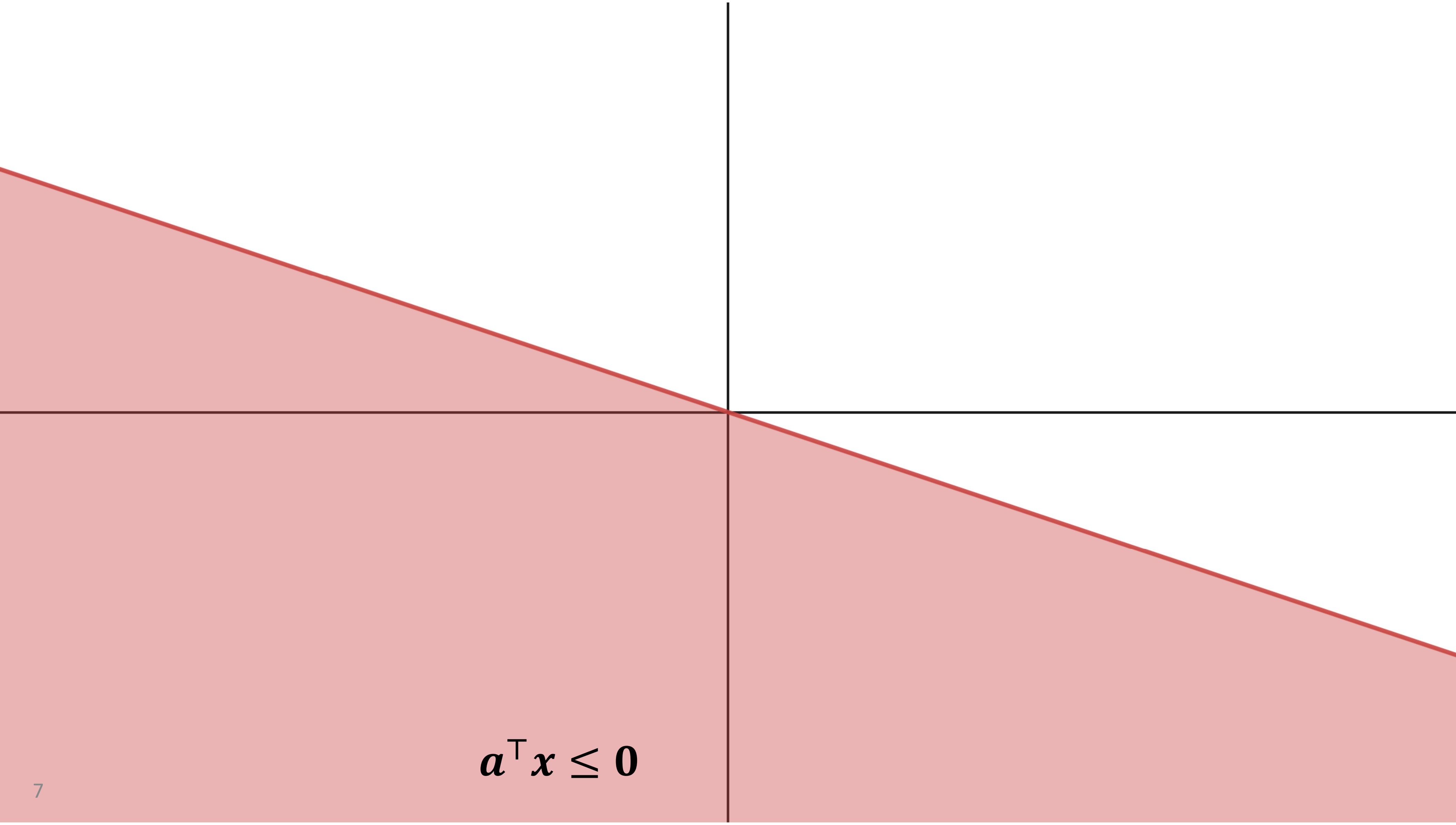
# Linear Algebra primer

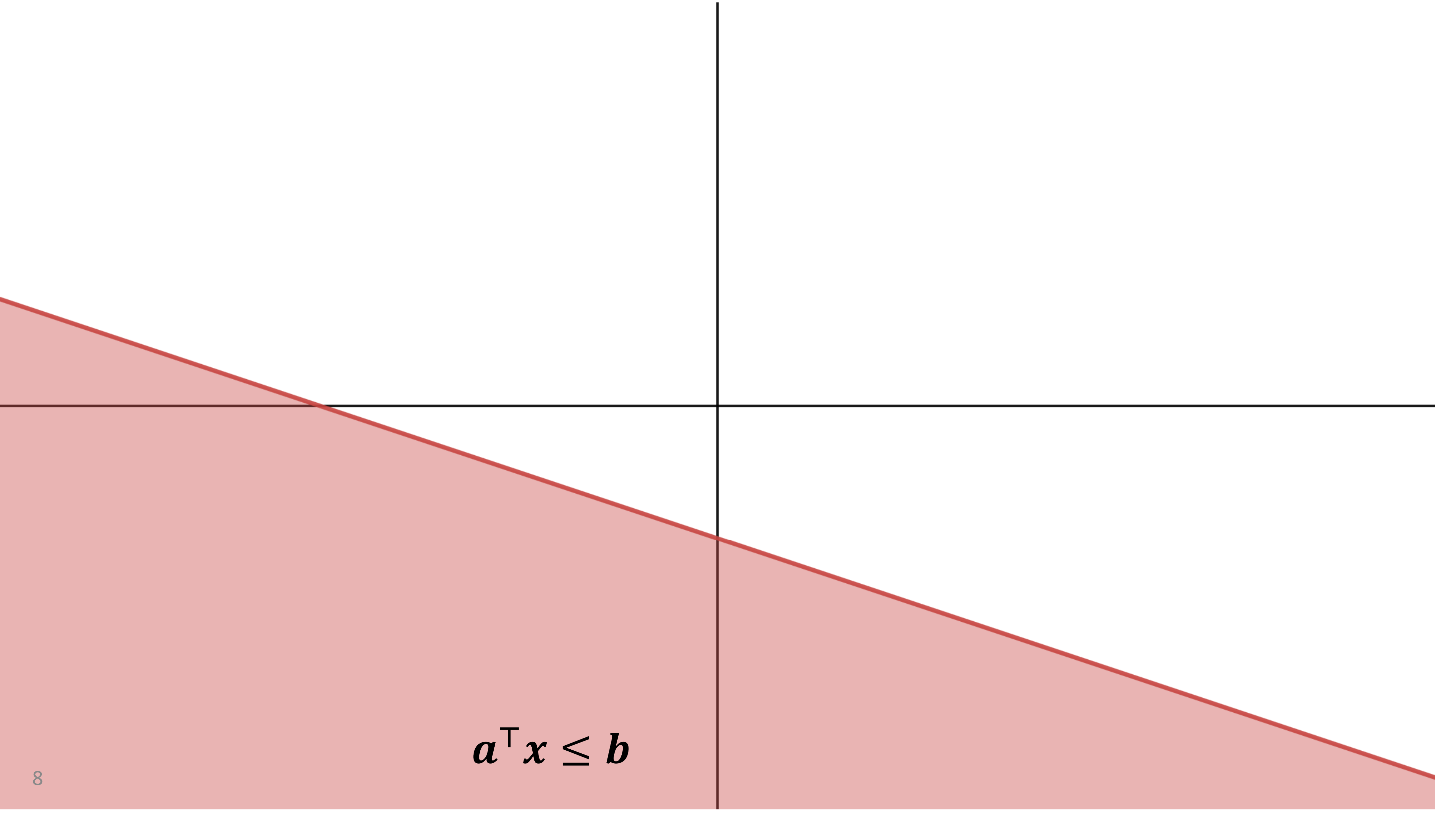
For $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ we think of $\boldsymbol{a}$ and $\boldsymbol{x}$ as column vectors

$$\boldsymbol{a}^\top \boldsymbol{x} = \boldsymbol{a}_1 \boldsymbol{x}_1 + \cdots + \boldsymbol{a}_n \boldsymbol{x}_n$$

The set of $\boldsymbol{x}$ satisfying $\boldsymbol{a}^\top \boldsymbol{x} = \mathbf{0}$ is *hyperplane*

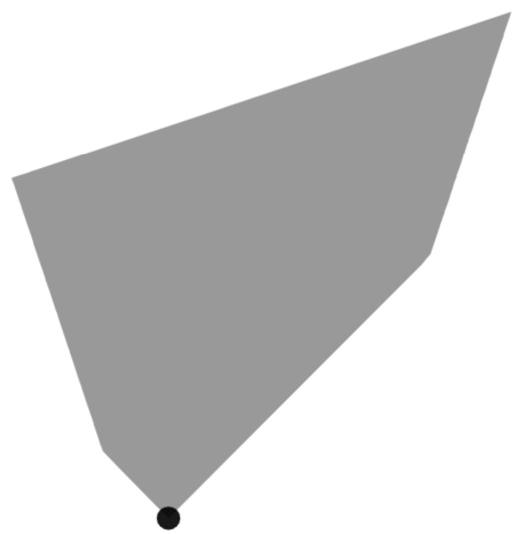

$\boldsymbol{a}^\top \boldsymbol{x} = \mathbf{0}$


$\boldsymbol{a}^\top \boldsymbol{x} \leq \mathbf{0}$


$\boldsymbol{a}^\top \boldsymbol{x} \le \boldsymbol{b}$

**Given**: a polytope

**Find**: the *lowest* point in the polytope



**Given**: a polytope

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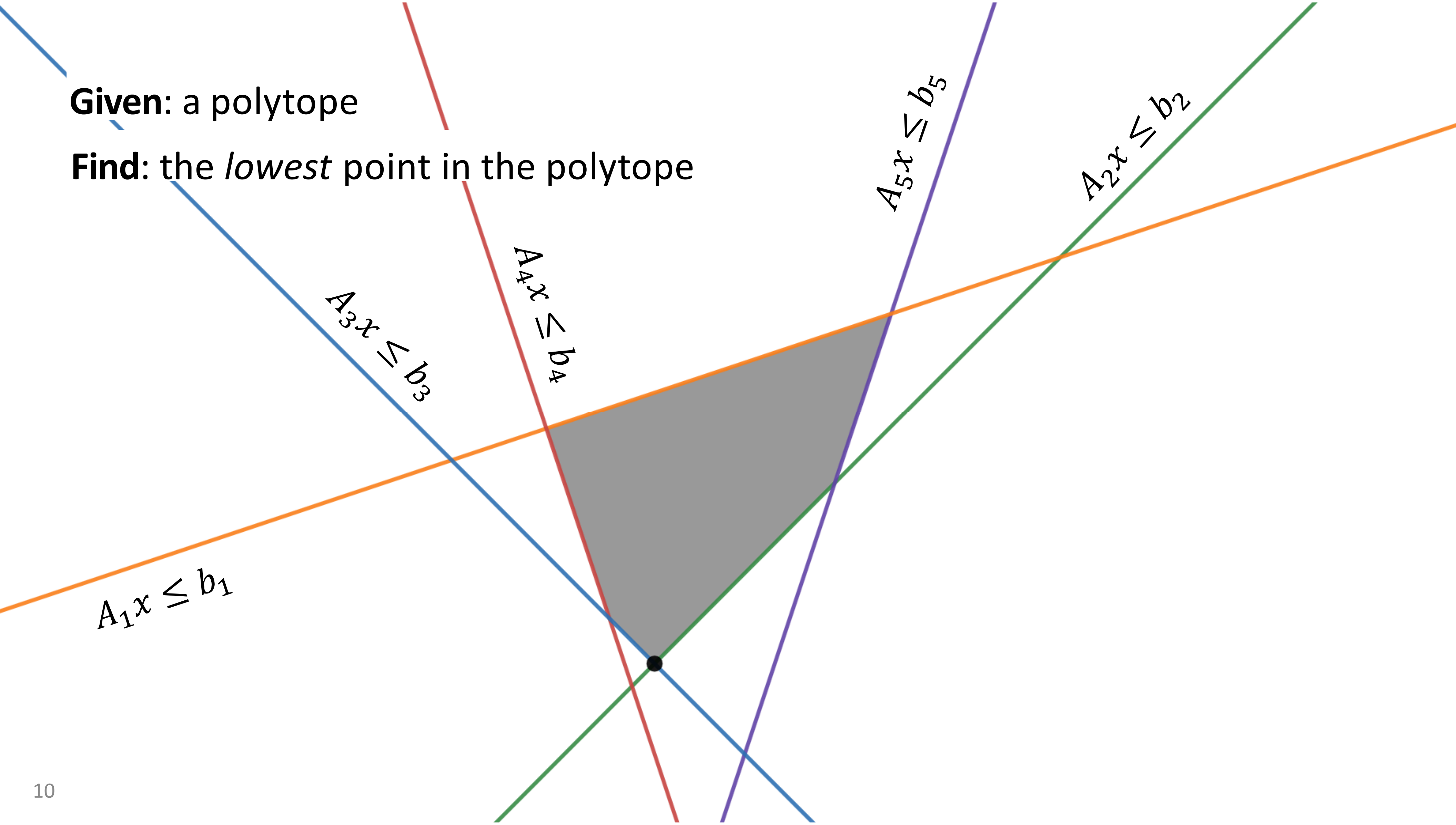


# Linear Algebra primer

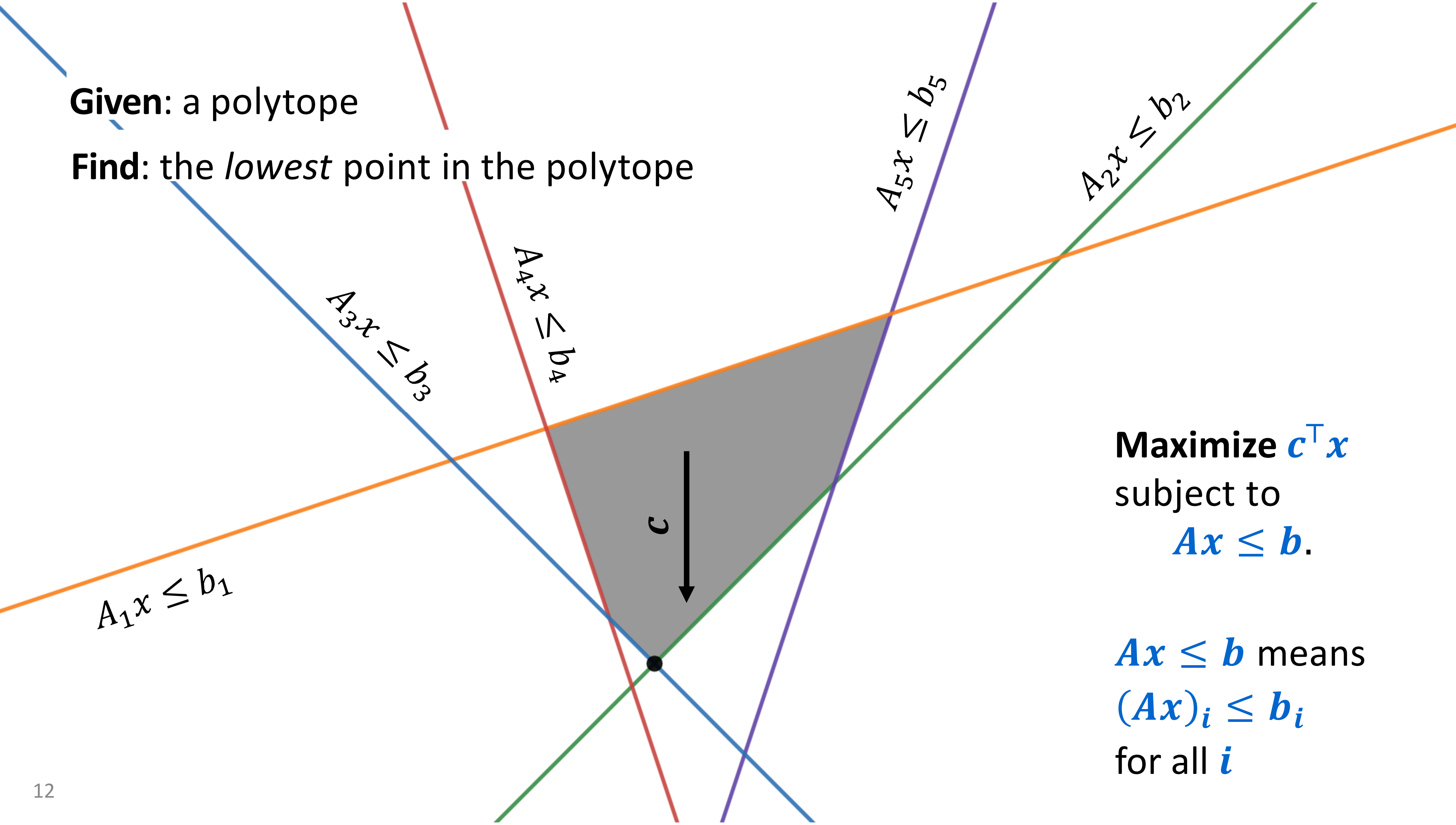
For $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ we think of $\boldsymbol{a}$ and $\boldsymbol{x}$ as column vectors

$$\boldsymbol{a}^{\top}\boldsymbol{x} = \boldsymbol{a_1 x_1} + \cdots + \boldsymbol{a_n x_n}$$

Write $\boldsymbol{m} \times \boldsymbol{n}$ matrix $\boldsymbol{A}$, for $\boldsymbol{Ax} = \begin{bmatrix} \boldsymbol{A_1 x} \\ \boldsymbol{A_2 x} \\ \boldsymbol{A_3 x} \\ \cdots \\ \boldsymbol{A_m x} \end{bmatrix}$ where $\boldsymbol{A_1}, \ldots, \boldsymbol{A_m}$ are rows of $\boldsymbol{A}$.

**Given**: a polytope

**Find**: the *lowest* point in the polytope

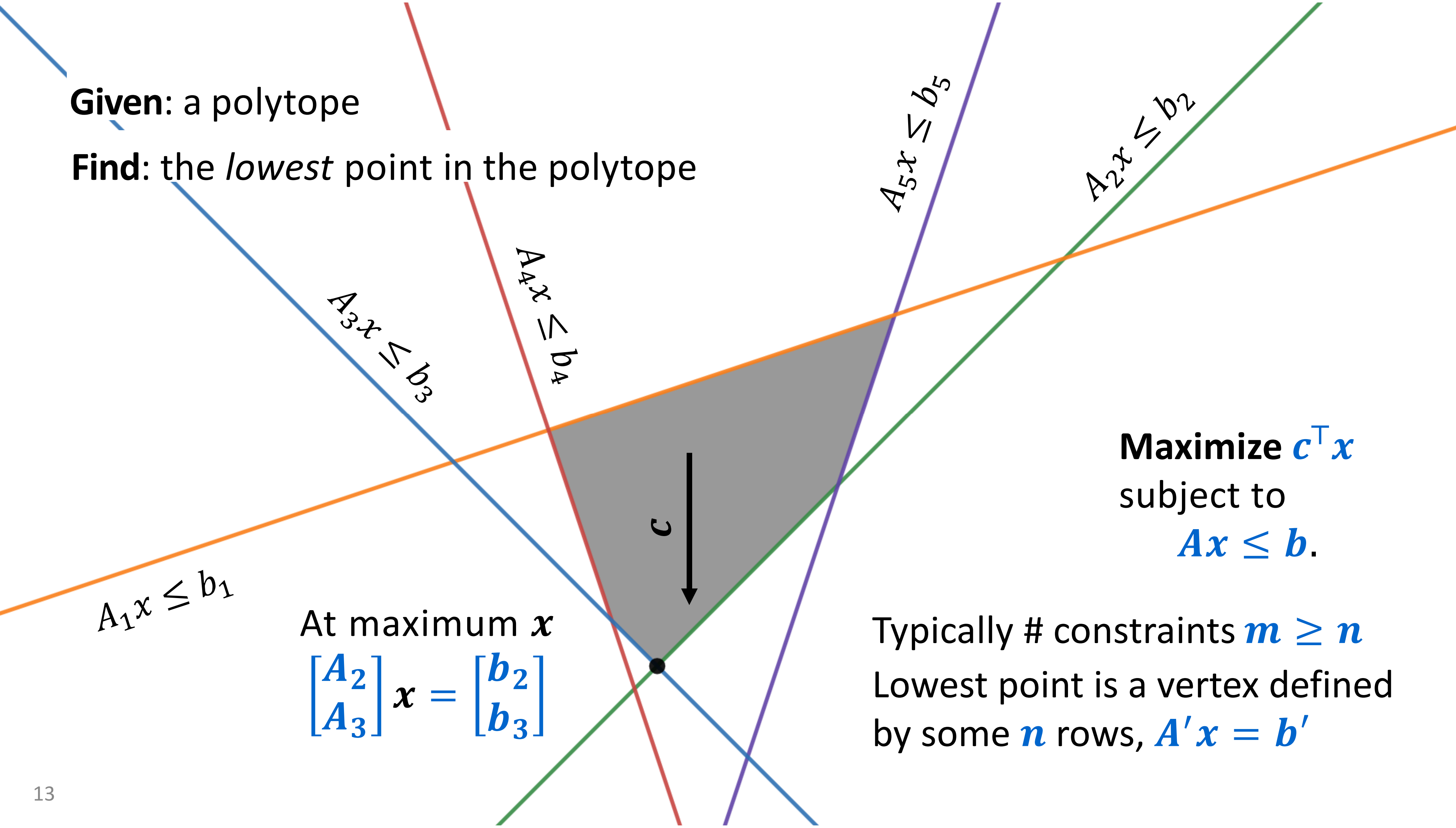



**Maximize** $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$\boldsymbol{Ax} \le \boldsymbol{b}$.

$\boldsymbol{Ax} \le \boldsymbol{b}$ means
$(\boldsymbol{Ax})_{\boldsymbol{i}} \le \boldsymbol{b}_{\boldsymbol{i}}$
for all $\boldsymbol{i}$

**Given**: a polytope

**Find**: the *lowest* point in the polytope




At maximum $\boldsymbol{x}$

$$\begin{bmatrix} \boldsymbol{A_2} \\ \boldsymbol{A_3} \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{b_2} \\ \boldsymbol{b_3} \end{bmatrix}$$

**Maximize** $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$\boldsymbol{Ax} \le \boldsymbol{b}$.

Typically # constraints $\boldsymbol{m} \ge \boldsymbol{n}$
Lowest point is a vertex defined by some $\boldsymbol{n}$ rows, $\boldsymbol{A'x} = \boldsymbol{b'}$

# Standard Form

**Maximize** $\boldsymbol{c^\top x}$
subject to
$$\boldsymbol{Ax \le b}$$
$$\boldsymbol{x \ge 0}$$

**Maximize** $\boldsymbol{z_1 + 2z_3}$
subject to
$$\begin{aligned} \boldsymbol{2z_1 - z_2 + 3z_3} &\boldsymbol{\le 1} \\ \boldsymbol{-z_1 + z_2 - \phantom{3}z_3} &\boldsymbol{\le 5} \end{aligned}$$

replace each $\boldsymbol{z_i}$ by
$\boldsymbol{x_{i,a} - x_{i,b}}$
for $\boldsymbol{x_{i,a}, x_{i,b} \ge 0}$

**Maximize** $\boldsymbol{(x_{1,a} - x_{1,b}) + 2(x_{3,a} - x_{3,b})}$
subject to
$$\begin{aligned} \boldsymbol{2(x_{1,a} - x_{1,b}) - (x_{2,a} - x_{2,b}) + 3(x_{3,a} - x_{3,b})} &\boldsymbol{\le 1} \\ \boldsymbol{-(x_{1,a} - x_{1,b}) + (x_{2,a} - x_{2,b}) - \phantom{3}(x_{3,a} - x_{3,b})} &\boldsymbol{\le 5} \\ \boldsymbol{x} &\boldsymbol{\ge 0} \end{aligned}$$

# Max Flow

**Given:** A Flow Network $G = (V, E)$ with source $s$, sink $t$, and $c: E \to \mathbb{R}^{\geq 0}$

**Maximize** flow out of $s$

subject to

- respecting capacities
- flow conservation at internal nodes

**LP Variables:**

$x_e$ for each $e \in E$ representing flow on edge $e$

**Maximize** $\sum_{e \text{ out of } s} x_e$

subject to

$0 \leq x_e \leq c(e)$ for every $e \in E$

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

# Max Flow

**Maximize** $\displaystyle\sum_{e \text{ out of } s} x_e$

subject to

$0 \le x_e \le c(e)$ for every $e \in E$

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

**Maximize** $c^\top x$
subject to

$$Ax \le b$$
$$x \ge 0$$

This is for the $c$ above. Nothing to do with capacities!

1. $c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise} \end{cases}$
2. $x_e \le c(e)$
3. $\sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e \le 0$
4. $\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e \le 0$
5. $x \ge 0$

Replace equality constraints by a pair of inequalities

# Minimization or Maximization

**Minimize** $\boldsymbol{c}^\top \boldsymbol{x}$
subject to

$$\boldsymbol{A x} \geq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

$\Longleftrightarrow$

**Maximize** $(-\boldsymbol{c})^\top \boldsymbol{x}$
subject to

$$(-\boldsymbol{A})\boldsymbol{x} \leq (-\boldsymbol{b})$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

# Shortest Paths

**Given:** Directed graph $G = (V, E)$
vertices $s$, $t$ in $V$

**Find:** (length of) shortest path from $s$ to $t$

**Claim:** *Length* $\ell$ of the shortest path is the solution (minimum value) for this program.

**Proof sketch:** A shortest path yields a solution of cost $\ell$. Optimal solution must be a combination of flows on shortest paths also cost $\ell$; otherwise there is a part of the $1$ unit of flow that gets counted on more than $\ell$ edges.

**Minimize** $\displaystyle\sum_{e} x_e$ Sum of flow on all edges

subject to

$$x \geq 0$$

$$\sum_{e \text{ out of } s} x_e = 1$$ Flow out of $s$ is $1$

$$\sum_{e \text{ into } t} x_e = 1$$ Flow into $t$ is $1$

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

Flow conservation

# Shortest Paths

**Given:** Directed graph $G = (V, E)$
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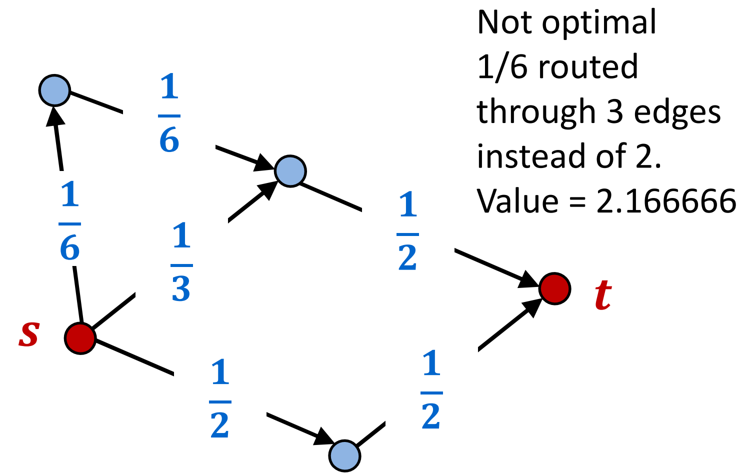
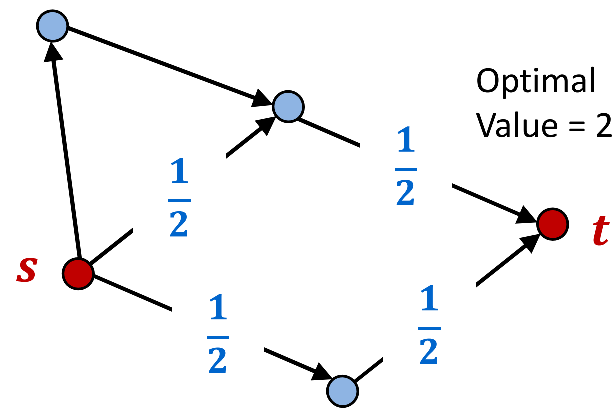


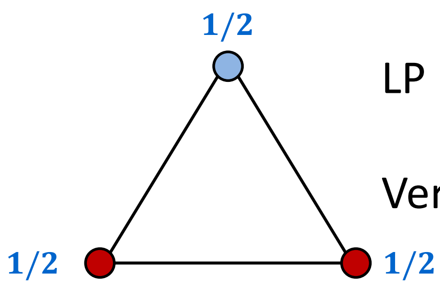



# Vertex Cover

**Given:** Undirected graph $G = (V, E)$

**Find:** smallest set of vertices touching all edges of $G$.

**Doesn't work:** To define a set we need $x_v = 0$ or $x_v = 1$




LP minimum $= \ 3/2$

Vertex Cover minimum $= \ 2$

**Natural Variables for LP:**

$x_v$ for each $v \in V$

**Minimize** $\sum_v x_v$

subject to

$0 \leq x_v \leq 1$ for each node $v \in V$

$x_u + x_v \geq 1$ for each edge $\{u, v\} \in E$

This LP optimizes for a different problem: "**fractional vertex cover**".

$x_v$ indicates the fraction of vertex $v$ that is chosen in the cover.

# What makes Max Flow different?

For Vertex Cover we only got a fractional optimum but for Max Flow can get integers.

- Why?
  - Ford-Fulkerson analysis tells us this for Max Flow.
  - Is there a reason we can tell just from the LP view?

**Recall:** Optimum is at some vertex $\boldsymbol{x}$ satisfying $\boldsymbol{A'x = b'}$ for some subset of exactly $\boldsymbol{n}$ constraints.

This means that $\boldsymbol{x = (A')^{-1}b'}$.

Entries of the matrix inverse are quotients of determinants of sub-matrices of $\boldsymbol{A'}$ so, for integer inputs, optimum is always rational.

**Fact:** Every full rank submatrix of MaxFlow matrix $\boldsymbol{A}$ has determinant $\boldsymbol{\pm 1}$

$\Rightarrow$ all denominators are $\boldsymbol{\pm 1}$ $\Rightarrow$ integers. $\boldsymbol{A}$ is "totally unimodular"

**Next:** How **MaxFlow=MinCut** is an example of a general "duality" property of LPs