

CSE 421

Introduction to Algorithms

Lecture 11: Dynamic Programming

Algorithmic Paradigms

Greedy: Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer: Break up a problem into sub-problems (each typically a constant factor smaller), solve each sub-problem *independently*, and combine solution to sub-problems to form solution to original problem.

Dynamic programming: Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

Algorithm Design Techniques

Dynamic Programming:

- Technique for making building solution to a problem based on solutions to smaller subproblems (recursive ideas).
- The subproblems just have to be smaller, but don't need to be a constant-factor smaller like divide and conquer.
- Useful when *the same subproblems show up over and over again*
- The final solution is simple iterative code when the following holds:
 - *The parameters to all the subproblems are predictable in advance*

Dynamic Programming History

Bellman. [1950s] Pioneered the systematic study of dynamic programming.

Etymology

- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.

"it's impossible to use dynamic in a pejorative sense"
"something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography*.

Dynamic Programming Applications

Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, compilers, systems,

Some famous dynamic programming algorithms.

- Unix `diff` for comparing two files.
- Viterbi for hidden Markov models.
- Smith-Waterman for genetic sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

Three Steps to Dynamic Programming

1. Formulate the answer as a recurrence relation or recursive algorithm
2. Figure out the possible values of parameters in the recursive calls.
 - This should be “small”, i.e., bounded by a low-degree polynomial
 - Can use memoization to store a cache of previously computing values
3. Specify an order of evaluation for the recurrence so that you already have the partial results stored in memory when you need them.

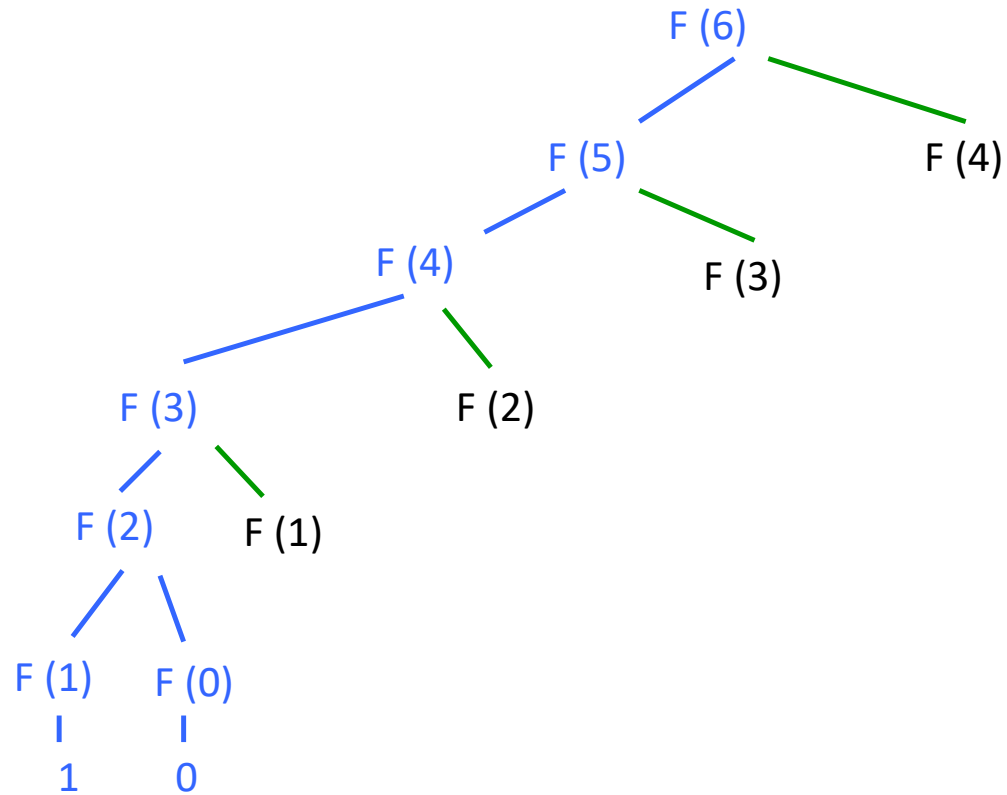
A Simple Case: Computing Fibonacci Numbers

Recall $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0, F_1 = 1$

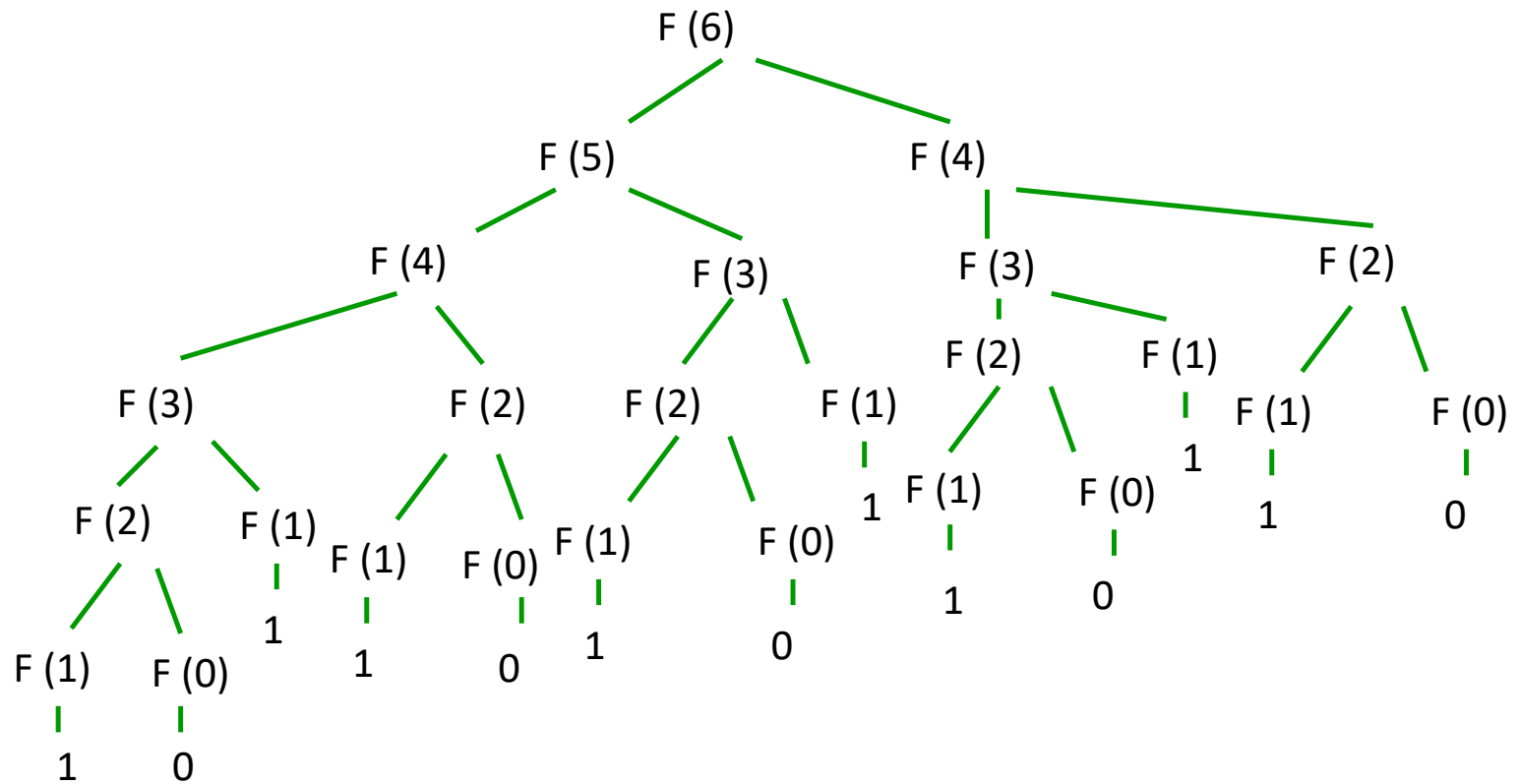
The obvious recursive algorithm direct from this recurrence is

```
F(n) {  
    if n=0 return(0)  
    else if n=1 return(1)  
    else return(F(n-1)+F(n-2))  
}
```

Let's start tracking the call tree...



The full call tree has $> F_n$ leaves (exponential in n)



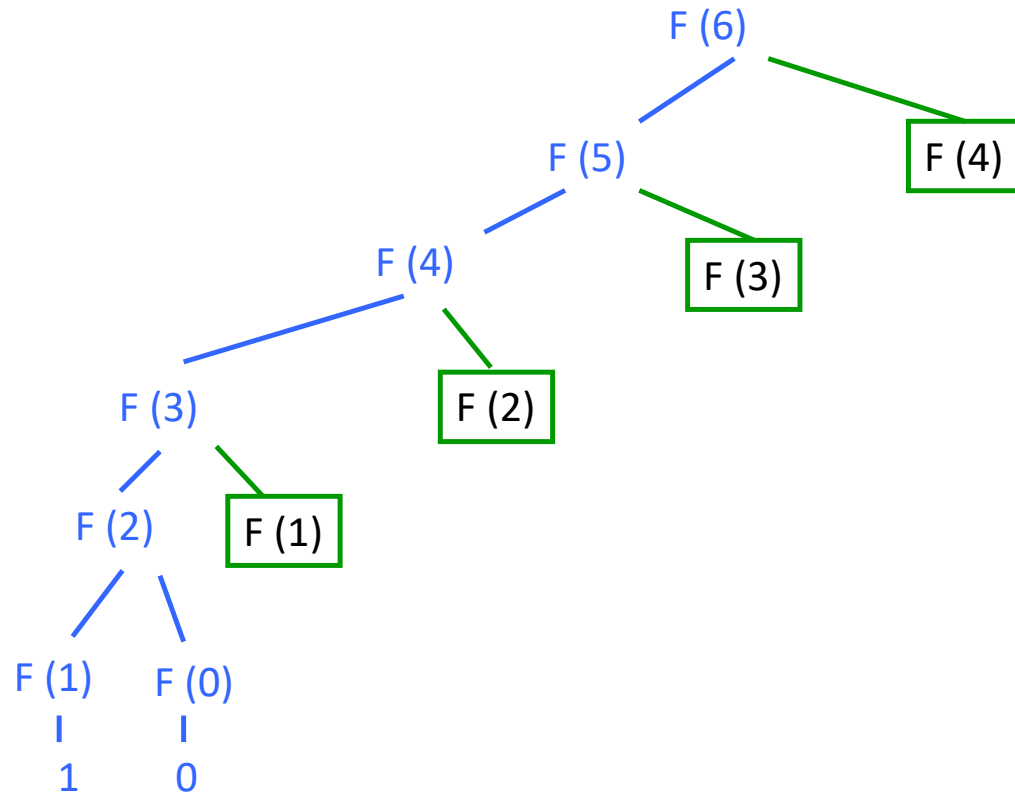
Memoization (Caching)

Remember all values from previous recursive calls in a cache

- the parameters and
- The values returned on those parameters

Before each recursive call, test to see if value has already been computed for those parameters

Memoization



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Fibonacci: Dynamic Programming Version

```
FiboDP (n) :  
    F[0] ← 0  
    F[1] ← 1  
    for i ← 2 to n {  
        F[i] ← F[i-1] + F[i-2]  
    }  
    return (F[n])
```

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Once you have an iterative DP solution: see if you can save space...

Fibonacci: Space-Saving Dynamic Programming

```
FiboDP (n) :  
    prev←0  
    curr←1  
    for i←2 to n {  
        temp←curr  
        curr←curr+prev  
        prev←temp  
    }  
    return (curr)
```


Dynamic Programming

When is dynamic programming useful?

- For optimization problems this typically requires that the “Principle of optimality” hold for the problem
 - “Optimal solutions to the sub-problems suffice for optimal solution to the whole problem”

Weighted Interval Scheduling

Input: Like interval scheduling each request i has start and finish times s_i and f_i . Each request i also has an associated **value** or **weight** v_i

v_i might be

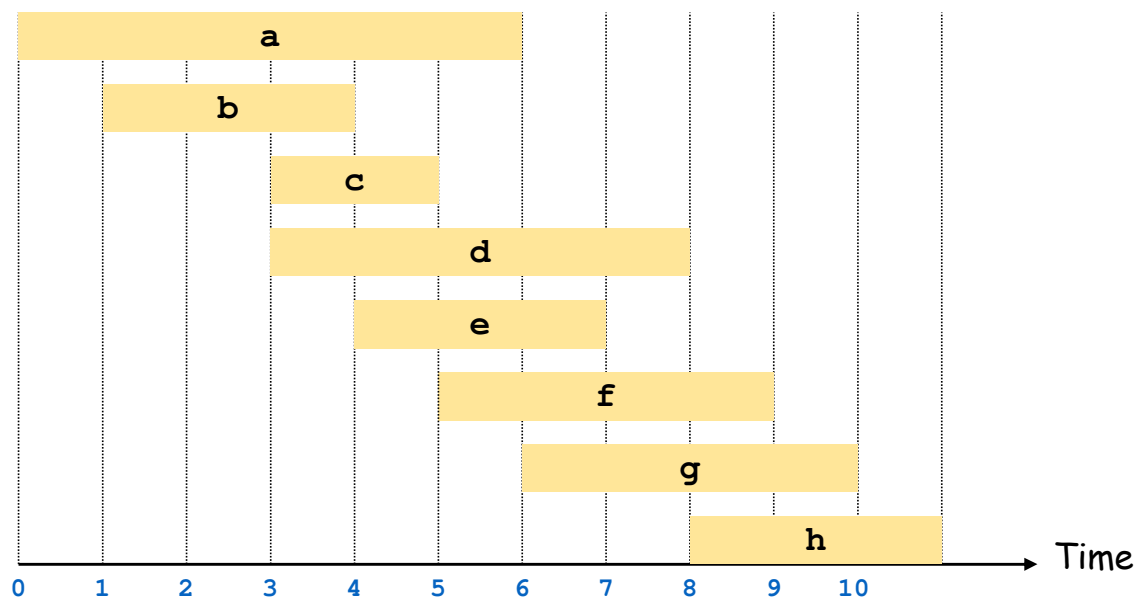
- the amount of money we get from renting out the resource
- the amount of time the resource is being used ($v_i = f_i - s_i$)

Find: A maximum-weight compatible subset of requests.

Weighted Interval Scheduling

Input: Set of jobs with start times, finish times, and **weights**

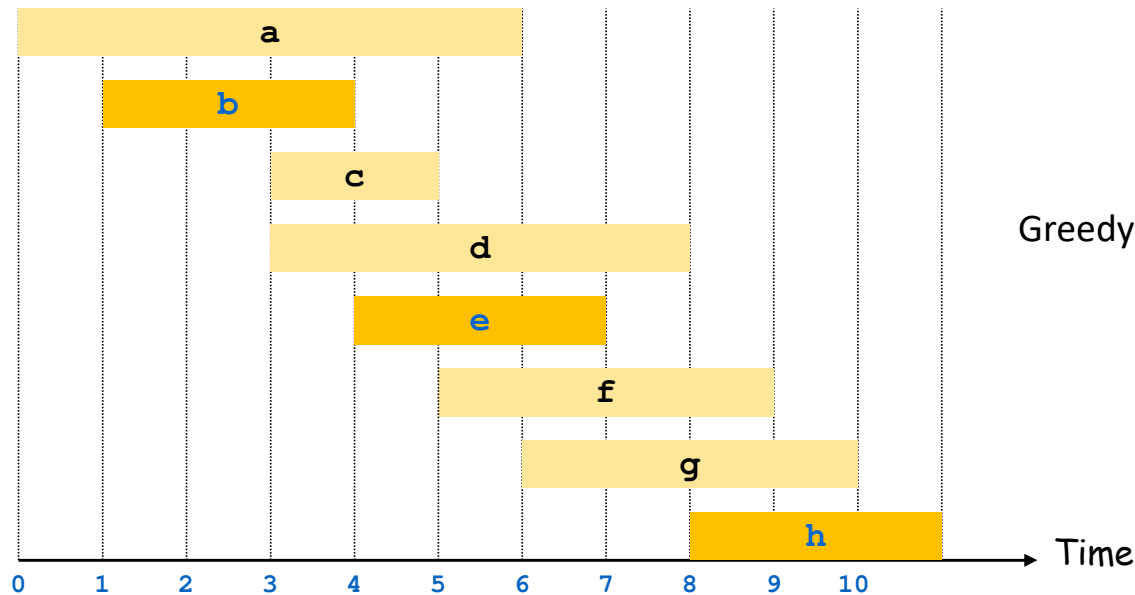
Goal: Find **maximum weight** subset of mutually compatible jobs.



Weighted Interval Scheduling

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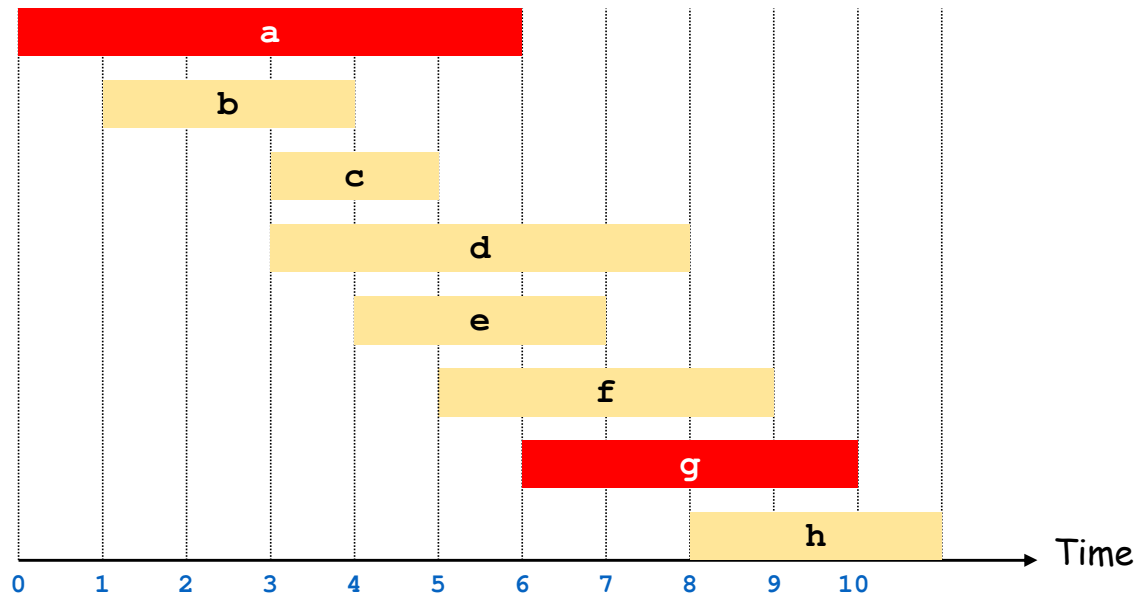


Greedy by finish times would give 9

Weighted Interval Scheduling

Input: Set of jobs with start times, finish times, and **weights**

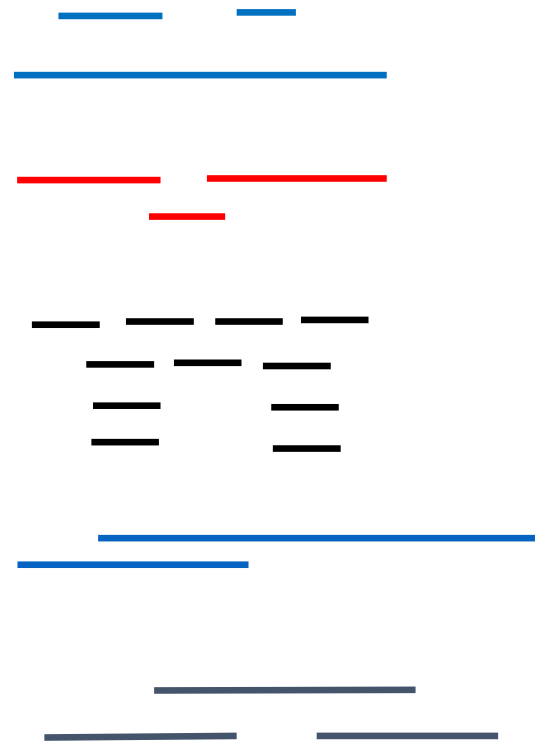
Goal: Find **maximum weight** subset of mutually compatible jobs.



Optimal yields 10

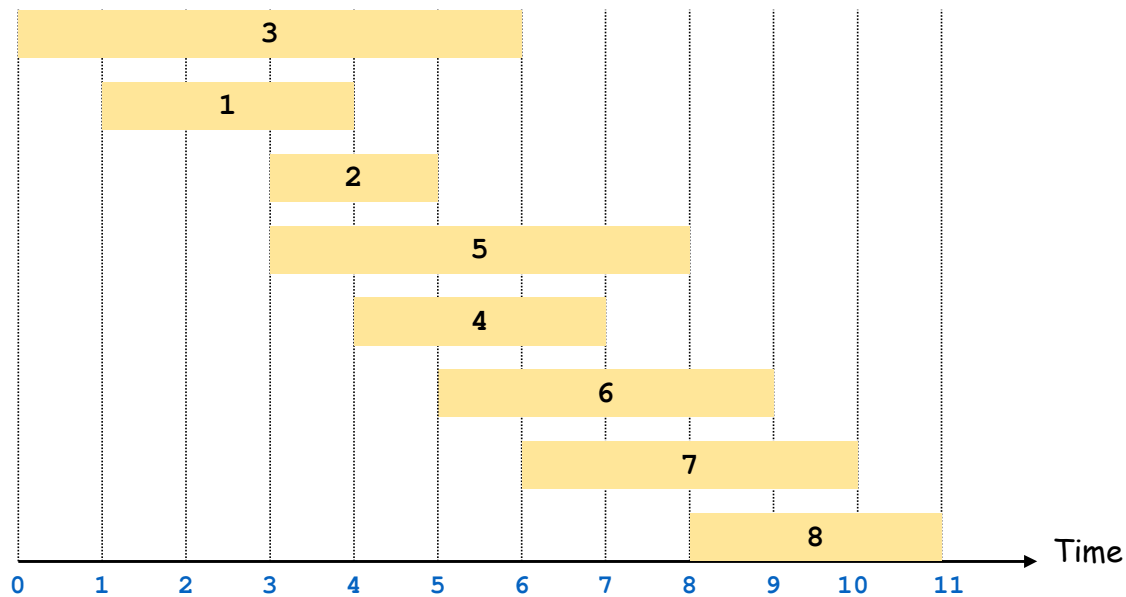
Greedy Algorithms for Weighted Interval Scheduling?

- What criterion should we try?
 - Earliest start time s_i
 - Doesn't work
 - Shortest request time $f_i - s_i$
 - Doesn't work
 - Fewest conflicts
 - Doesn't work
 - Earliest finish time f_i
 - Doesn't work
 - Largest value/weight v_i
 - Doesn't work



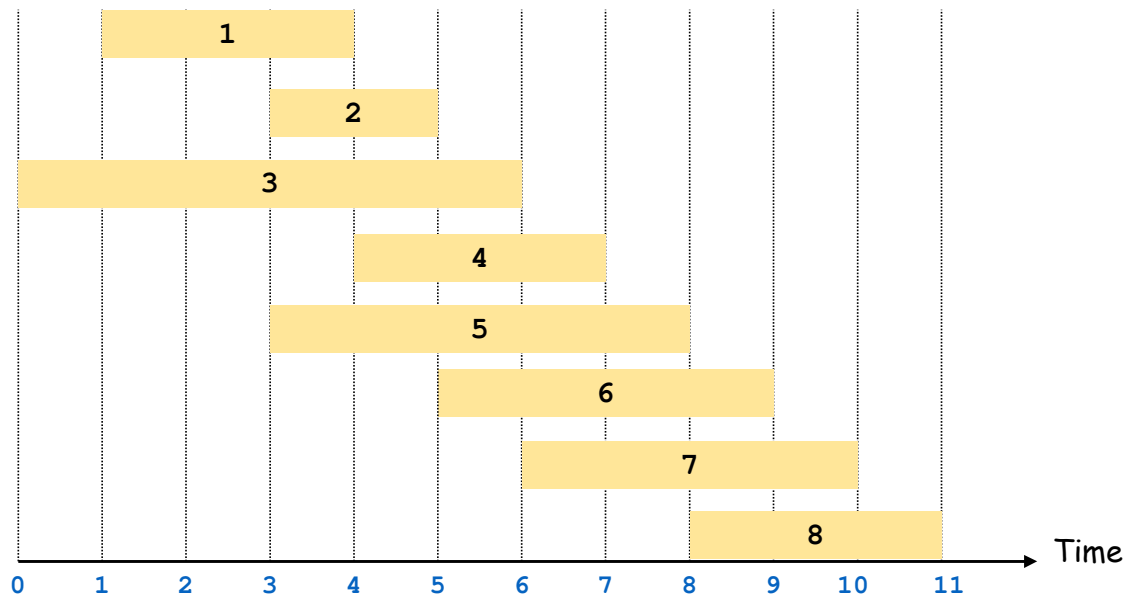
Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.



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Towards Dynamic Programming: Step 1 – Recursive Algorithm

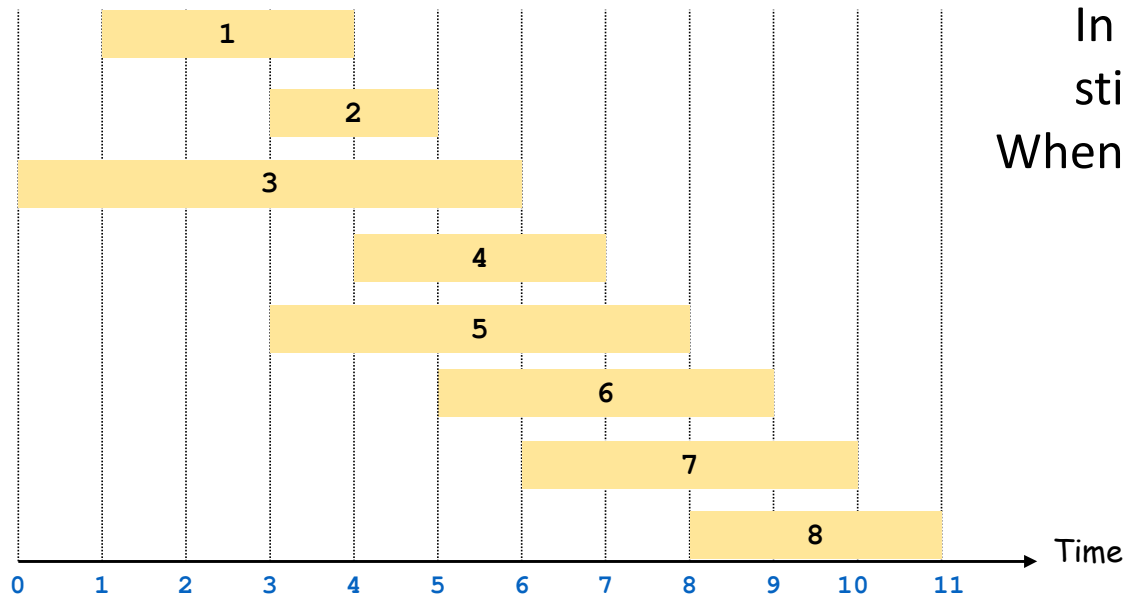
Suppose that we first sort the requests by finish time f_i so $f_1 \leq f_2 \leq \dots \leq f_n$.

We now want

- a recursive solution that makes calls to smaller problems and
- the indices for those smaller problems to be convenient,
so we first focus on the options for the *last* request, request n .

Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.



There are two cases we need to compare:

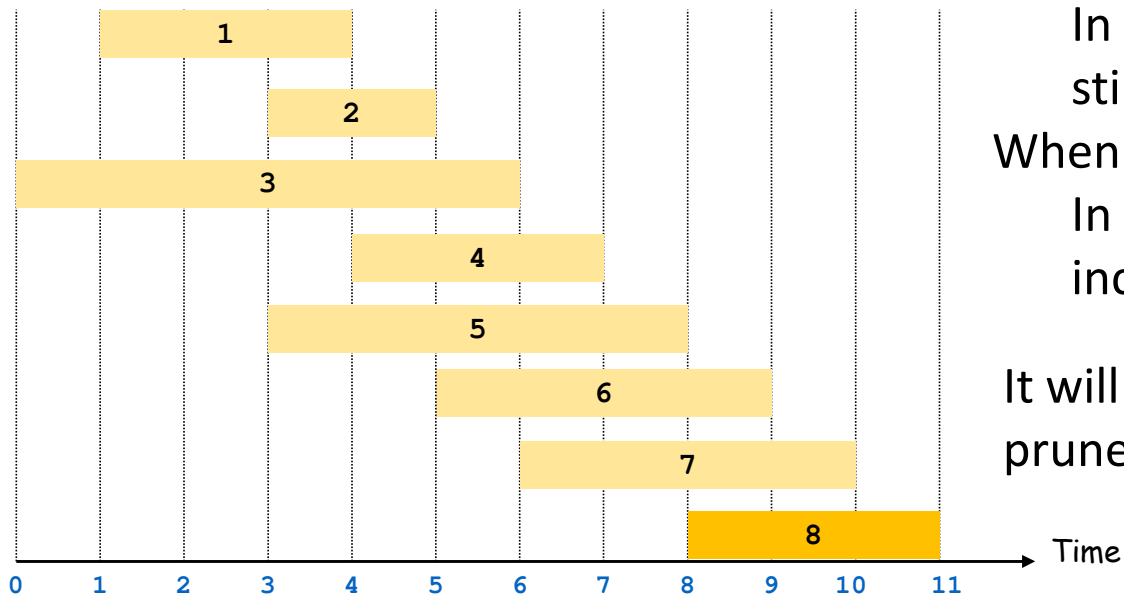
When we don't include request n .

In this case all the other requests are still fair game

When we do include request n .

Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.



There are two cases we need to compare:

When we don't include request n .

In this case all the other requests are still fair game

When we do include request n .

In this case we need to rule out some incompatible requests.

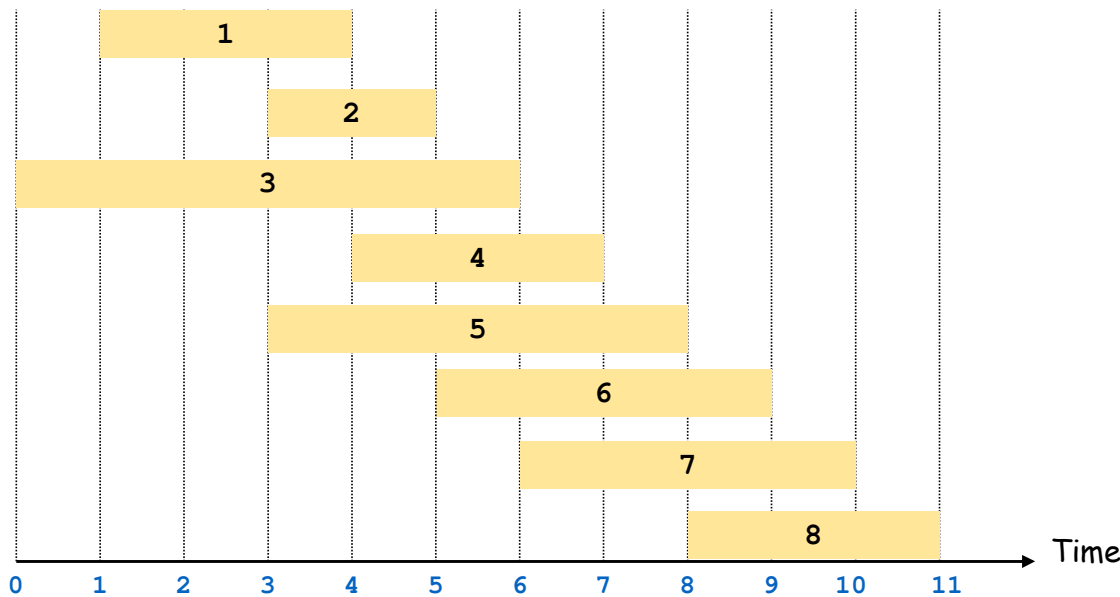
It will be convenient to be able to prune incompatible requests quickly...

Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j .

Example: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$



j	$p(j)$
1	0
2	0
3	0
4	1
5	0
6	2
7	3
8	5

$O(n \log n)$

Computing all the $p(j)$ values:

For $j \leftarrow 1$ to n

Run binary search for s_j in the list $f_1 \leq f_2 \leq \dots \leq f_n$ to find largest i with $f_i \leq s_j$

$p(j) \leftarrow i$

Structure of the subproblems

Notation: $\text{OPT}(j)$ = value of optimal solution to the problem consisting of job requests $1, 2, \dots, j$.

Case 1: OPT selects job j

- It can't use incompatible jobs $p(j) + 1, \dots, j - 1$
- It must include an optimal solution to problem consisting of remaining compatible jobs $1, \dots, p(j)$.

Case 2: OPT doesn't select job

- It must include an optimal solution to problem consisting of remaining compatible jobs $1, \dots, j - 1$

Optimal substructure



$$\text{OPT}(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\{v_j + \text{OPT}(p(j)), \text{OPT}(j - 1)\} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Recursive Solution

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

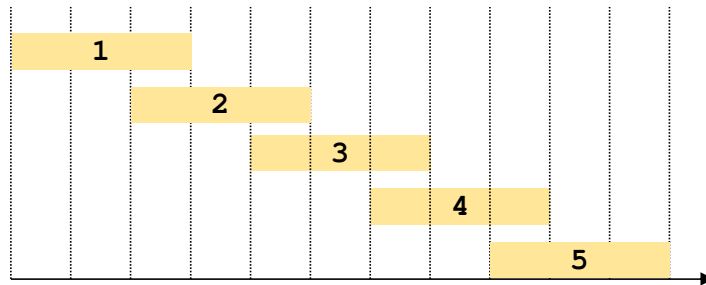
Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

```
Compute-Opt(j) {  
    if (j = 0)  
        return 0  
    else  
        return max( $v_j + \text{Compute-Opt}(p(j))$ ,  $\text{Compute-Opt}(j-1)$ )  
}
```

Weighted Interval Scheduling: Recursive Solution

This recursive algorithm can be very bad...



Suppose that $p(j) = j - 2$ for every $j \geq 2$.

- Then **Compute-Opt**(j) calls **Compute-Opt**($j - 1$) and **Compute-Opt**($j - 2$)
- This is the same exponential run-time as the recursive Fibonacci code!

Weighted Interval Scheduling: Step 2 Memoization


Memoization: Store results of each sub-problem in a cache; lookup as needed.

```
Input:  $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$ 
```

```
Sort jobs by finish times so that  $f_1 \leq f_2 \leq \dots \leq f_n$ .
```

```
Compute  $p(1), p(2), \dots, p(n)$ 
```

```
for  $j = 1$  to  $n$ 
```

```
     $M[j] = \text{empty}$   global array
```

```
 $M[0] = 0$ 
```

```
M-Compute-Opt( $j$ ) {
```

```
    if ( $M[j]$  is empty)
```

```
         $M[j] = \max(v_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j-1))$ 
```

```
    return  $M[j]$ 
```

```
}
```


Weighted Interval Scheduling: Step 3

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 - Produces iterative code

Recursion for $\text{OPT}[j]$ only needs values of $\text{OPT}[i]$ for $0 \leq i < j$.

- So we can evaluate them in order $j = 0, 1, 2, \dots, n$

Weighted Interval Scheduling: Iterative Solution

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

```
Iterative-Compute-Opt {  
    OPT[0] = 0  
    for j = 1 to n  
        OPT[j] = max(vj + OPT[p(j)], OPT[j-1])  
}
```

$O(n \log n)$

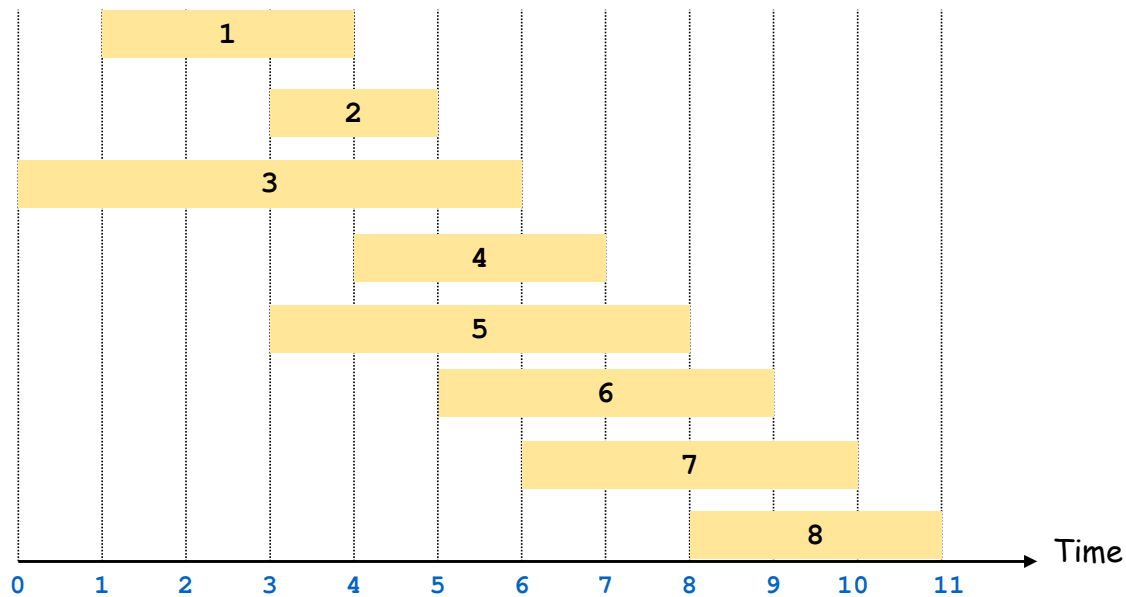
$O(n)$

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

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Example: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$



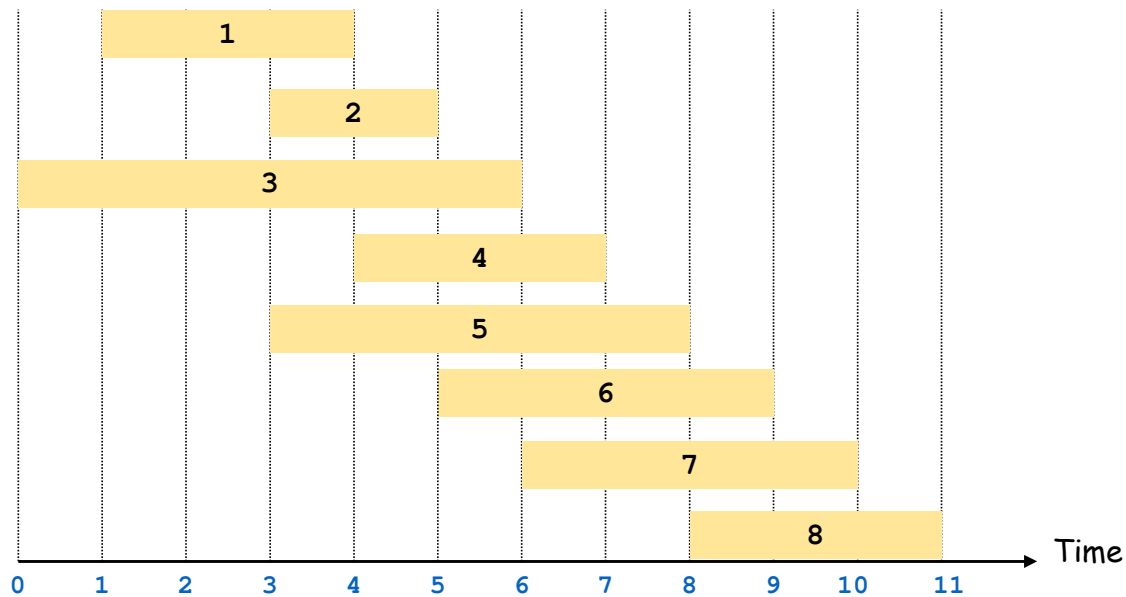
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	
2	2	0	
3	6	0	
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

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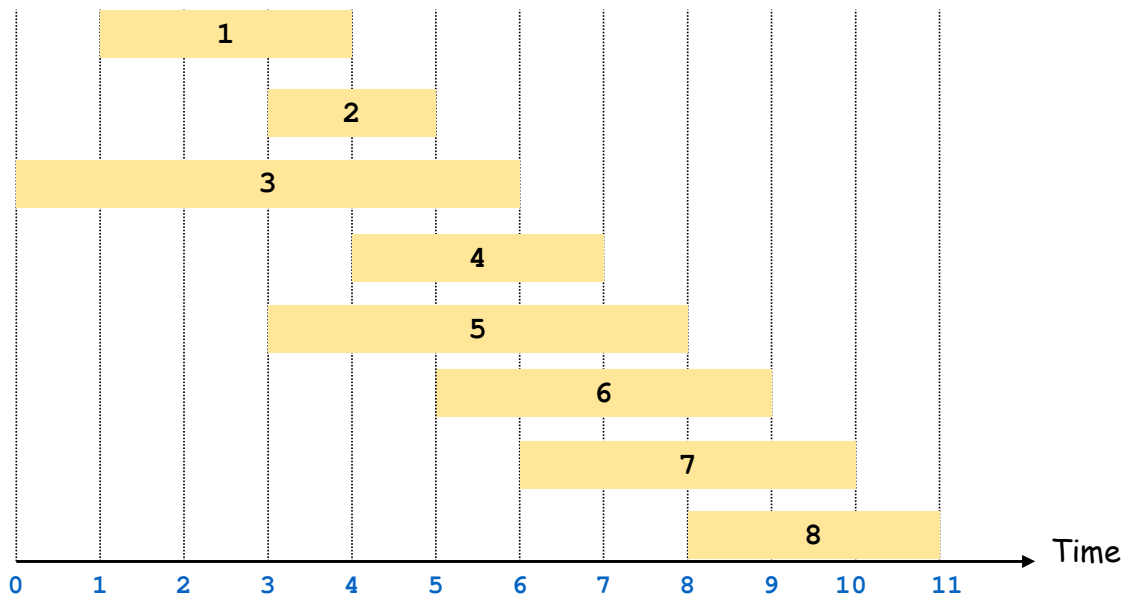
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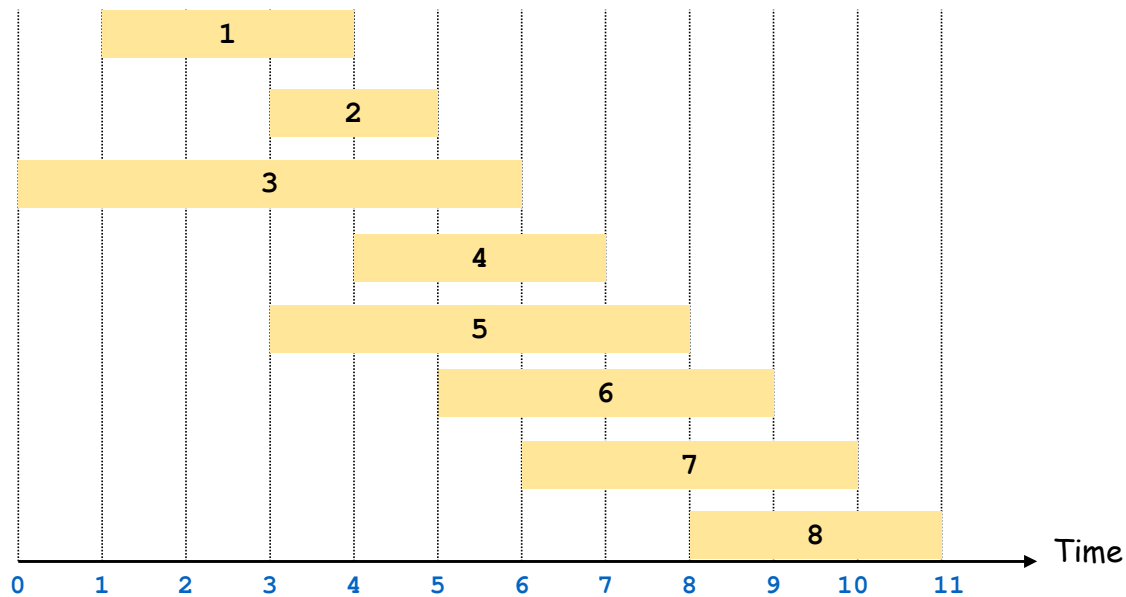
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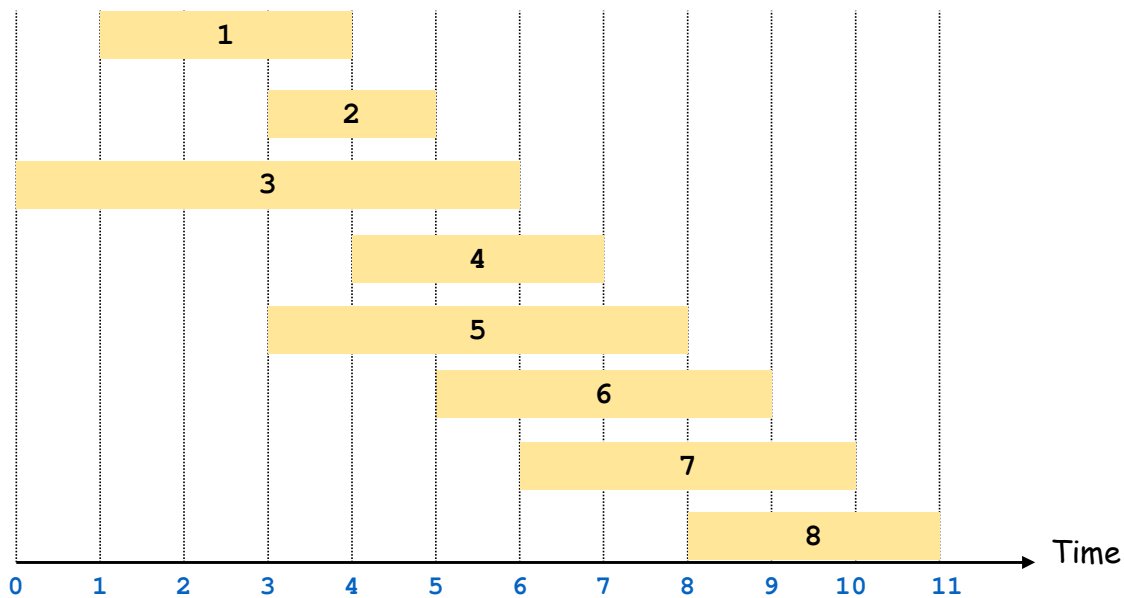
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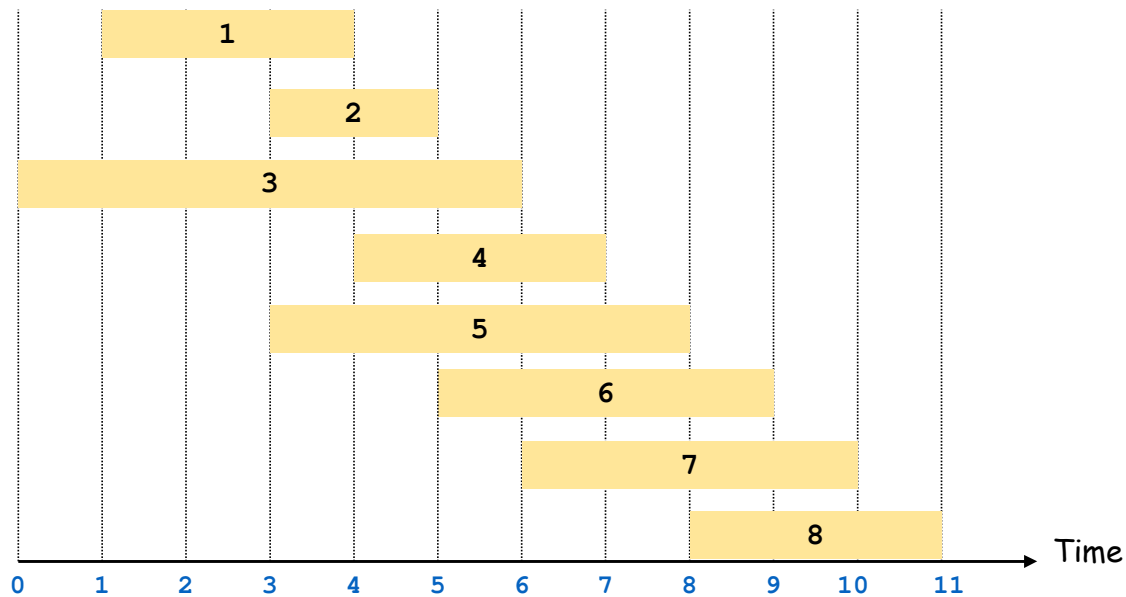
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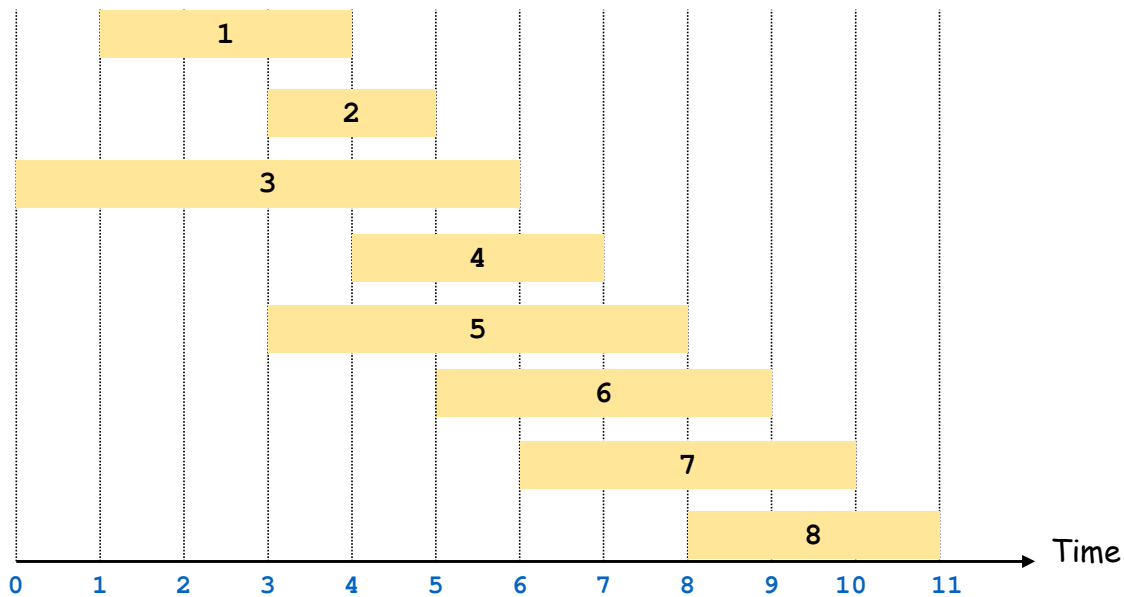
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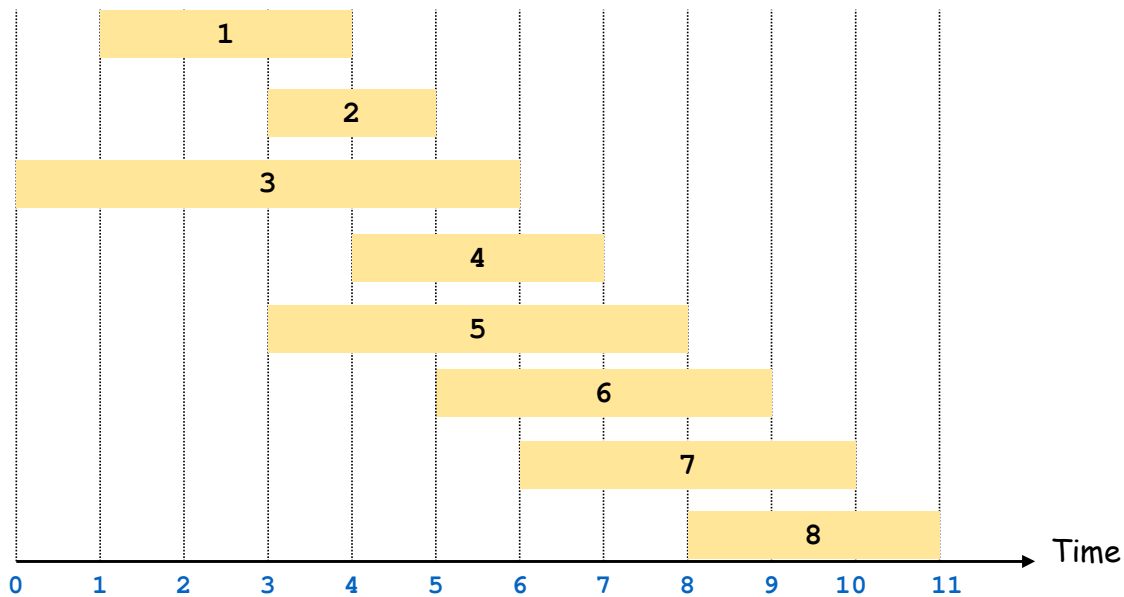
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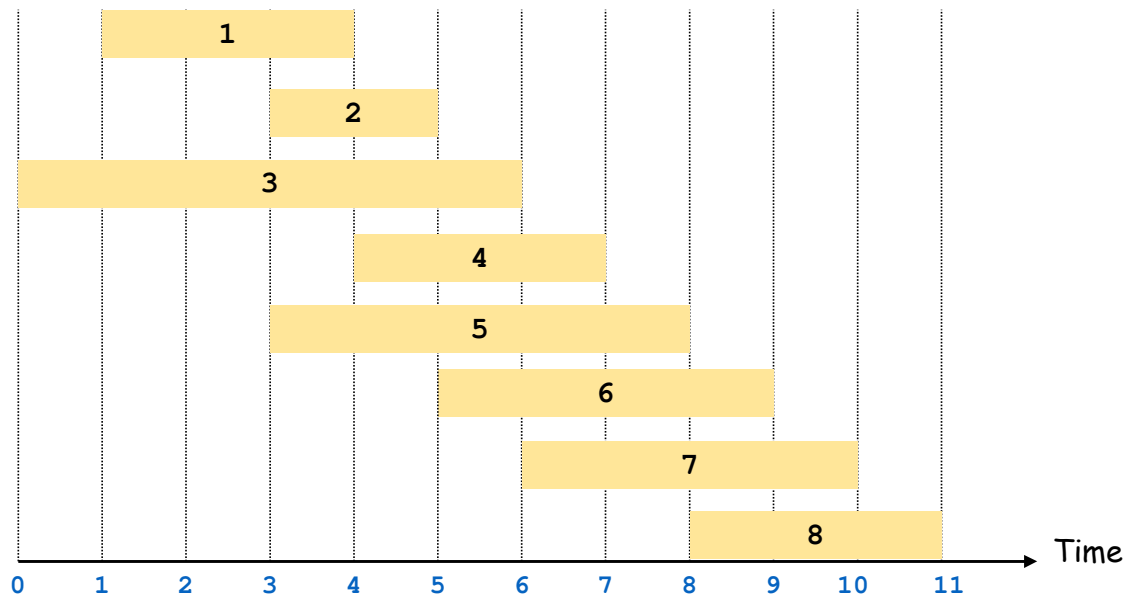
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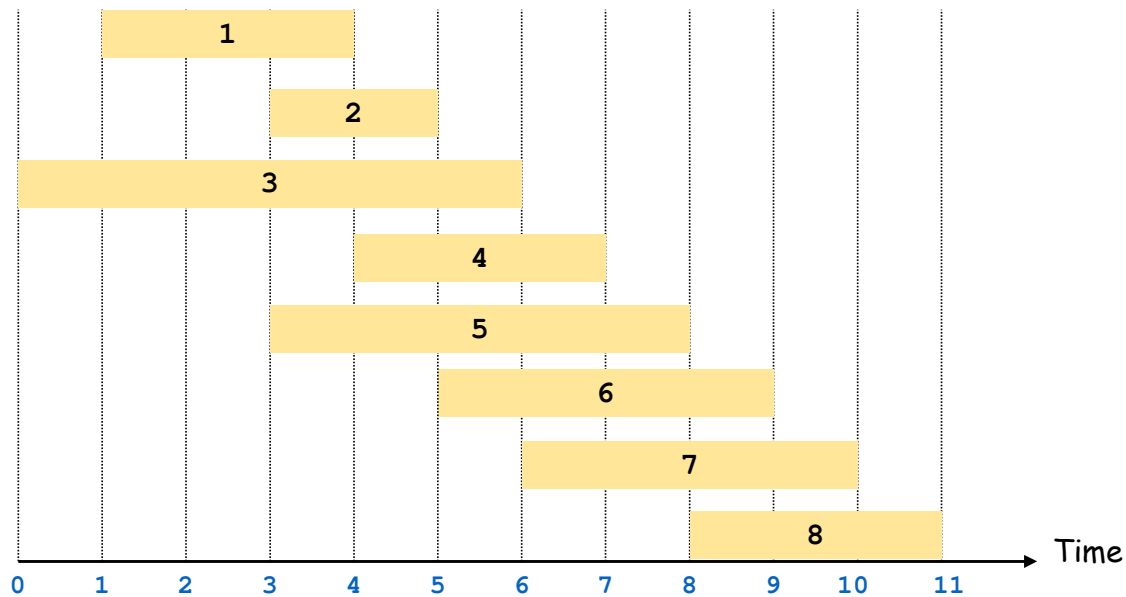
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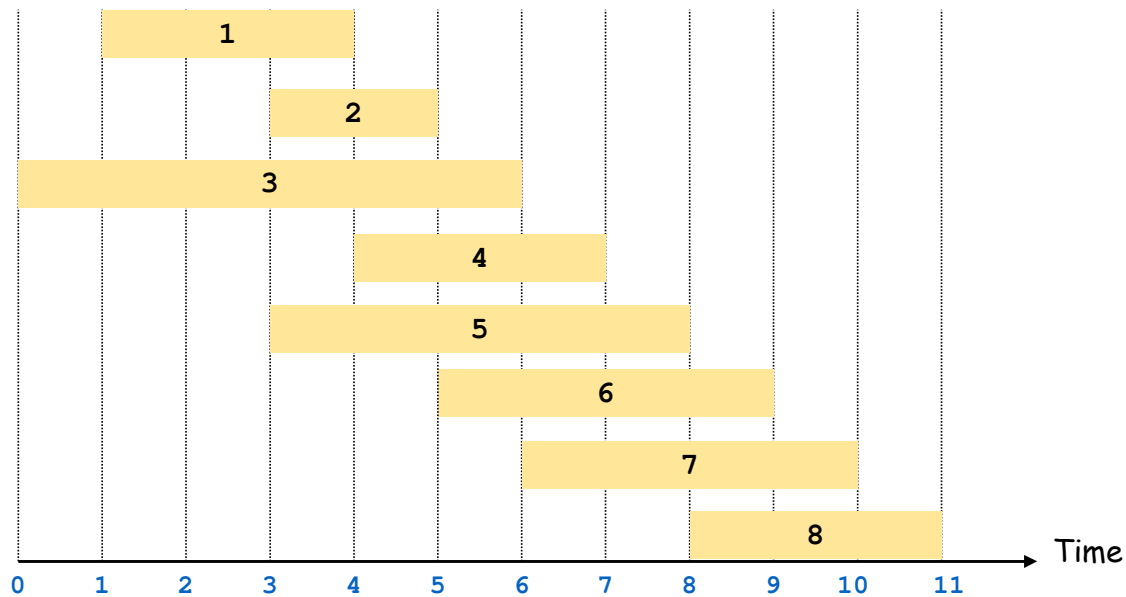
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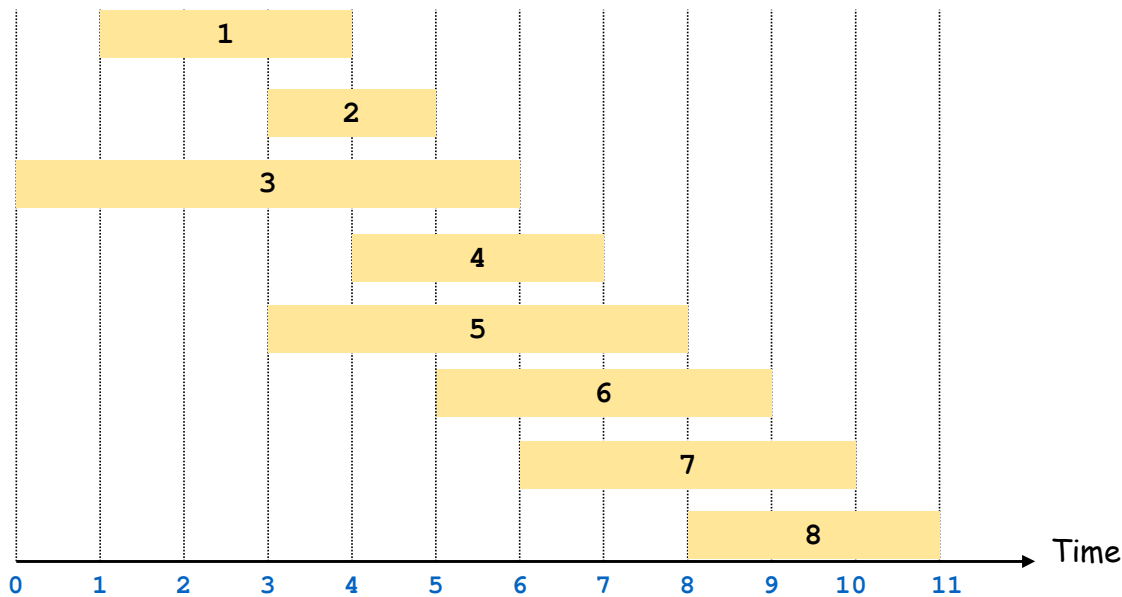
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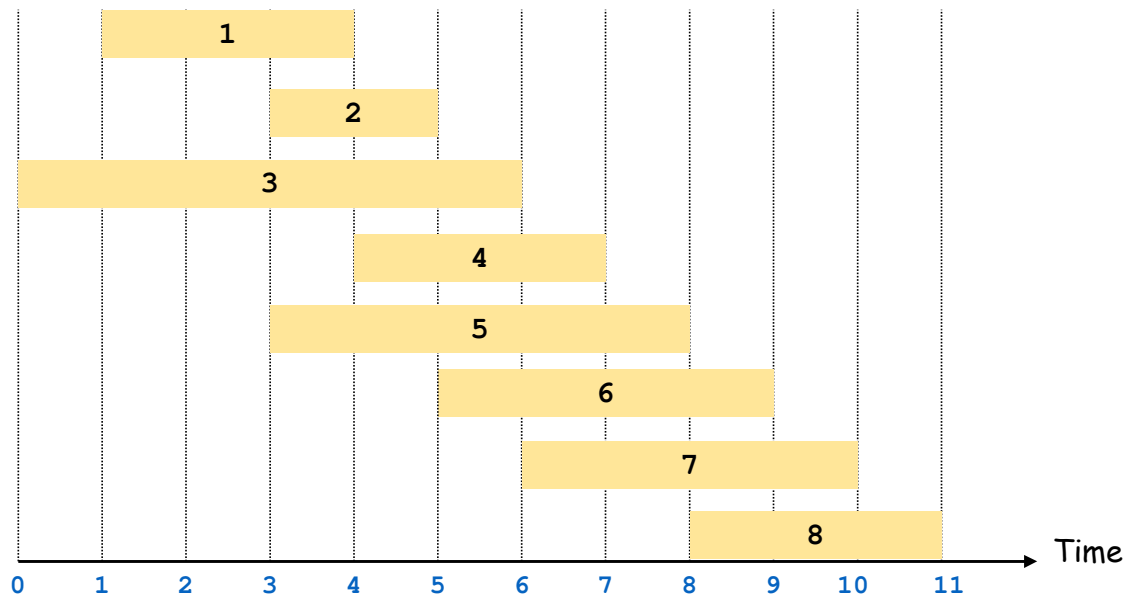
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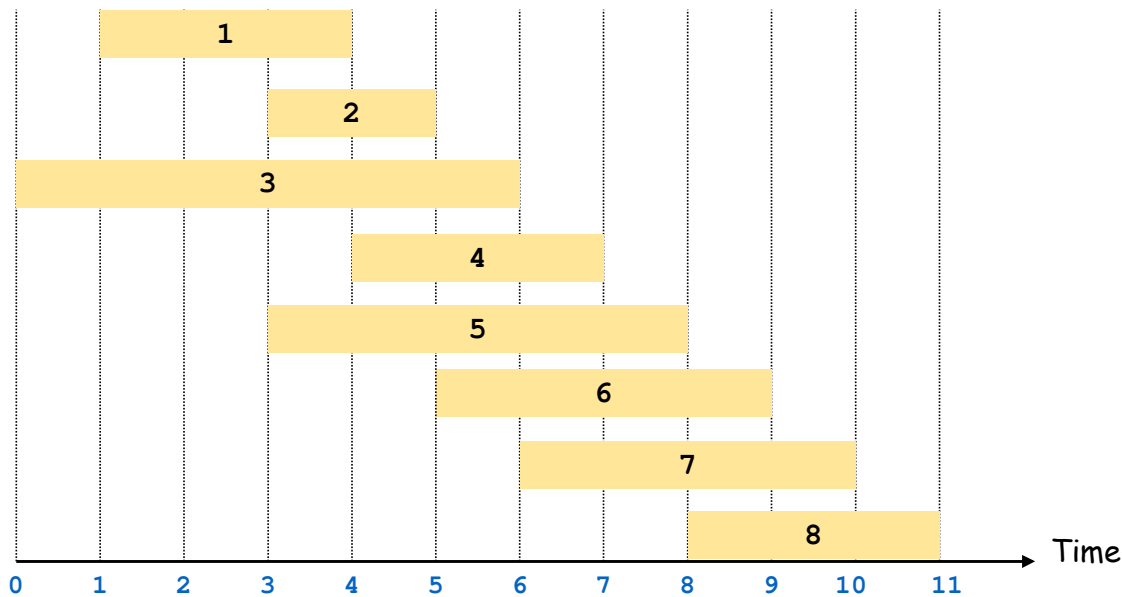
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0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	10
8	3	5	

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j .

Example: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$



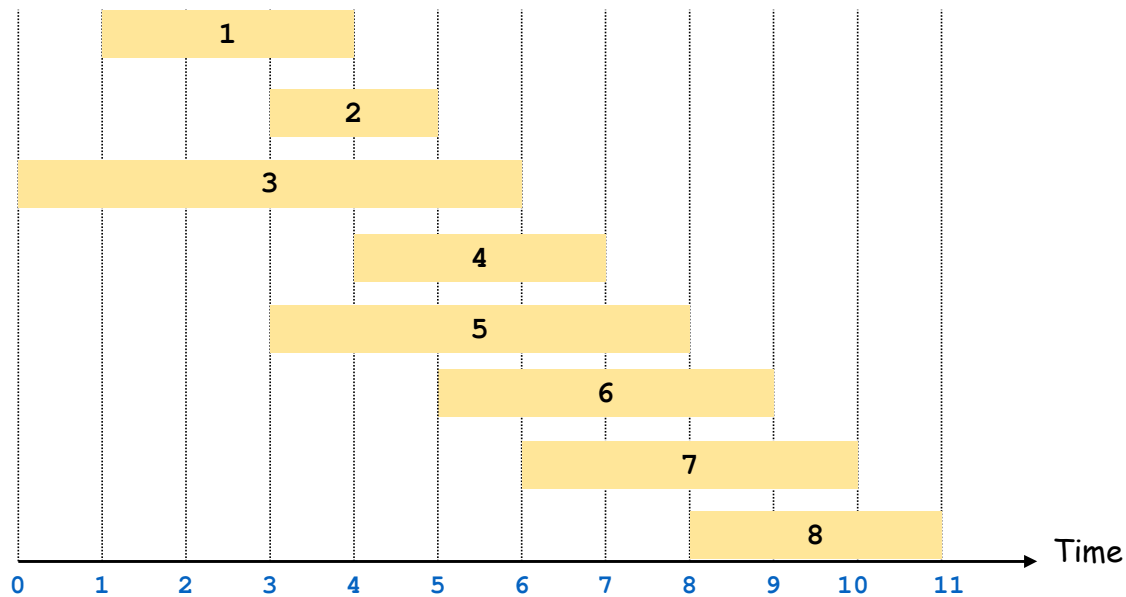
j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
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1	3	0	3
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3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	10
8	3	5	10

Weighted Interval Scheduling: Finding the Solution

So far we have computed the value $\text{OPT}(n)$ but we probably want to know what that solution OPT actually is!

We can do this, too, by keeping track of which option was better at each step.

Define $\text{Used}[j] = \begin{cases} 1 & \text{solution with value } \text{OPT}(j) \text{ includes request } j \\ 0 & \text{otherwise} \end{cases}$

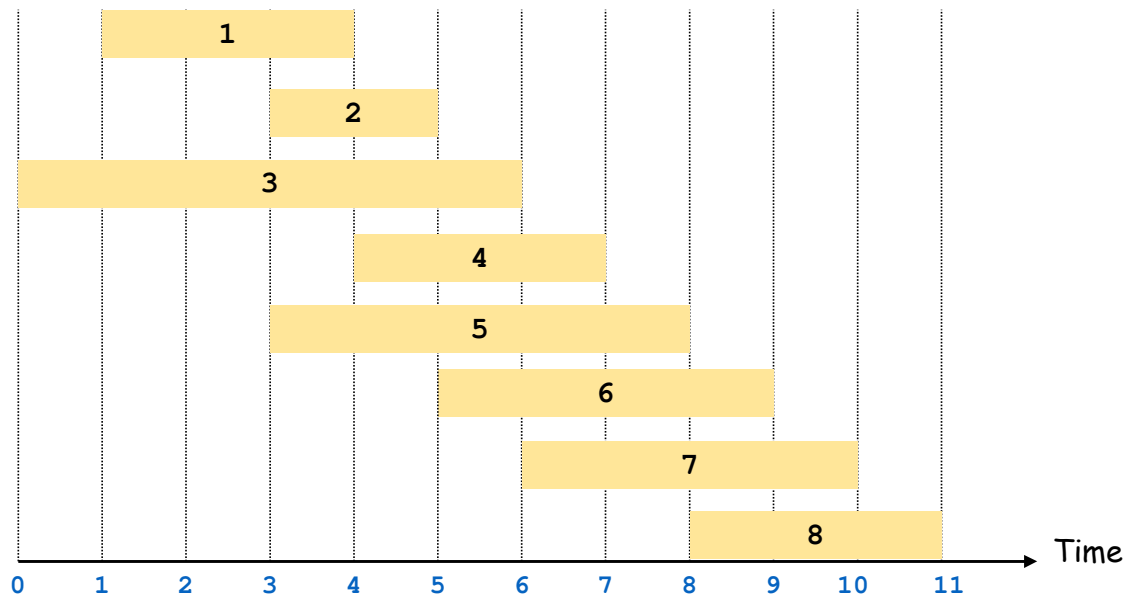
This gives a “pointer” that leads the way along a path to the optimal solution...

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j .

Example: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$



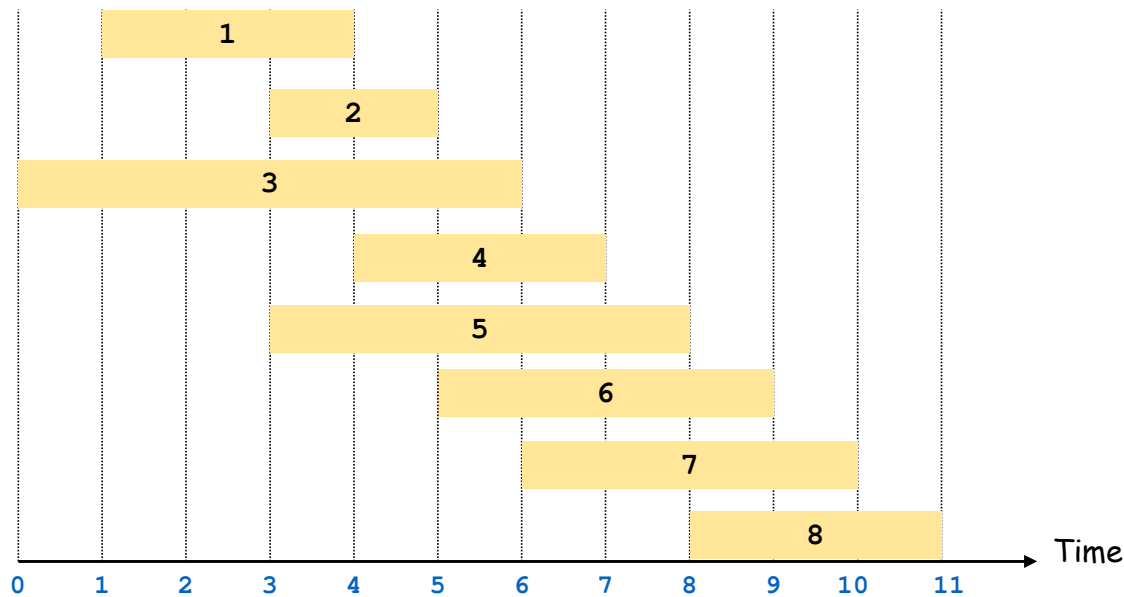
j	v_j	$p(j)$	$OPT[j]$	$Used[j]$
0	-	-	0	-
1	3	0	3	1
2	2	0	3	0
3	6	0	6	1
4	3	1	6	1
5	5	0	6	0
6	4	2	7	1
7	4	3	10	1
8	3	5	10	0

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j .

Example: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$



j	v_j	$p(j)$	$OPT[j]$	Used[j]
0	-	-	0	-
1	3	0	3	1
2	2	0	3	0
3	6	0	6	1
4	3	1	6	1
5	5	0	6	0
6	4	2	7	1
7	4	3	10	1
8	3	5	10	0

Weighted Interval Scheduling: Finding the Solution

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

```
Iterative-Compute-Opt {
  OPT[0] = 0
  for j = 1 to n
    if  $v_j + \text{OPT}[p(j)] > \text{OPT}[j-1]$  {
      OPT[j] =  $v_j + \text{OPT}[p(j)]$ 
      Used[j] = 1
    } else {
      OPT[j] = OPT[j-1]
      Used[j] = 0
    }
  }
```

```
Find-Opt {
  j = n
  OPTSol =  $\emptyset$ 
  while j > 0
    if Used[j] == 0 {
      j = j-1
    } else {
      OPTSol = OPTSol  $\cup$  {j}
      j = p(j)
    }
  }
```

Three Steps to Dynamic Programming

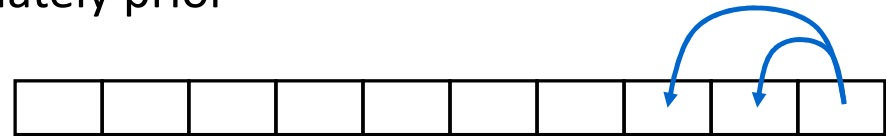
1. Formulate the answer as a recurrence relation or recursive algorithm
2. Figure out the possible values of parameters in the recursive calls.
 - This should be “small”, i.e., bounded by a low-degree polynomial
 - Can use memoization to store a cache of previously computing values
3. Specify an order of evaluation for the recurrence so that you already have the partial results stored in memory when you need them.
 - Produces iterative code

Once you have an iterative DP solution: see if you can save space...

Dynamic Programming Patterns

Fibonacci pattern:

- 1-dimensional, $O(1)$ values immediately prior
- Space saving possible



Weighted interval scheduling pattern:

- 1-dimensional, $O(1)$ values arbitrarily far back
- No space saving possible

