CSE 421
Introduction to Algorithms

Lecture 10: Divide and Conquer
Median, Quicksort

Today

Divide and conquer examples

- Simple, randomized median algorithm
  - Expected $O(\boldsymbol{n})$ time
- Surprising deterministic median algorithm
  - Worst case $O(\boldsymbol{n})$ time
- Expected time analysis for randomized QuickSort
  - Expected $O(\boldsymbol{n} \log \boldsymbol{n})$ time

Order problems: Find the k^{th} smallest

Runtime measures

- # of machine instructions
- # of comparisons

- 1^{st} Smallest = Minimum
  - $O(n)$ time
  - $n-1$ comparisons

- 2^{nd} Smallest
  - Still $O(n)$ time and comparisons...

Median and Selection

Median: k^{th} smallest for $k = n/2$

- Easily computed in $O(n \log n)$ time with sorting.

Q: How can Median be solved in $O(n)$ time?

A: Use divide and conquer ...

- But Median for a smaller set isn't a natural subproblem for Median.
- ***Idea***: Generalize Median so natural subproblems are of the same type.

Selection:

Given: A (multi-)set S of n numbers, and an integer k.

Find: The k^{th} smallest number in S.

Linear Time Divide and Conquer for Selection

General idea:

- Use a linear amount of work to reduce* Selection for a set of size $\boldsymbol{n}$ to Selection for a set that is a *constant factor smaller* than $\boldsymbol{n}$.

Recurrence

- $\boldsymbol{T(n) = T(n/b) + O(n)}$ for some $\boldsymbol{b > 1.}$

Apply the Master Theorem for $\boldsymbol{a = 1}$, $\boldsymbol{k = 1}$, and $\boldsymbol{b > 1}$

- Since $\boldsymbol{a^k = 1 < b}$ solution is $\boldsymbol{O(n)}$.

*The value of $\boldsymbol{k}$ will also change to some $\boldsymbol{k'}$ for the recursive call.

General Recursive Selection

Select(k, S)
  Choose element x from S "pivot"
  $S_L \leftarrow \{y \in S \mid y < x\}$
  $S_E \leftarrow \{y \in S \mid y = x\}$ $O(n)$ time to partition
  $S_G \leftarrow \{y \in S \mid y > x\}$
  if $|S_L| \geq k$
    return **Select**(k, S_L)
  else if $|S_L| + |S_E| \geq k$
    return x
  else
    return **Select**($k - |S_L| - |S_E|$, S_G)

Implementing: "Choose element x ..."

Select(k, S)
  Choose element x from S
  $S_L \leftarrow \{y \in S \mid y < x\}$
  $S_E \leftarrow \{y \in S \mid y = x\}$
  $S_G \leftarrow \{y \in S \mid y > x\}$
  if $|S_L| \geq k$
    return **Select**(k, S_L)
  else if $|S_L| + |S_E| \geq k$
    return x
  else
    return **Select**$(k - |S_L| - |S_E|, S_G)$

Want to choose an x so that $\max(|S_L|, |S_G|)$ is as small as possible. That is, want x near the middle.

Two algorithms:

- QuickSelect
  - Choose x at random
  - Good average case performance
- BFPRT Algorithm
  - Choose x by a complicated, but linear time method guaranteeing good split
  - Good worst case performance

QuickSelect: Random Choice of Pivot

QuickSelect:

- Run **Select** always choosing the pivot element $\boldsymbol{x}$ *uniformly at random* from among the elements of $\boldsymbol{S}$.

Theorem: **QuickSelect** has expected runtime $O(\boldsymbol{n})$.

Proof: Let $\boldsymbol{T}(\boldsymbol{n})$ be the expected runtime of **QuickSelect** on worst-case input sets $\boldsymbol{S}$ of size $\boldsymbol{n}$ and integer $\boldsymbol{k}$.

(The only randomness in the expectation is in the random choices of the algorithm.)

QuickSelect: Random Choice of Pivot

Consider a call to **Select**($\boldsymbol{k}$, $\boldsymbol{S}$) and sorted order of elements in $\boldsymbol{S}$

Elements of $\boldsymbol{S}$ listed in sorted order

Say x is "good" iff
it is in the middle half

With probability $\geq \mathbf{1/2}$ pivot x is good

- For any good pivot the recursive call has subproblem size $\leq \mathbf{3}n/\mathbf{4}$
- After 2 calls QuickSelect has expected problem size $\leq \mathbf{3}n/\mathbf{4}$

So $T(n) = T(n/b) + O(n)$ for $b = 4/3 > 1$ $\Rightarrow$ Expected $O(n)$ time ■

Blum-Floyd-Pratt-Rivest-Tarjan Algorithm

QuickSelect requires randomness to find a good pivot and is only good on the average.

The **BFPRT Algorithm** *always* finds a good pivot that will guarantee to leave a sub-problem of size $\leq 3n/4$. Here is how it works…

- Split S into $n/5$ sets of size 5.
- Sort each set of size 5 and choose the median of that set as its representative.
- Compute the median of those $n/5$ representatives. *Another recursion!*
- Let the pivot x be that median.

Why does it work…?

BFPRT, Step 1: Construct sets of size 5, sort each set

Input: 13, 15, 32, 14, 95, 5, 16, 45, 86, 65, 62, 41, 81, 52, 32, 32, 12, 73, 25, 81, 47, 8, 69, 9, 7, 81, 18, 25, 42, 91, 64, 98, 96, 91, 6, 51, 21, 12, 36, 11, 11, 9, 5, 17, 77

Group:

13	5	62	32	47	81	64	51	11
15	16	41	12	8	18	98	21	9
32	45	81	73	69	25	96	12	5
14	86	52	25	9	42	91	36	17
95	65	32	81	7	91	6	11	77

Sort each group:

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5

$O(\boldsymbol{n})$

BFPRT, Step 2: Find median of column medians

Column medians:

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
14	16	41	25	8	25	64	12	9
13	5	32	12	7	18	6	11	5

$T(n/5)$

Imagining rearranging columns by column median

95	86	81	81	69	91	98	51	77
32	65	62	73	47	81	96	36	17
15	45	52	32	9	42	91	21	11
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13	5	32	12	7	18	6	11	5

$T(n/5)$

Choose x to be that median of medians

95	51	77	69	81	91	98	86	81
32	36	17	47	73	81	96	65	62
15	21	11	9	32	42	91	45	52
14	12	9	8	25	25	64	16	41
13	11	5	7	12	18	6	5	32

Not in S_G

Size $\geq n/4$

BFPRT, Step 2: Find median of column medians

Column medians:

95	86	81	81	69	91	98	51	77
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$T(n/5)$

Choose x to be that median of medians

95	51	77	69	81	91	98	86	81
32	36	17	47	73	81	96	65	62
15	21	11	9	32	42	91	45	52
14	12	9	8	25	25	64	16	41
13	11	5	7	12	18	6	5	32

Not in S_L

Size $\geq n/4$

$|S_L|, |S_G| \leq \frac{3n}{4}$

BPFRT Recurrence

Choose partitioning element $\boldsymbol{x}$

- $\boldsymbol{T(n/5) + O(n)}$

Partitioning based on $\boldsymbol{x}$

- $\boldsymbol{O(n)}$

Cost of recursive subproblem

- $\boldsymbol{T(3n/4)}$

Recurrence

- $\boldsymbol{T(n) = T(3n/4) + T(n/5) + O(n)}$

Why is the solution $\boldsymbol{O(n)}$?

Solution to $T(n) = T(3n/4) + T(n/5) + cn$ is $O(n)$

Key property of recurrence:

- $\mathbf{3/4 + 1/5 < 1}$
- Sum is $\mathbf{19/20}$

Cost at top level is $\boldsymbol{cn}$; so at other levels, linear in the sum of problem sizes

- Sum of problem sizes decreases by $\mathbf{19/20}$ factor per level of recursion
- Total cost is geometric series with ratio $< \mathbf{1}$ and largest term $\boldsymbol{cn}$
- Solution is $O(\boldsymbol{n})$.

QuickSort

QuickSort(S)

if $|S| \leq 1$ return S

Choose element x from S "pivot"

$S_L \leftarrow \{y \in S \mid y < x\}$

$S_E \leftarrow \{y \in S \mid y = x\}$

$S_G \leftarrow \{y \in S \mid y > x\}$

return [QuickSort(S_L), S_E, QuickSort(S_G)]

QuickSort

Pivot selection

- Choose the median
  - $T(n) = 2\,T(n/2) + O(n)$ $\qquad O(n \log n)$
- Choose arbitrary element
  - Worst case – $O(n^2)$
    - Element might be smallest, so one subproblem has size $n - 1$
  - Average case – $O(n \log n)$ similar to QuickSelect analysis
- Choose random pivot
  - Expected time – $O(n \log n)$

We'll give an analysis for this bound ...

Expected Runtime for QuickSort: "Global analysis"

Runtime is proportional to # of comparisons

- Count comparisons for simplicity

Master theorem kind of analysis won't work ...

Instead, use a clever global analysis:

- Number elements $a_1, a_2, \ldots, a_n$ based on **final** sorted order
- Let p_{ij} = Probability that QuickSort compares a_i and a_j

Expected number of comparisons:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_{ij}$$

Expected Runtime for QuickSort: "Global analysis"

Lemma: For $i < j$ we have $p_{ij} \leq \frac{2}{j-i+1}$.

Proof: If a_i and a_j are compared then it must be during the call when they end up in different subproblems

- Before that, they aren't compared to each other
- After they aren't compared to each other

During this call they are only compared if one of them is the pivot

All elements between a_i and a_j are also in the call:

- $\Rightarrow$ set has size at least $j - i + 1$ in this call
- Probability one of the 2 is chosen as pivot is $\leq 2/(j - i + 1)$. ■

Expected Runtime for QuickSort: "Global analysis"

Lemma: For $i < j$ we have $p_{ij} \leq \frac{2}{j-i+1}$.

Harmonic series sum:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

Fact: $H_n = \ln n + O(1)$

Expected number of comparisons:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} p_{ij} \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i+1} \frac{2}{k+1} \qquad \text{for } k = j - i$$

$$< 2 \sum_{i=1}^{n-1} \boxed{\sum_{k=1}^{n} \frac{1}{k}}$$

$$< 2\, n\, H_n$$

$$= 2\, n \ln n + O(n) \leq 1.387\, n \log_2 n \qquad \blacksquare$$

QuickSort in Practice (Nonrandom)

Separating out set S_E of elements equal to the pivot is important

- Use 4-finger algorithm instead of 2-finger algorithm for partitioning
  - Collect equal elements at each end and swap to middle at end of partitioning (saves a lot on size of recursive set sizes)

- If n is very small use InsertionSort instead (also good if set is nearly sorted)
- Small n
  - choose middle element of subarray as pivot
- Medium n
  - choose median of 3 elements as pivot
- Large n
  - consider 9 elements in 3 groups of 3; choose median of medians as pivot