CSE 421
Introduction to Algorithms

Lecture 14: Dynamic Programming
Bellman-Ford

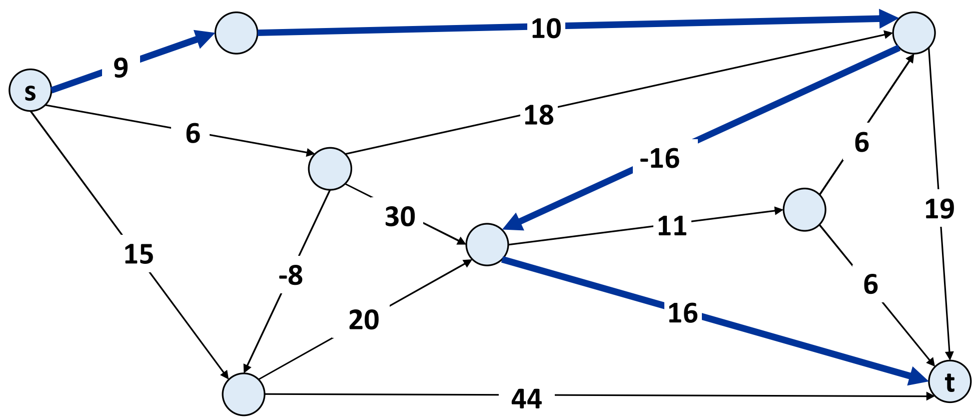
Shortest Paths allowing negative-cost edges

Shortest path problem:

Given: a directed graph $G = (V, E)$ with edge weights c_{vw} (possibly negative) and vertices $s, t \in V$.

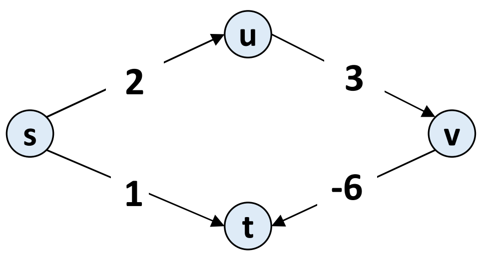
Find: a shortest path in G from s to node t.

Sample Application: Nodes represent agents in a financial setting and c_{vw} is cost of a transaction in which we buy from agent v and sell immediately to w.



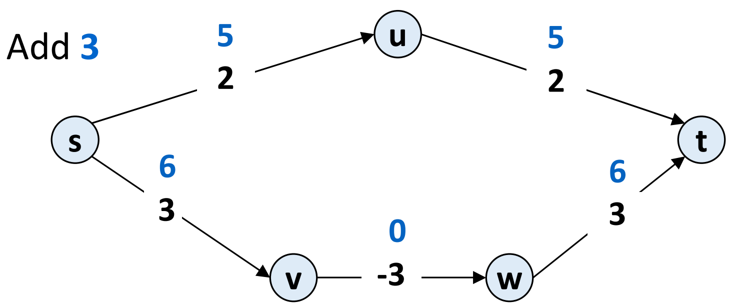
Shortest Paths: Failed Attempts

Why not Dijkstra's Algorithm? Can fail if negative edge costs.



Dijkstra begins with $S = \{s\}$ and $d(s) = 0$.
Next step would add t to s at distance 1, though actual minimum distance from s to t is -1.

Adding a constant to every edge cost to make them ≥ 0? Also fails.



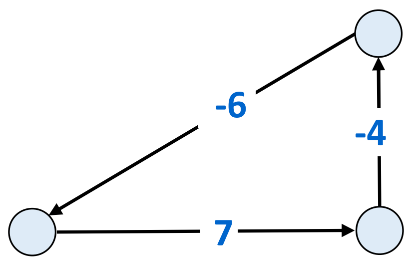
Problem: Paths can have different lengths so adding a fixed amount per edge changes relative costs.

Original shortest path is **s-v-w-t** with cost **3**.

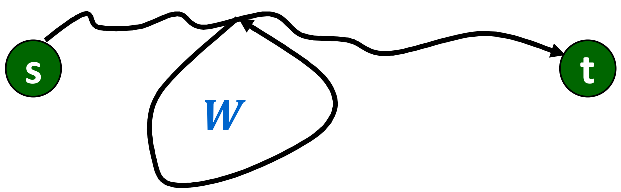
After adjustment, shortest path is **s-u-t**.

Shortest Paths: Negative Cost Cycles

Negative cost cycle:



Observation: (1) If some path from s to t contains a negative cost cycle, there does not exist a shortest s-t path.



$c(W) < 0$

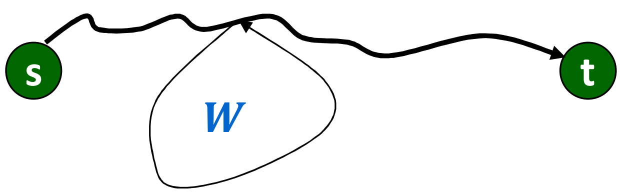
The path can go around the cycle W more times and get even lower cost, the limit of path costs is $-\infty$.

Shortest Paths: Negative Cost Cycles

Observation: (1) If some path from s to t contains a negative cost cycle, there does not exist a shortest s-t path.

(2) If the graph G has no negative cycles then a shortest s-t path must have at most $n - 1$ edges.

If not, there would be a repeated vertex which would create a cycle that could be removed without decreasing the cost.



$c(W) \geq 0$

Shortest Paths: Dynamic Programming

Defn: $\text{OPT}(i, v)$ = length of shortest v-t path P using at most i edges.

Case 1: P uses at most $i - 1$ edges.

- In this case $\text{OPT}(i, v) = \text{OPT}(i - 1, v)$

Case 2: P uses exactly i edges.

- if (v, w) is first edge, then OPT uses (v, w), and then selects the best w-t path using at most $i - 1$ edges

$$\text{OPT}(i, v) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v = t \\ \infty & \text{if } i = 0 \text{ and } v \neq t \\ \min\left(\text{OPT}(i-1, v),\ \min_{(v,w)\in E} c_{vw} + \text{OPT}(i-1, w)\right) & \text{otherwise} \end{cases}$$

By observation: if no negative cost cycles, $\text{OPT}(n - 1, v)$ = length of shortest v-t path.

Shortest Paths: Implementation

```
Shortest-Path(G, t) {
   foreach node v ∈ V
      OPT[0, v] ← ∞
   OPT[0, t] ← 0

   for i = 1 to n-1
      foreach node v ∈ V
         OPT[i, v] ← OPT[i-1, v]
          foreach edge (v, w) ∈ E
               OPT[i, v] ← min { OPT[i, v], c_vw + OPT[i-1, w] }
}
```

$n-1$ iterations of outer loop

Two inner loops together touch each directed edge once

Total: $O(nm)$ time
$O(n^2)$ space

To find the shortest paths, maintain a "successor" pointer for each vertex that gives the next vertex on the current shortest path to t.

Shortest Paths: Practical Improvements

Practical improvements:

- Maintain only one array **OPT**$[v]$ = shortest v-t path that we have found so far.
- No need to check edges of the form (v, w) unless **OPT**$[w]$ changed in previous iteration.

Theorem: Throughout the algorithm, **OPT**$[v]$ is length of some v-t path, and after i rounds of updates, the value **OPT**$[v]$ is no larger than the length of shortest v-t path using at most i edges.

Overall impact.

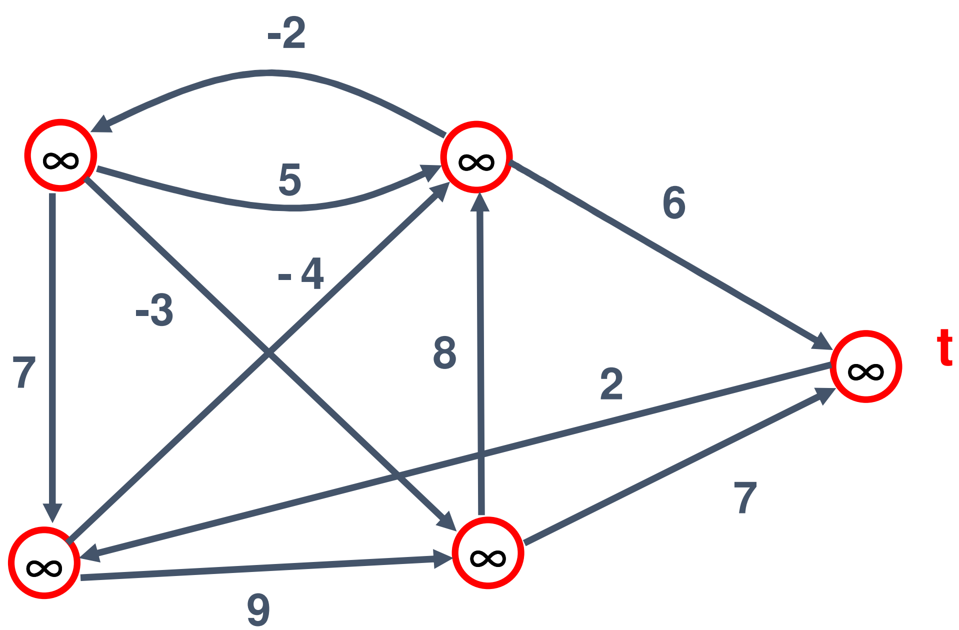
Space: $O(m + n)$.

Running time: Still $O(mn)$ worst case, but substantially faster in practice.

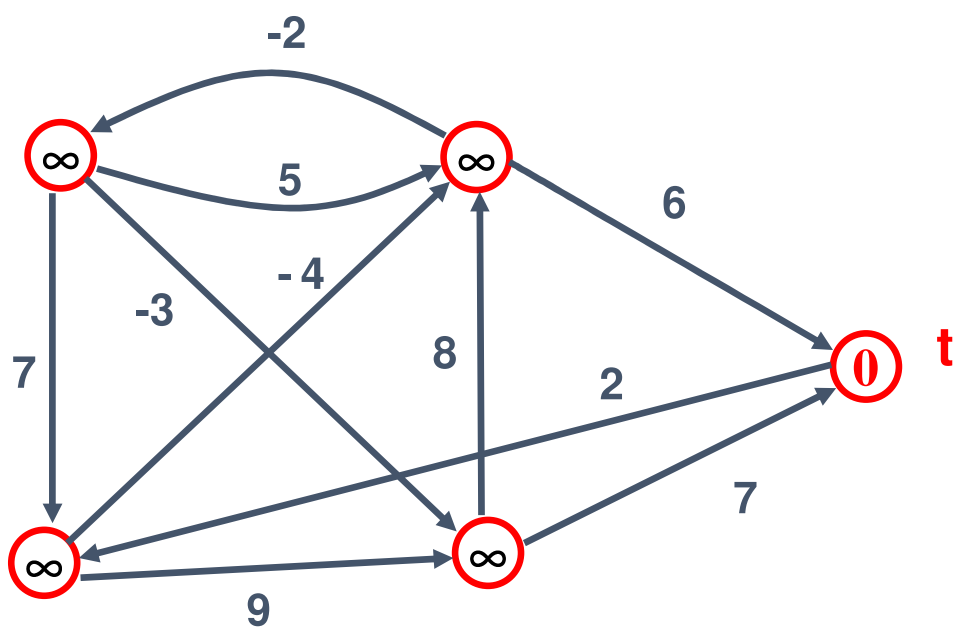
Bellman-Ford: Efficient Implementation

```
Push-Based-Shortest-Path(G, s, t) {
   foreach node v ∈ V {
      OPT[v] ← ∞
      successor[v] ← φ
   }
   OPT[t] = 0; oldupdated ← {t}
   for i = 1 to n-1 {
      updated ← φ
      foreach node w ∈ V {
      if (w is in oldupdated) {
         foreach node v such that (v, w) ∈ E {
            if (OPT[v] > c_vw + OPT[w]) {
               OPT[v] ← c_vw + OPT[w]
               successor[v] ← w
               updated ← updated ∪ {v}
            }
         }
      }
      if updated = φ, stop.
      else oldupdated ← updated
   }
}
```

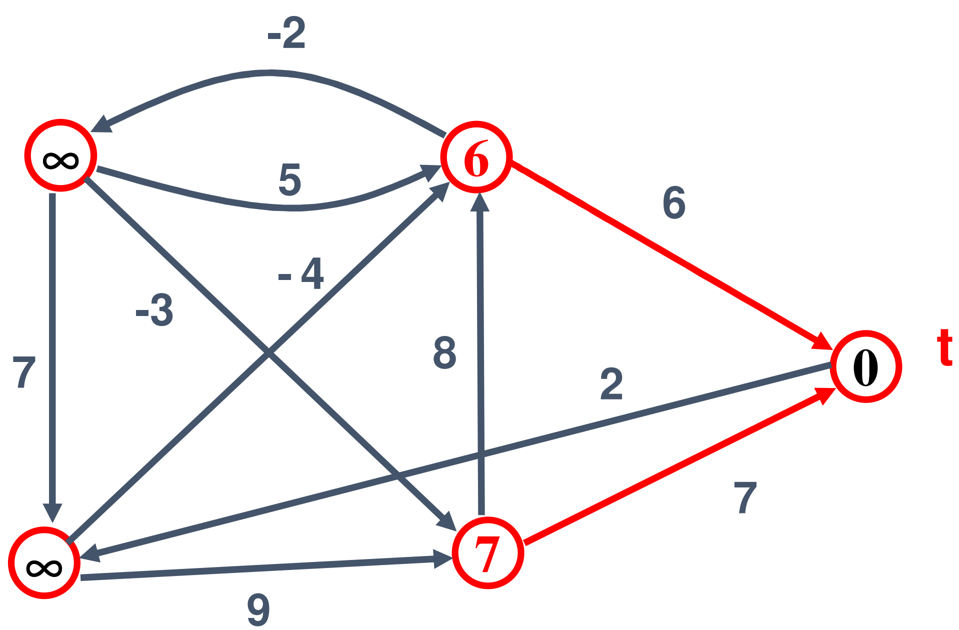
Bellman-Ford



Bellman-Ford



Bellman-Ford



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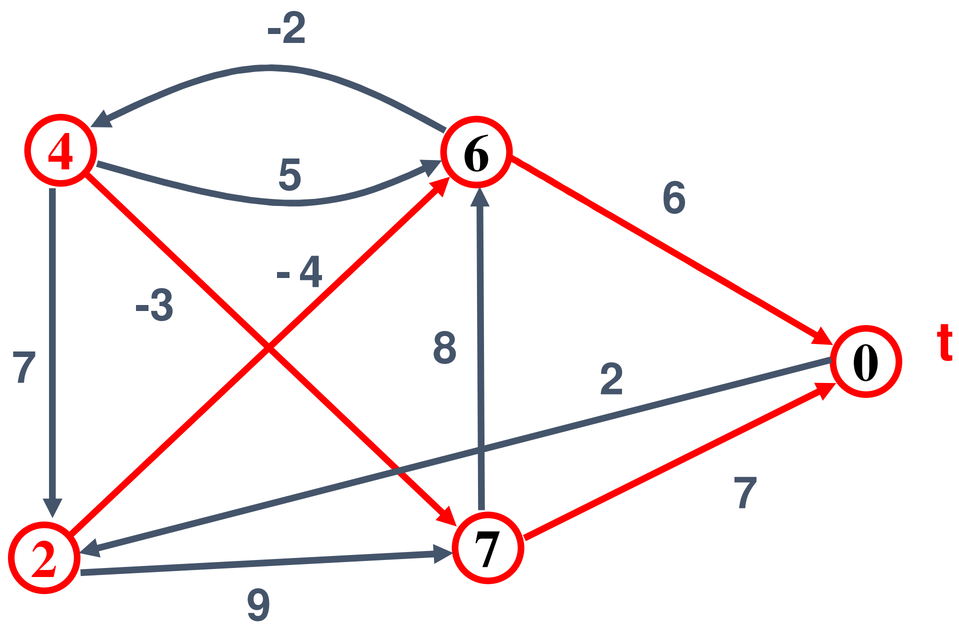

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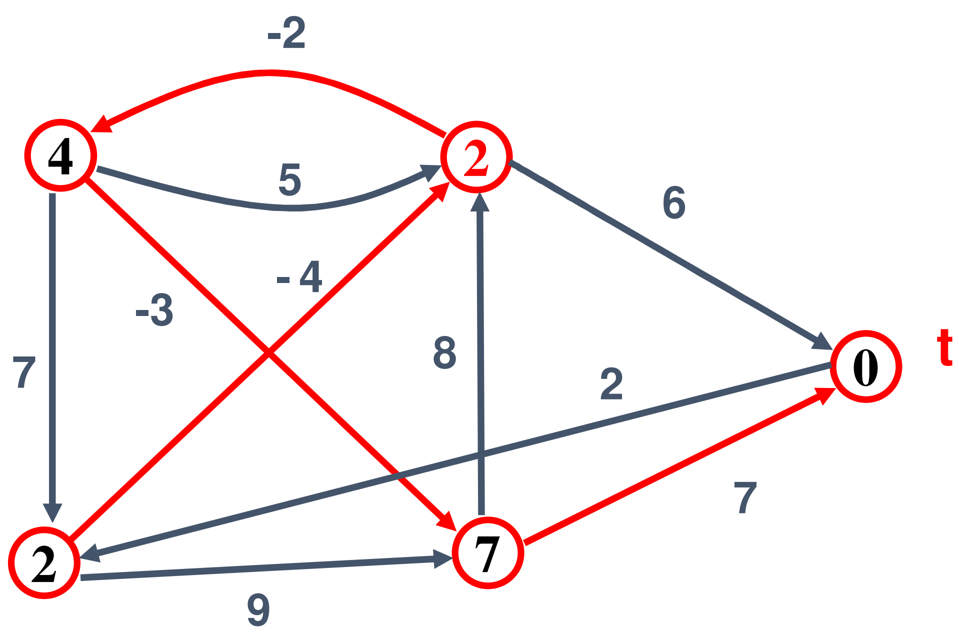

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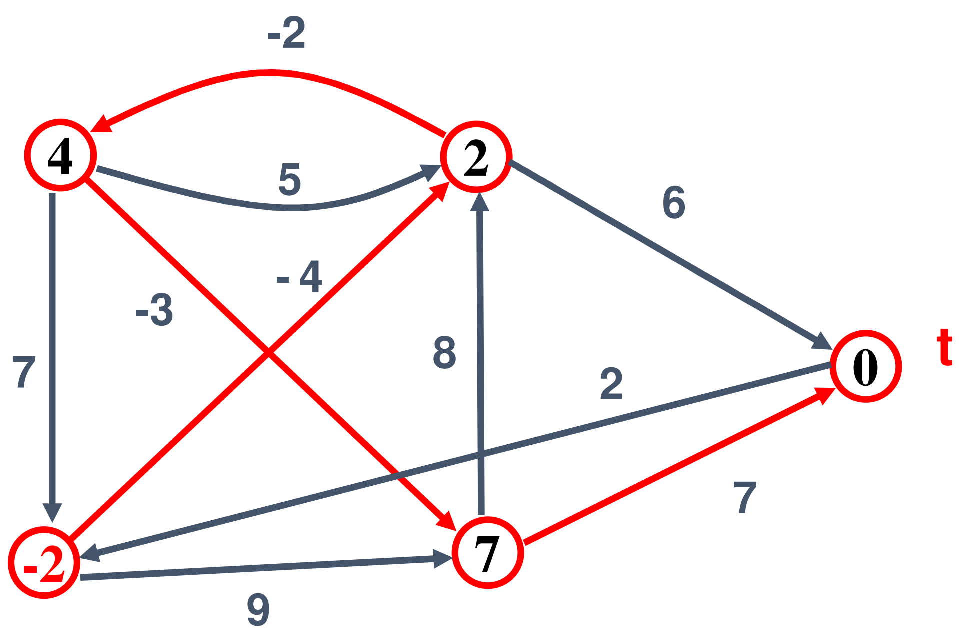

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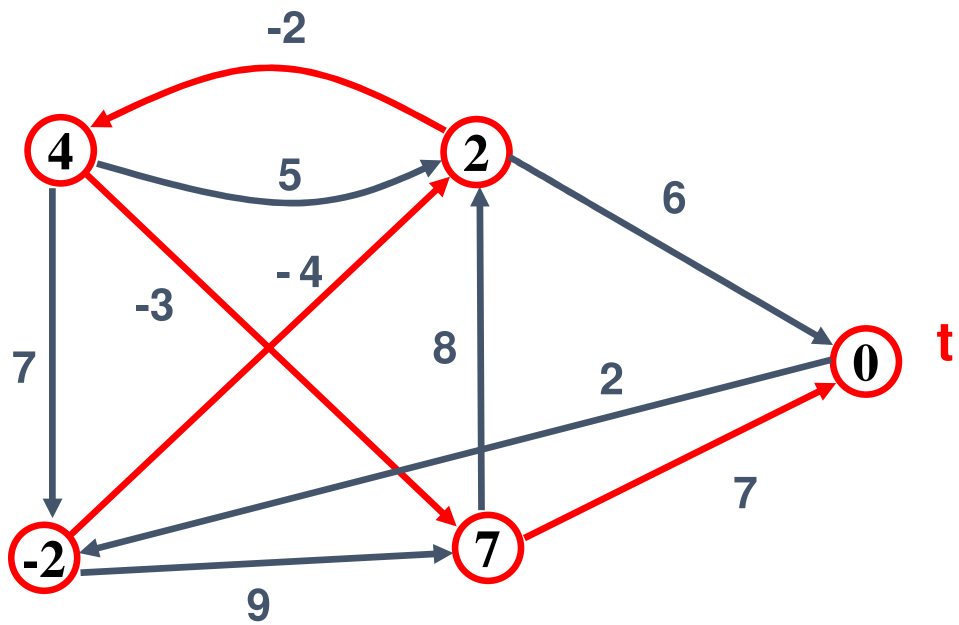
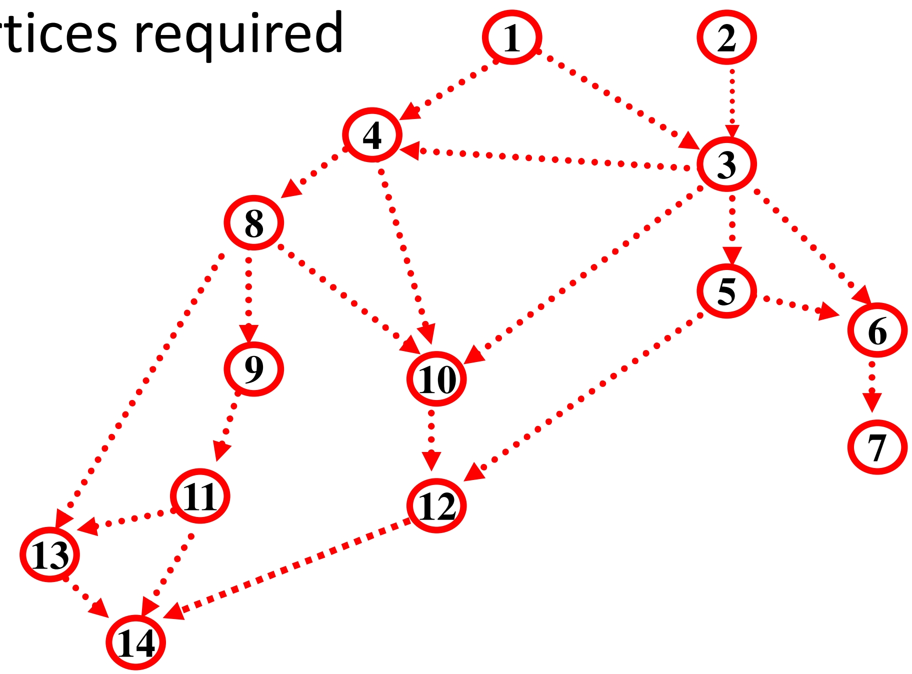


Shortest paths with negative costs on a DAG

Edges only go from lower to higher-numbered vertices

- Update distances in reverse order of topological sort
- Only one pass through vertices required
- $O(n + m)$ time



Distance Vector Protocol

Bellman-Ford Application: Distance Vector Protocol

Application domain: Communication networks

- Node ≈ router
- Edge ≈ direct communication link
- Cost of edge ≈ delay on link.

Edge costs are non-negative, why not use Dijkstra's algorithm?

- Dijkstra's algorithm requires global information in the network

Advantages of Bellman-Ford approach:

- It only uses only local knowledge of neighboring nodes.
- No need for synchronization: We don't expect routers to run in lockstep. The order in which each **`foreach`** loop executes in not important. Moreover, the Bellman-Ford algorithm still converges even if updates are asynchronous!

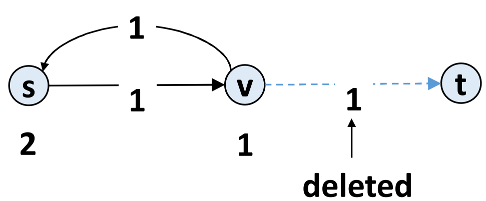
Distance Vector Protocol

Distance vector protocol:

- Each router maintains a vector of shortest path lengths to every other node (distances) and the first hop on each path (directions).
- **Algorithm:** each router performs n separate computations, one for each potential destination node.
- "Routing by rumor."

Examples: RIP, Xerox XNS RIP, Novell's IPX RIP, Cisco's IGRP, DEC's DNA Phase IV, AppleTalk's RTMP.

Caveat: Edge costs may *change* during algorithm (or fail completely).



"counting to infinity" problem

Path Vector Protocols

Link state routing:

- Each router also stores the entire path.
- Based on Dijkstra's algorithm.
- Avoids "counting-to-infinity" problem and related difficulties.
- Requires significantly more storage.

Examples: Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).

Negative Cycles in a Graph

Detecting Negative Cycles

Lemma: If every vertex in G can reach t and $\text{OPT}(n, v) = \text{OPT}(n-1, v)$ for all v, then G has no negative cycles.

Proof: This would be a fixed point of recurrence that computes $\text{OPT}(i, v)$ correctly for every i.
Vertices on negative cycles that can reach t couldn't possibly have a fixed point. ■

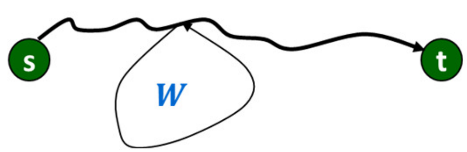
Lemma: If $\text{OPT}(n, v) < \text{OPT}(n-1, v)$ for some v, then shortest path from v to t with length $\leq n$ contains a cycle W. Moreover W has negative cost.

Proof: (By contradiction)

Since $\text{OPT}(n, v) < \text{OPT}(n-1, v)$, paths P with cost $\text{OPT}(n, v)$ have exactly n edges.

By pigeonhole principle, such a P must contain a directed cycle W.

Deleting W yields a v-t path with $< n$ edges $\Rightarrow$ W has negative cost.



■

c(W) < 0

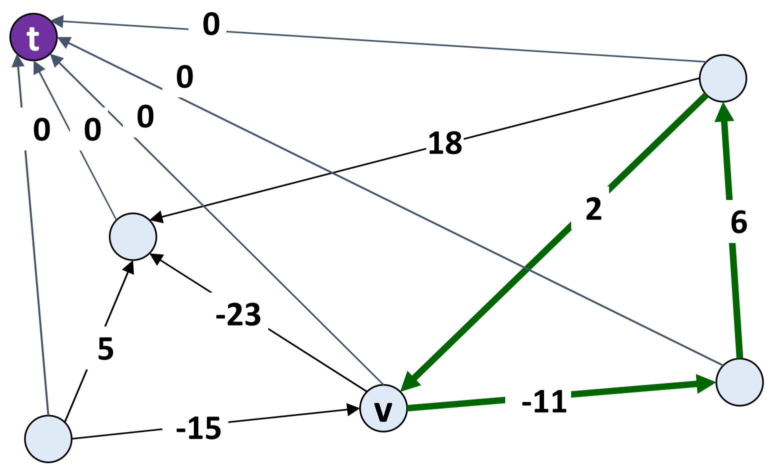
Detecting Negative Cycles

Theorem: Can detect negative cost cycles in $O(mn)$ time.

Algorithm: Add new node t and connect all nodes to t with $\mathbf{0}$-cost edge.

Check if $\text{OPT}(n, v) = \text{OPT}(n-1, v)$ for all vertices v

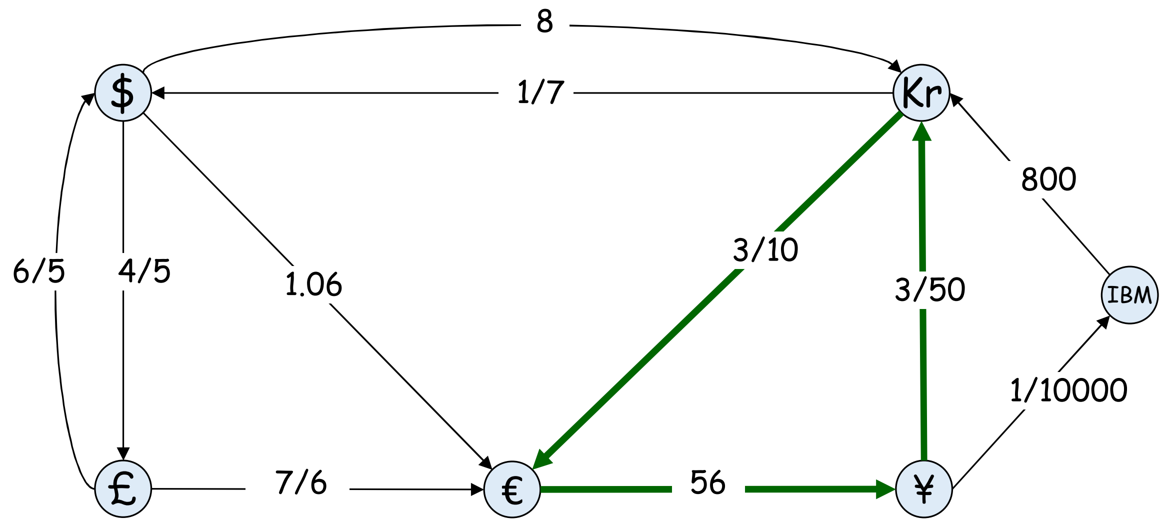
- if yes, then no negative cycles
- if no, then extract cycle from shortest path from v to t



Detecting Negative Cycles: Application

Currency conversion: Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark: High speed trading makes fastest algorithm very valuable!



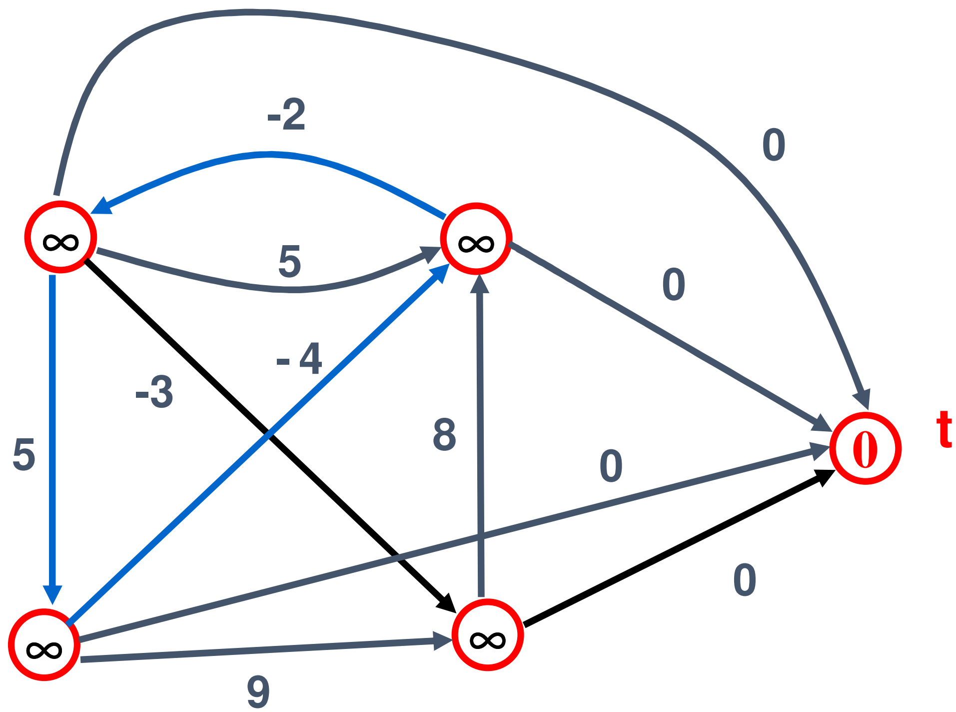
Detecting Negative Cycles: Summary

Run Bellman-Ford on graph with

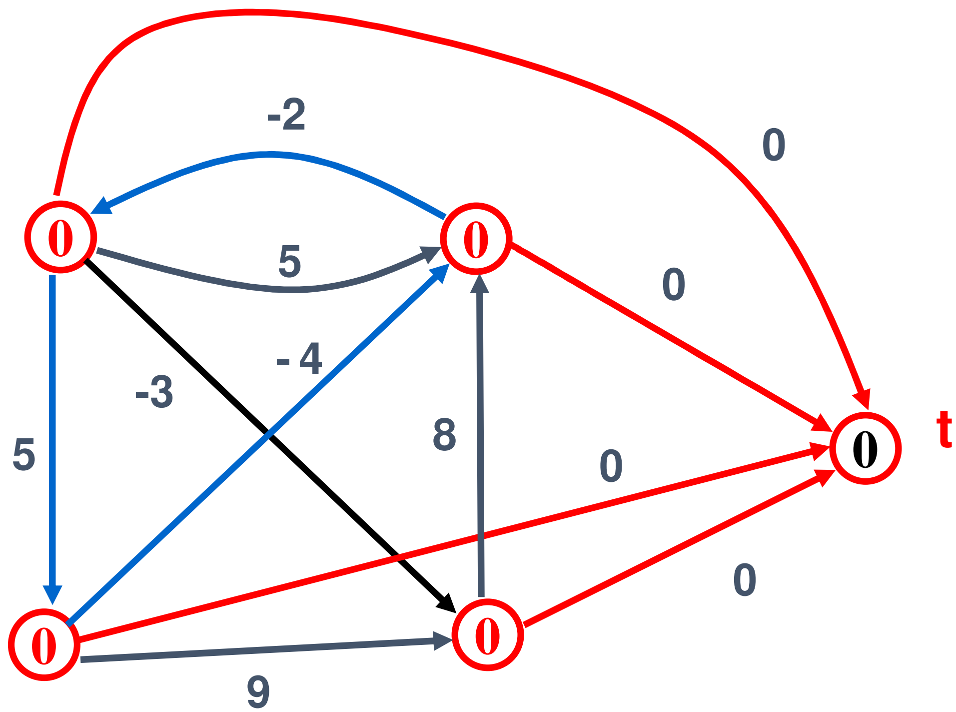
- extra node t.
- early stopping for up to n iterations (instead of $n - 1$).
- successor variables

Fact: upon termination, successor variables trace a negative cycle if one exists...

Bellman-Ford for Negative Cycles



Bellman-Ford for Negative Cycles



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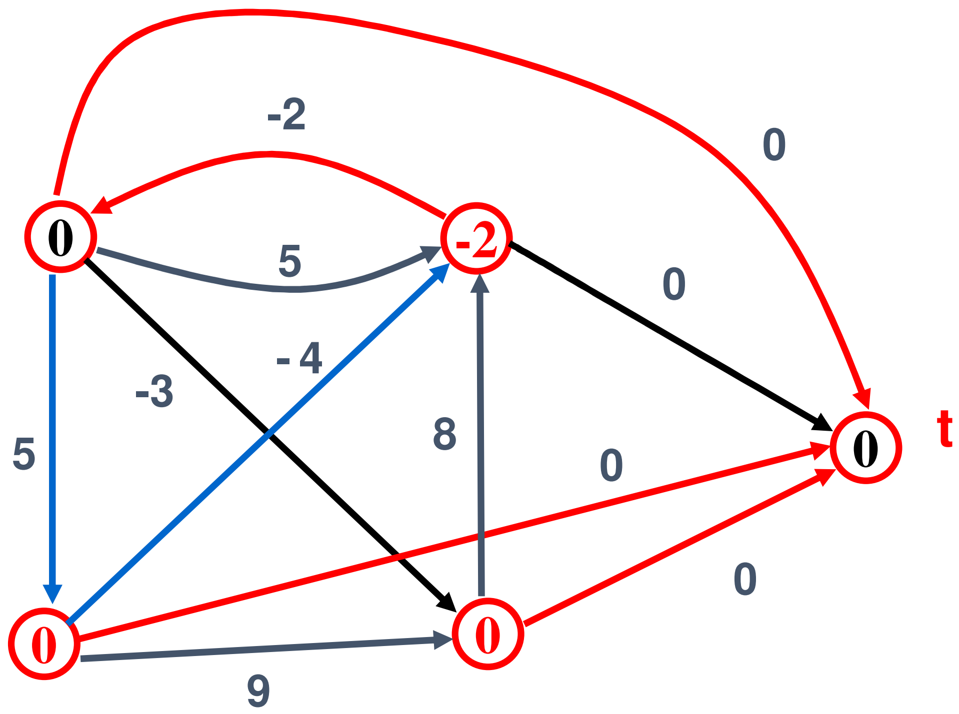

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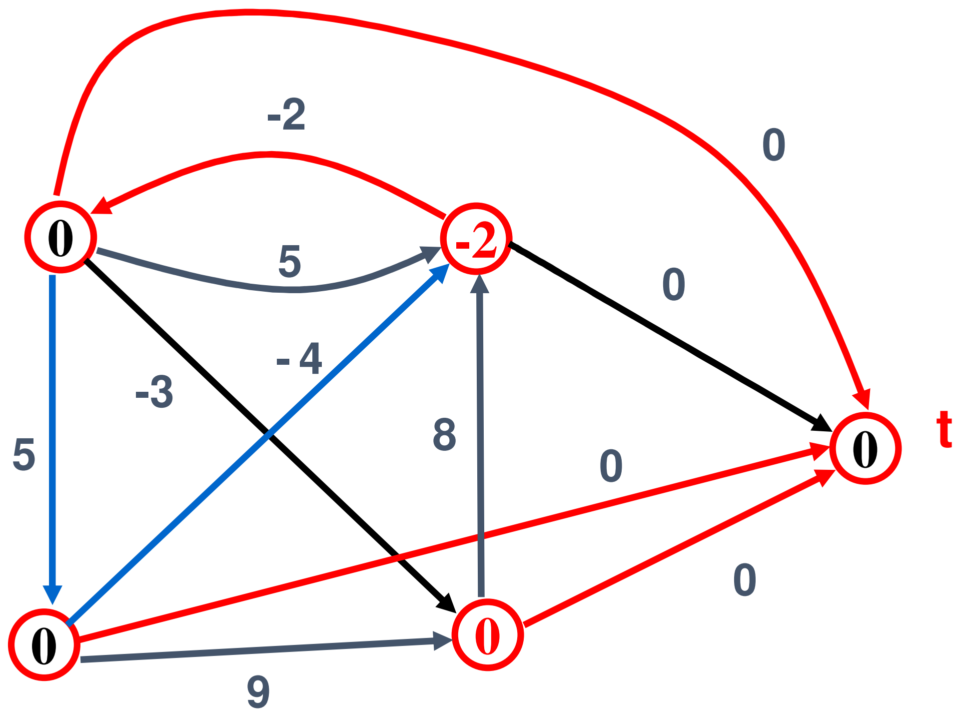

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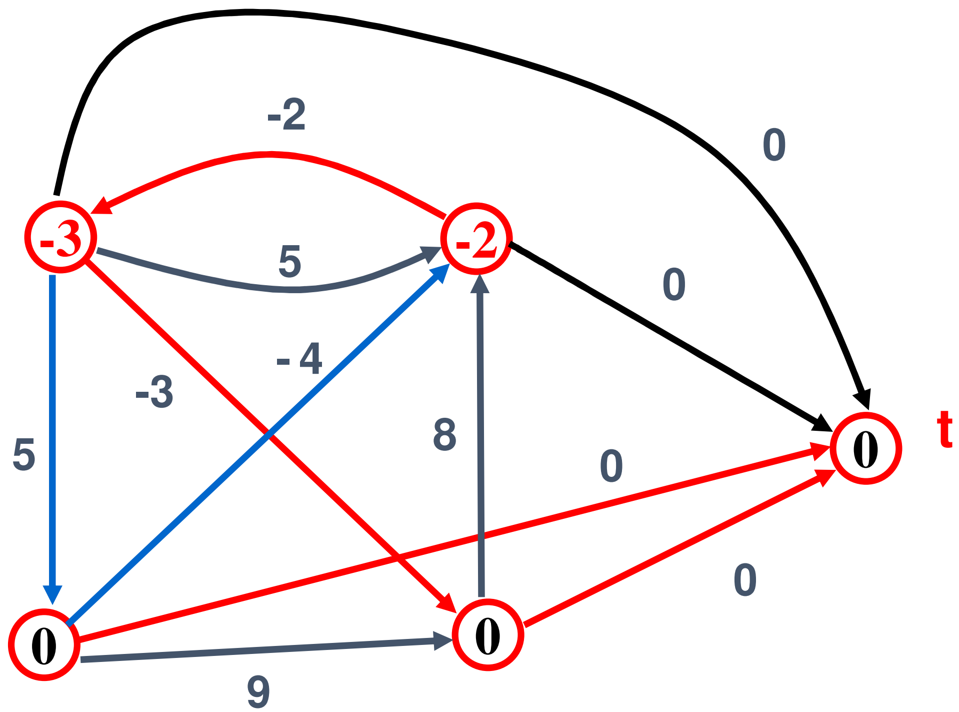

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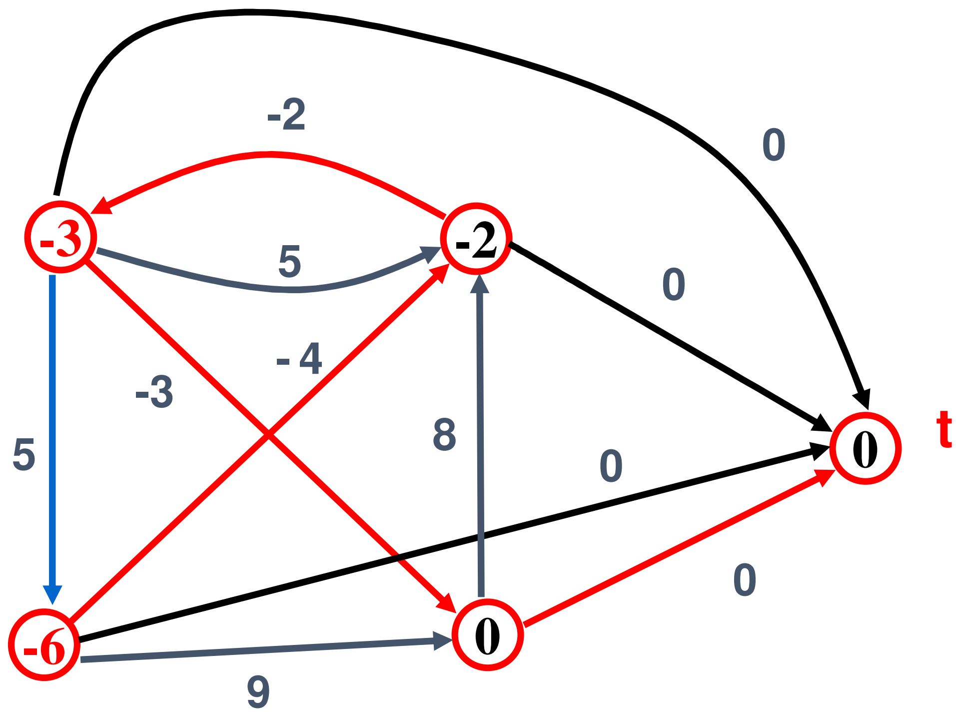

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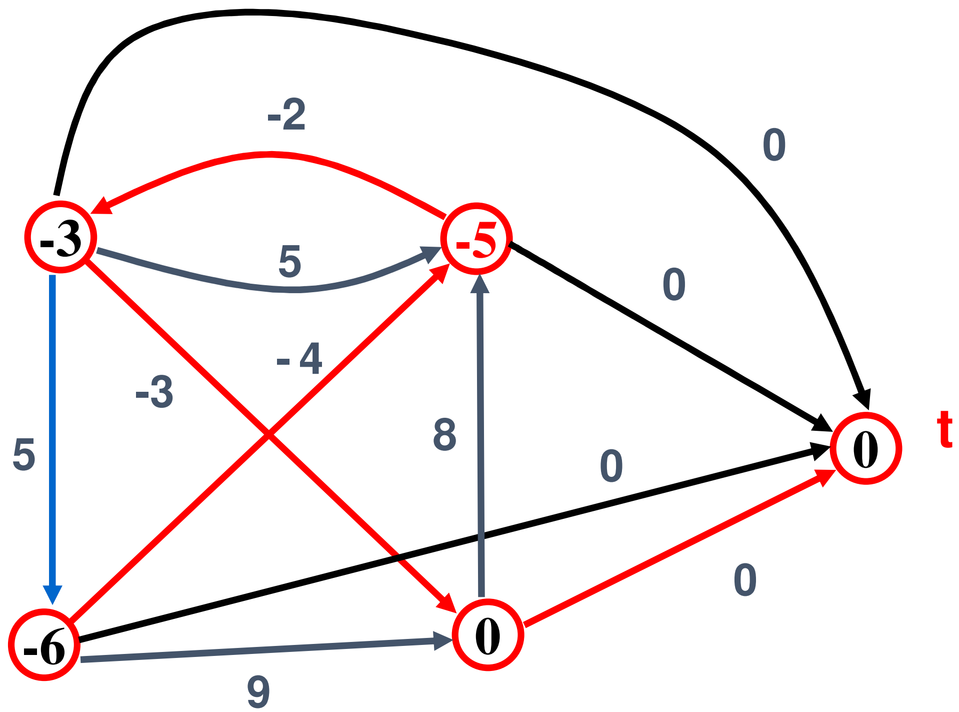

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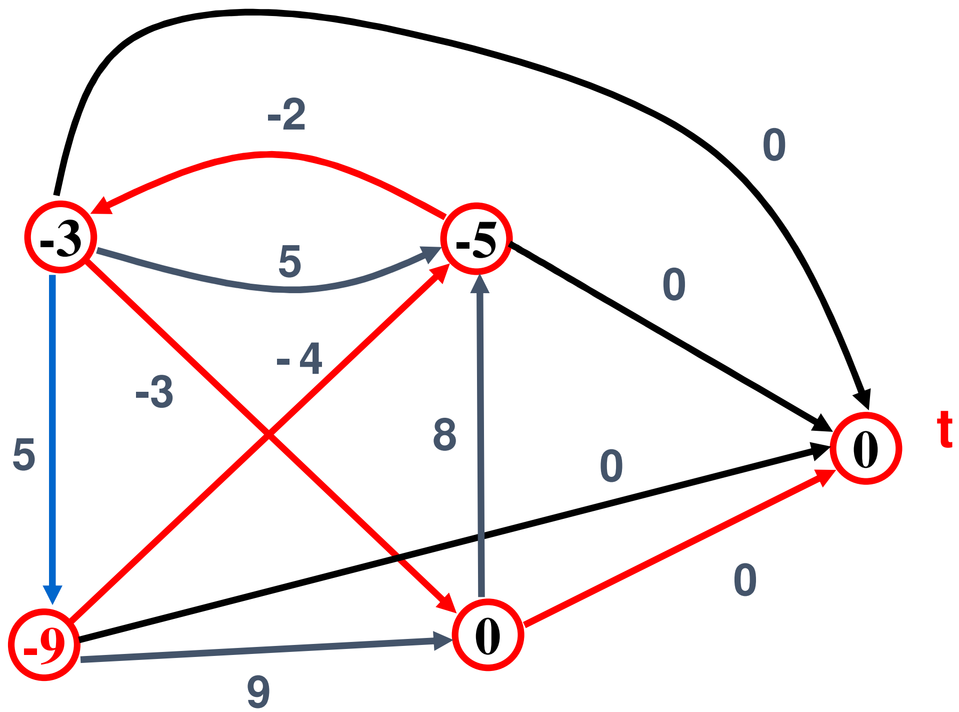

Bellman-Ford for Negative Cycles