CSE 421

Introduction to Algorithms

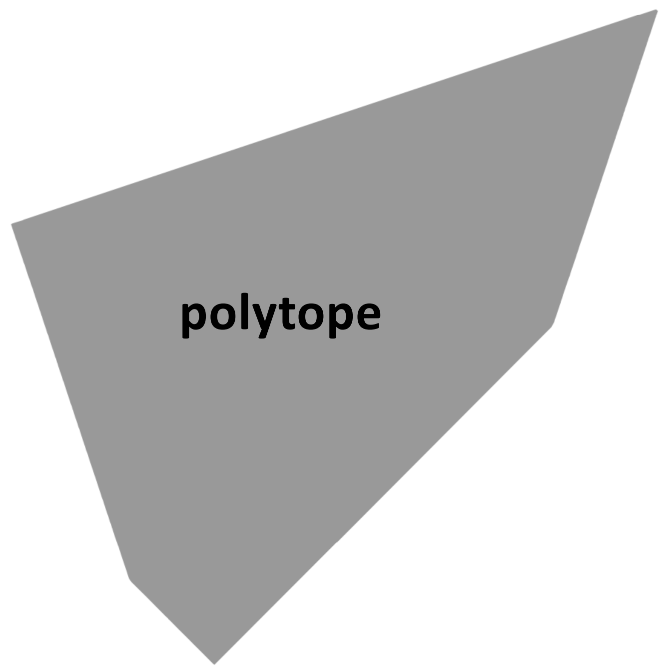
Lecture 20: Linear Programming:

A really very extremely big hammer

Will discuss midterm 1st

Given: a polytope

Find: the *lowest* point in the polytope



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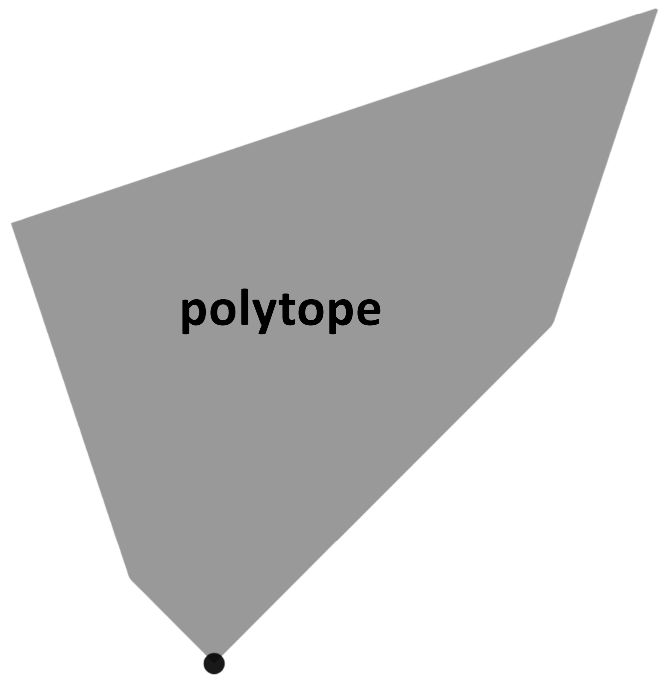
Given: a polytope

Find: the *lowest* point in the polytope



We have fast algorithms for this!

Maximize $z_1 + 2z_3$

subject to:

$2z_1 - z_2 + 3z_3 \leq 1$

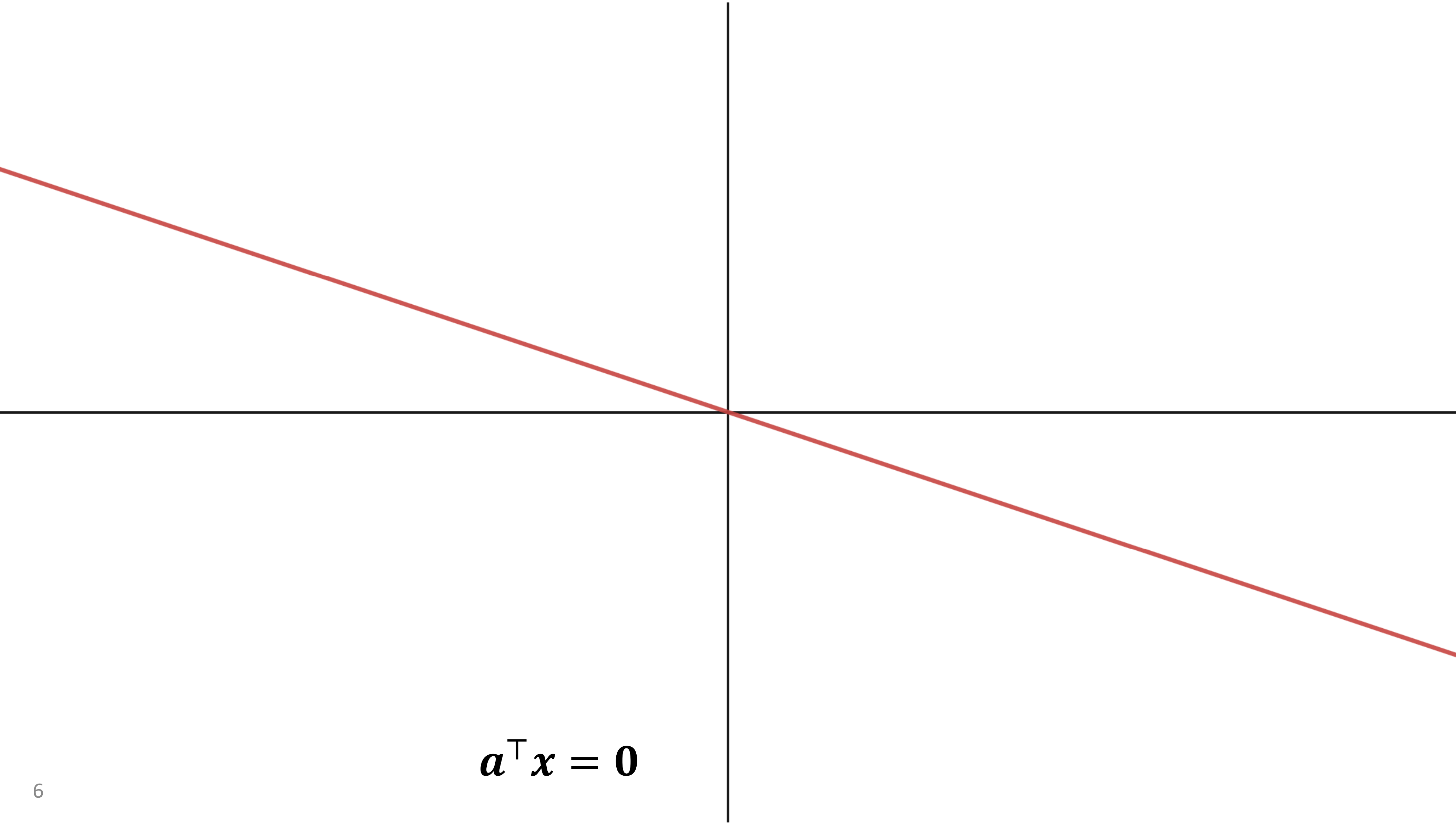
$-z_1 + z_2 - z_3 \leq 5$

Linear Algebra primer

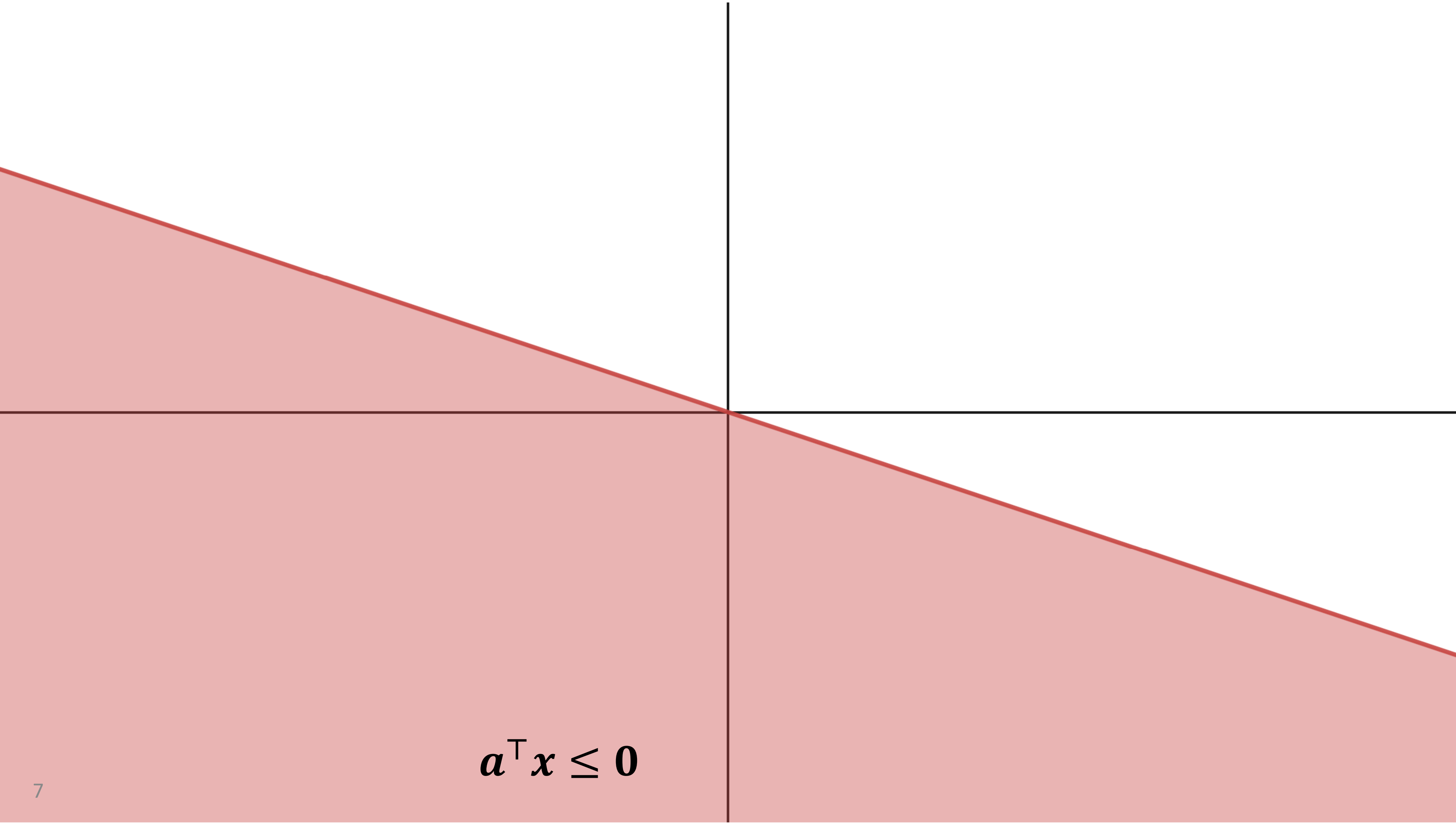
For $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ we think of $\boldsymbol{a}$ and $\boldsymbol{x}$ as column vectors

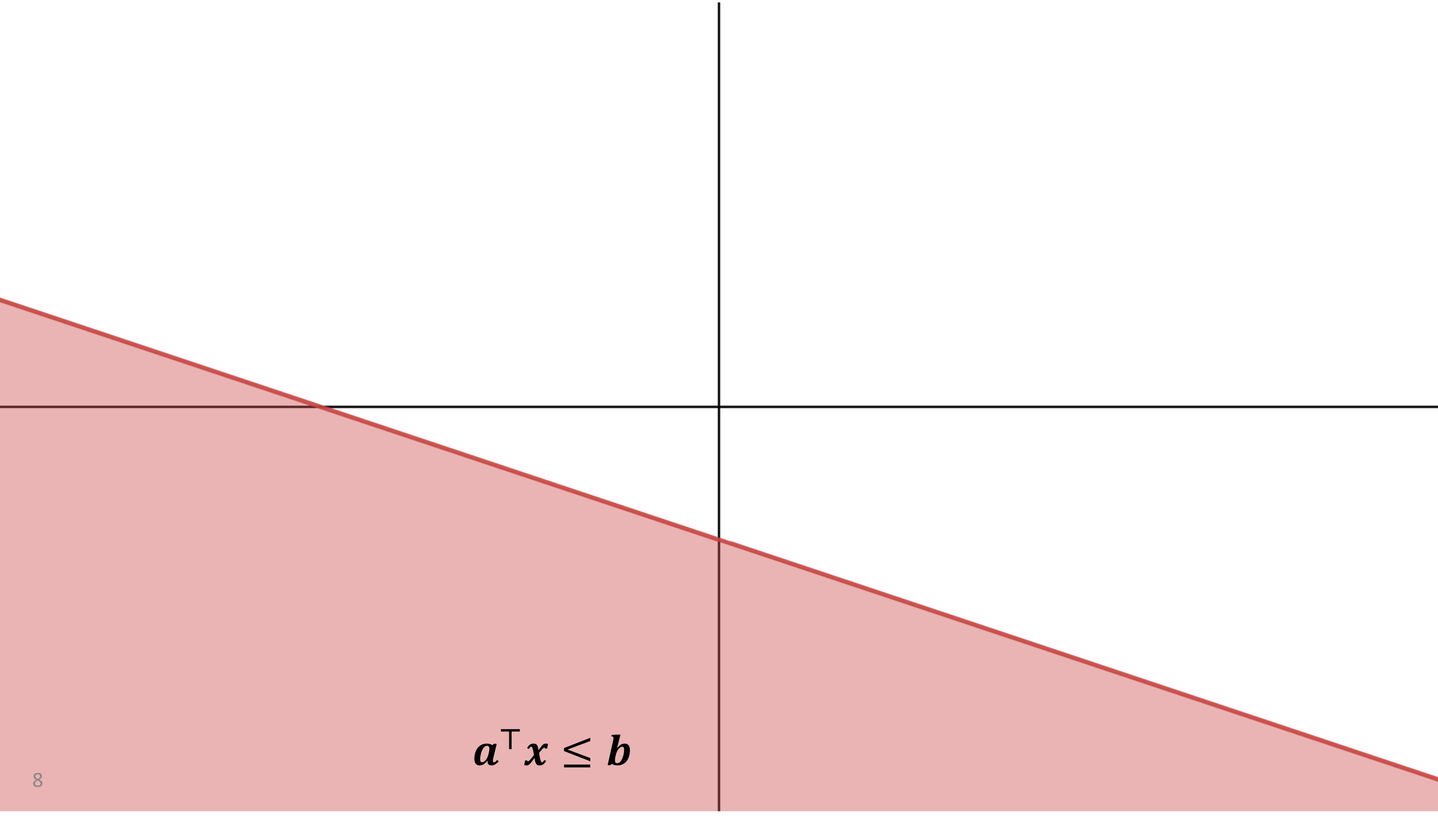
$$\boldsymbol{a}^\top \boldsymbol{x} = \boldsymbol{a_1 x_1} + \cdots + \boldsymbol{a_n x_n}$$

The set of $\boldsymbol{x}$ satisfying $\boldsymbol{a}^\top \boldsymbol{x} = \boldsymbol{0}$ is *hyperplane*



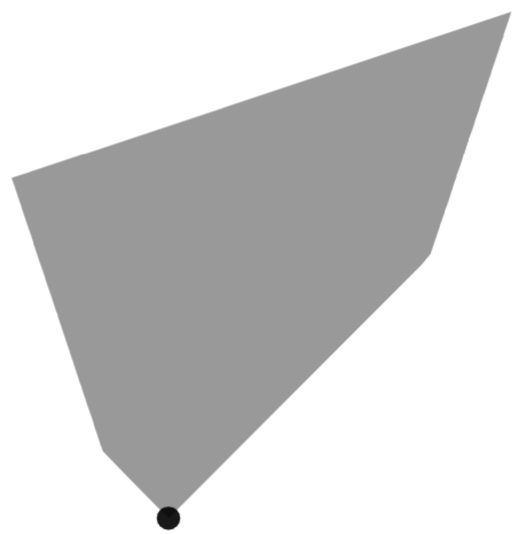
$$\boldsymbol{a}^\top \boldsymbol{x} = \boldsymbol{0}$$


$\boldsymbol{a}^\top \boldsymbol{x} \leq \mathbf{0}$


$\boldsymbol{a}^\top \boldsymbol{x} \leq \boldsymbol{b}$

Given: a polytope

Find: the *lowest* point in the polytope



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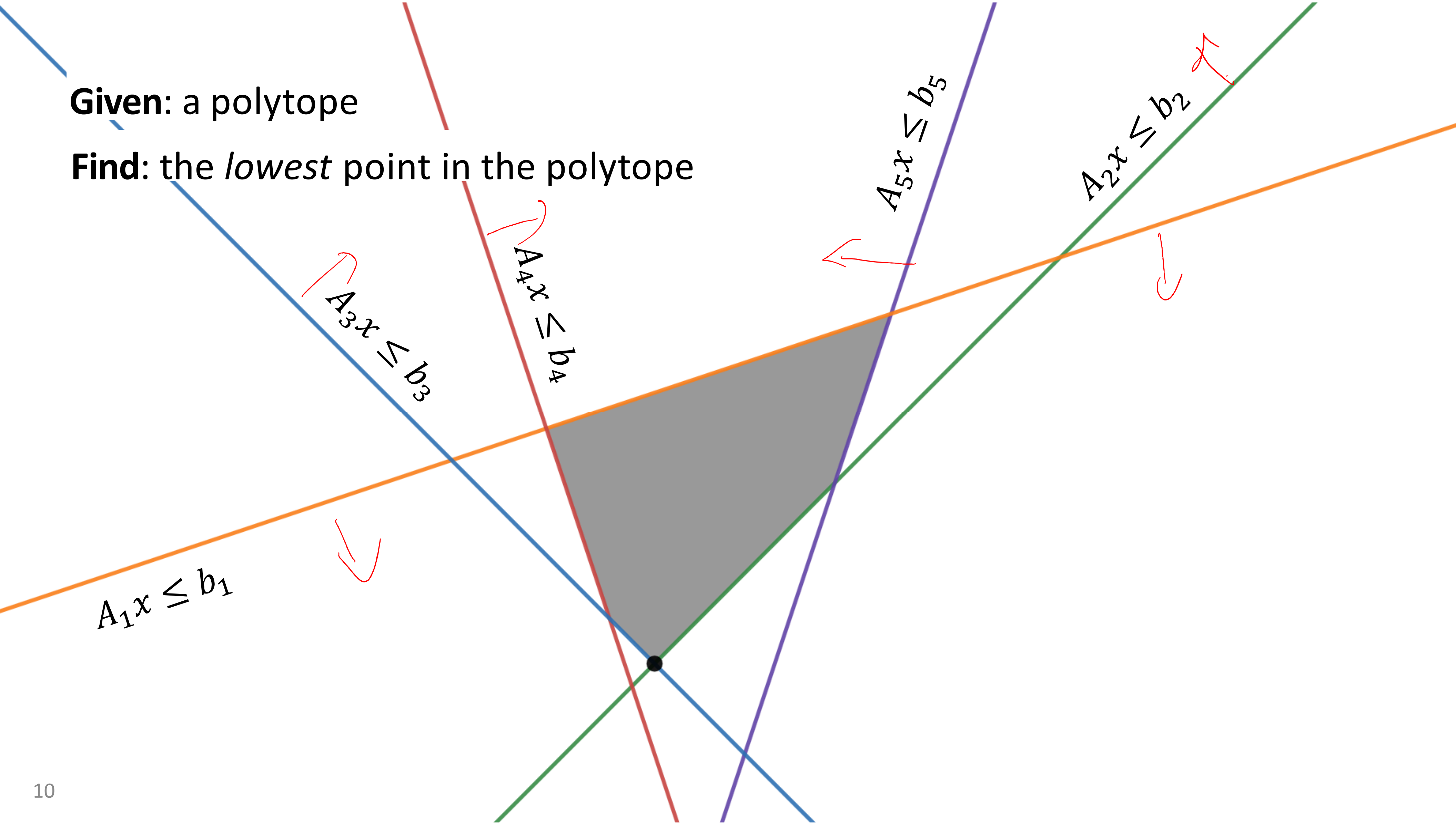


Linear Algebra primer

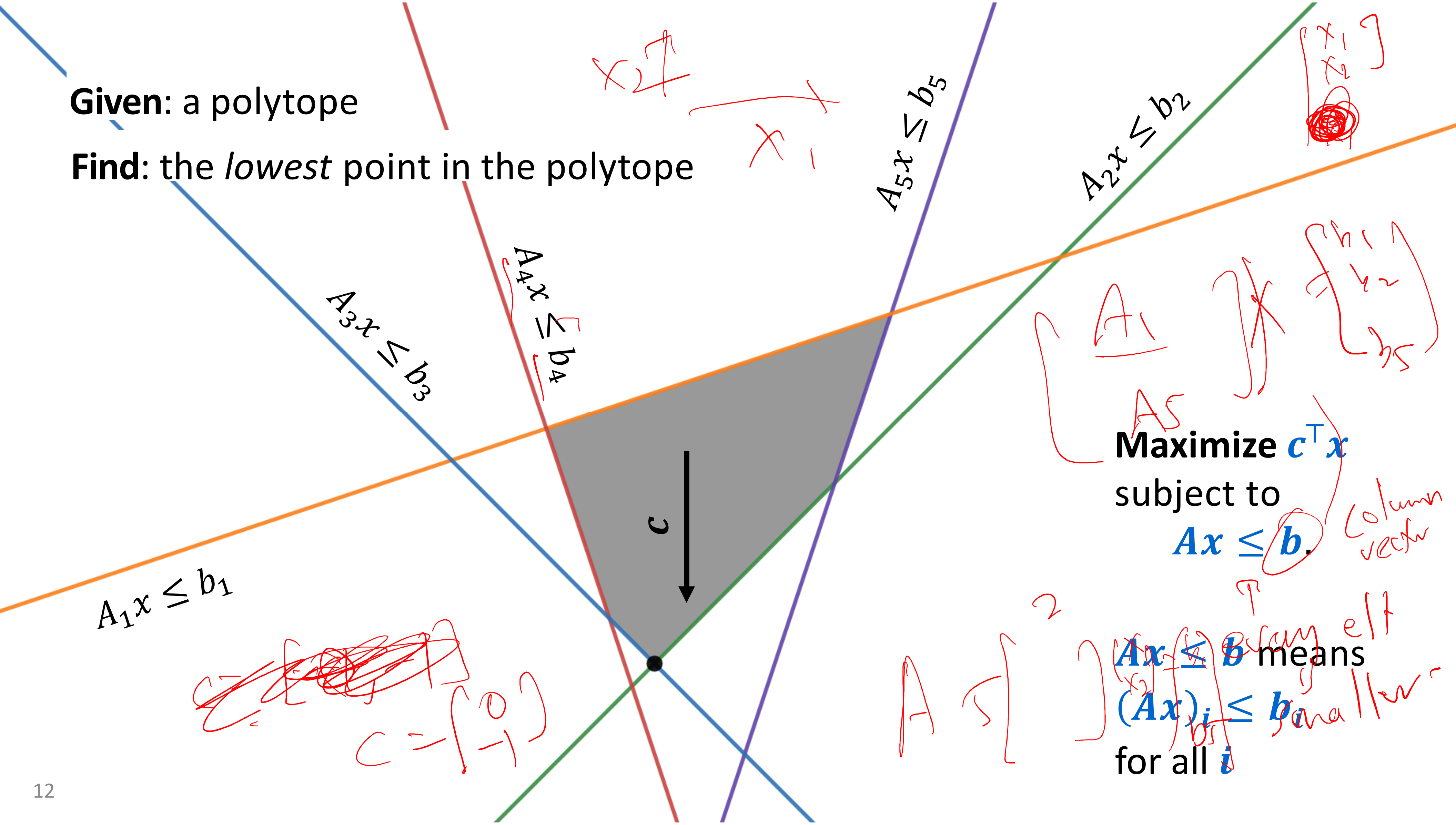
For $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ we think of $\boldsymbol{a}$ and $\boldsymbol{x}$ as column vectors

$$\boldsymbol{a}^\top \boldsymbol{x} = \boldsymbol{a_1 x_1} + \cdots + \boldsymbol{a_n x_n}$$

Write $\boldsymbol{m} \times \boldsymbol{n}$ matrix $\boldsymbol{A}$, for $\boldsymbol{Ax} = \begin{bmatrix} \boldsymbol{A_1 x} \\ \boldsymbol{A_2 x} \\ \boldsymbol{A_3 x} \\ \cdots \\ \boldsymbol{A_m x} \end{bmatrix}$ where $\boldsymbol{A_1}, \ldots, \boldsymbol{A_n}$ are rows of $\boldsymbol{A}$.

Given: a polytope

Find: the *lowest* point in the polytope

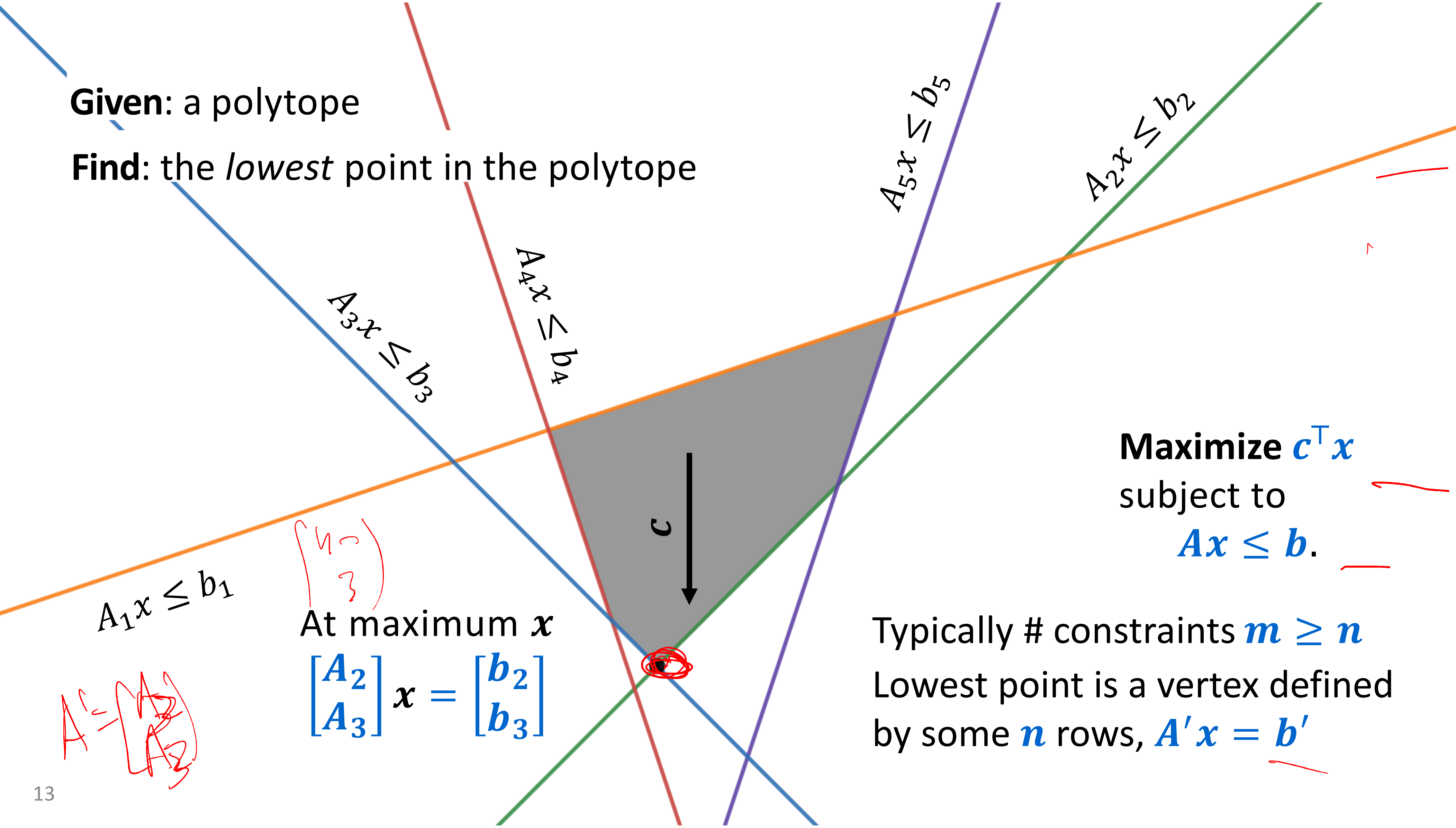


Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$\boldsymbol{Ax} \leq \boldsymbol{b}$.

$\boldsymbol{Ax} \leq \boldsymbol{b}$ means
$(\boldsymbol{Ax})_i \leq \boldsymbol{b}_i$
for all $\boldsymbol{i}$

Given: a polytope

Find: the *lowest* point in the polytope



At maximum $\boldsymbol{x}$

$$\begin{bmatrix} \boldsymbol{A}_2 \\ \boldsymbol{A}_3 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{bmatrix}$$

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$.

Typically # constraints $\boldsymbol{m} \geq \boldsymbol{n}$

Lowest point is a vertex defined by some $\boldsymbol{n}$ rows, $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$

Standard Form

Maximize $c^{\top}x$
subject to

$$Ax \le b$$
$$x \ge 0$$

Maximize $z_1 + 2z_3$
subject to

$$2z_1 - z_2 + 3z_3 \le 1$$
$$-z_1 + z_2 - z_3 \le 5$$

replace each z_i by
$x_{i,a} - x_{i,b}$
for $x_{i,a}, x_{i,b} \ge 0$

Maximize $(x_{1,a} - x_{1,b}) + 2(x_{3,a} - x_{3,b})$
subject to

$$2(x_{1,a} - x_{1,b}) - (x_{2,a} - x_{2,b}) + 3(x_{3,a} - x_{3,b}) \le 1$$
$$-(x_{1,a} - x_{1,b}) + (x_{2,a} - x_{2,b}) - (x_{3,a} - x_{3,b}) \le 5$$
$$x \ge 0$$

Max Flow

Given: A Flow Network $G = (V, E)$ with source s, sink t, and $c: E \to \mathbb{R}^{\geq 0}$

Maximize flow out of s

subject to

- respecting capacities
- flow conservation at internal nodes

LP Variables:

x_e for each $e \in E$ representing flow on edge e

Maximize $\sum_{e \text{ out of } s} x_e$

subject to

$0 \leq x_e \leq c(e)$ for every $e \in E$

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

Max Flow

Maximize $\sum_{e \text{ out of } s} x_e$

subject to

$0 \leq x_e \leq c(e)$ for every $e \in E$

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

Maximize $c^\top x$
subject to
$Ax \leq b$
$x \geq 0$

This is for the c above. Nothing to do with capacities!

1. $c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise} \end{cases}$
2. $x_e \leq c(e)$
3. $\sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e \leq 0$
4. $\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e \leq 0$
5. $x \geq 0$

Replace equality constraints by a pair of inequalities