CSE 421
Introduction to Algorithms

Lecture 3: Overview, Graph Search

Measuring efficiency: The RAM model

- RAM = Random Access Machine

- Time ≈ # of instructions executed in an ideal assembly language
  - each simple operation (+,*,-,=,if,call) takes one time step
  - each memory access takes one time step

Complexity analysis

- Problem size n
  - **Worst-case complexity**:
    maximum # steps algorithm takes on any input of size n
  - **Best-case complexity:**
    minimum # steps algorithm takes on any input of size n
  - **Average-case complexity**:
    average # steps algorithm takes on inputs of size n

Complexity

- The complexity of an algorithm associates a number $T(n)$, the worst/average-case/best time the algorithm takes, with each problem size **n**.

- Mathematically,
  - T is a function that maps positive integers giving problem size to positive real numbers giving number of steps.

- Sometimes we have more than one size parameter
  - e.g. n=# of vertices, m=# of edges in a graph.

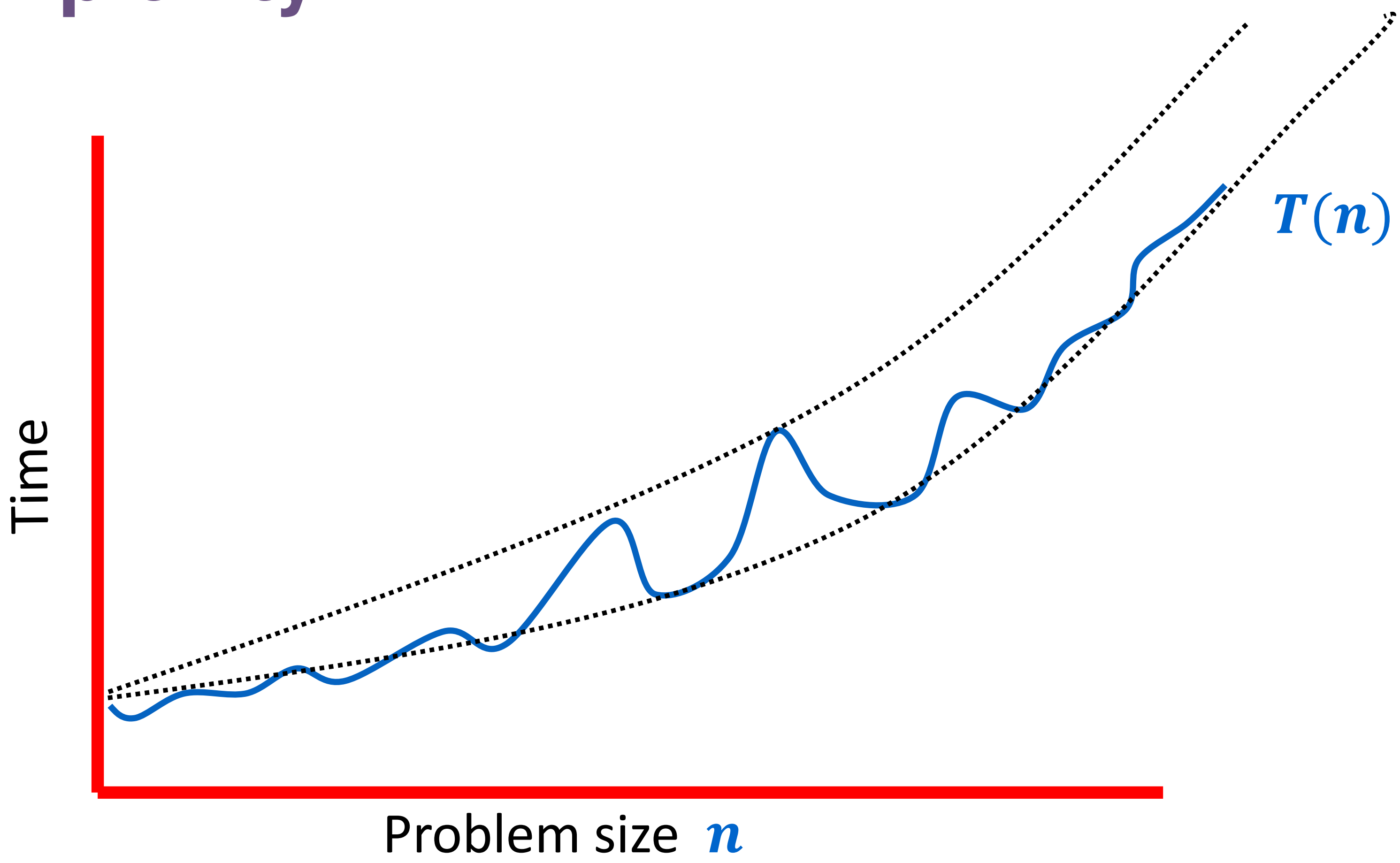
Efficient = Polynomial Time

- Polynomial time
  - Running time $T(n) \leq cn^k + d$ for some $c, d, k \geq 0$

- Why polynomial time?
  - If problem size grows by at most a constant factor then so does the running time
    - e.g. $T(2n) \leq c\,(2n)^k + d = 2^k cn^k + d \leq 2^k\left(cn^k + d\right) = 2^k\,T(n)$
    - polynomial-time is exactly the set of running times that have this property

  - Typical running times are small degree polynomials, mostly less than n^3, at worst n^6, not n^{100}

Complexity



O-notation etc

- Given two positive functions f and g
  - $f(n)$ is $O(g(n))$ iff there is a constant $c > 0$ so that $f(n)$ is eventually always $\leq c \cdot g(n)$
  - $f(n)$ is $o(g(n))$ iff the ratio $f(n)/g(n)$ goes to 0 as n gets large
  - $f(n)$ is $\Omega(g(n))$ iff there is a constant $\varepsilon > 0$ so that $f(n) \geq \varepsilon \cdot g(n)$ for infinitely many values of n
  - $f(n)$ is $\Theta(g(n))$ iff $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$

Note: The definition of $f(n)$ is $\Omega(g(n))$ is the same as "$f(n)$ is **not** $o(g(n))$ "

O, o, Ω, Θ-notation intuition

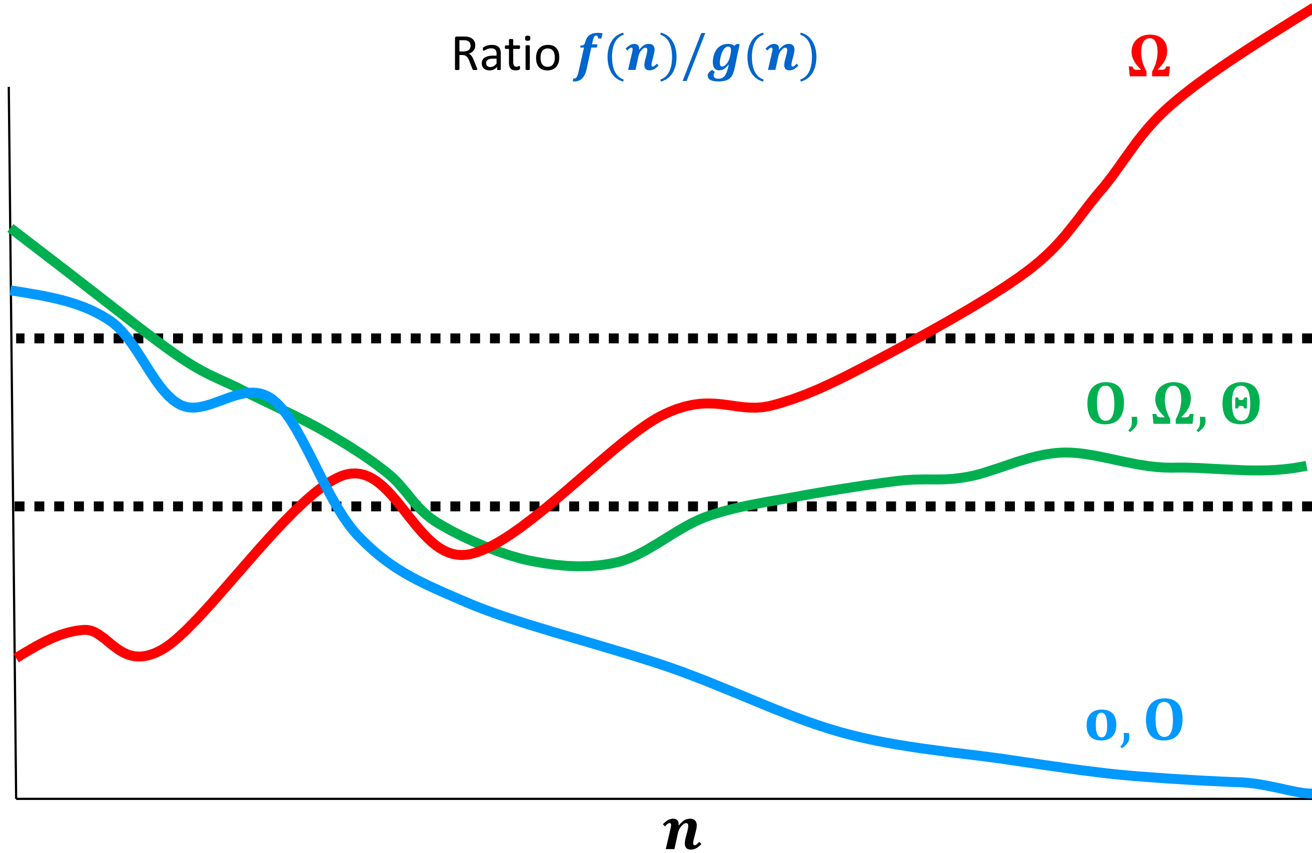
$f(n)$ is...

$O(g(n))$: ratio eventually below a line forever

$o(g(n))$: ratio goes to 0

$\Omega(g(n))$: ratio eventually above a line forever

$\Theta(g(n))$: both O and Ω



Introduction to Algorithms

- **Some representative problems**
  - Variety of techniques we'll cover
  - Seemingly small changes in a problem can require big changes in how we solve it

Some Representative Problems

Interval Scheduling:

- Single resource
- Reservation requests of form:

  "Can I reserve it from start time s to finish time f?"

  $$s < f$$

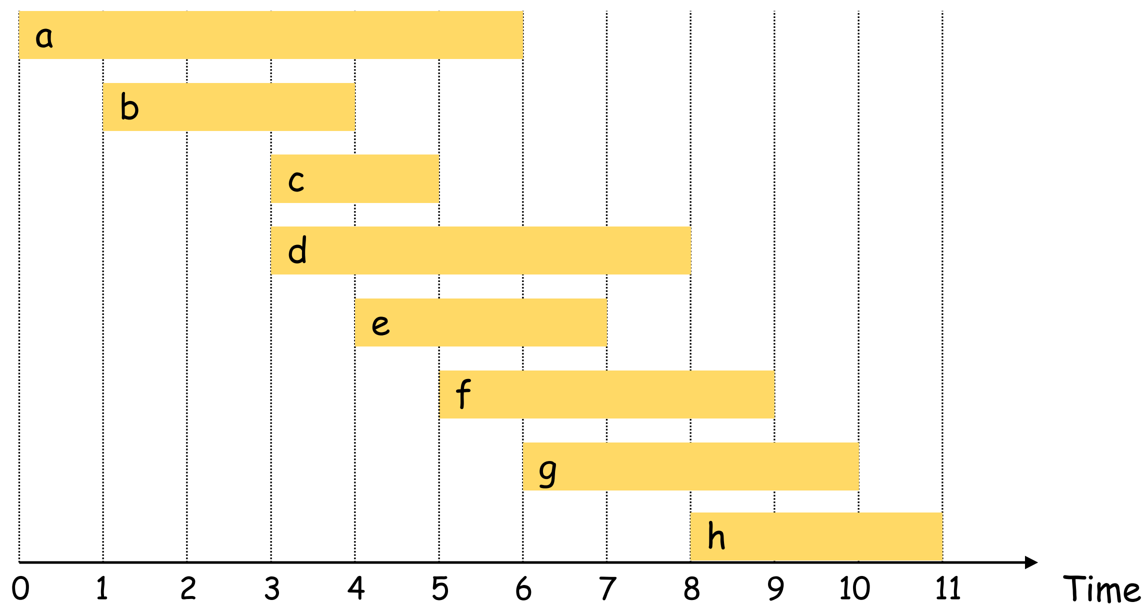
Interval Scheduling

Interval scheduling:

Input: set of jobs with start times and finish times

Goal: find maximum size subset of mutually compatible jobs.

jobs don't overlap



Interval Scheduling

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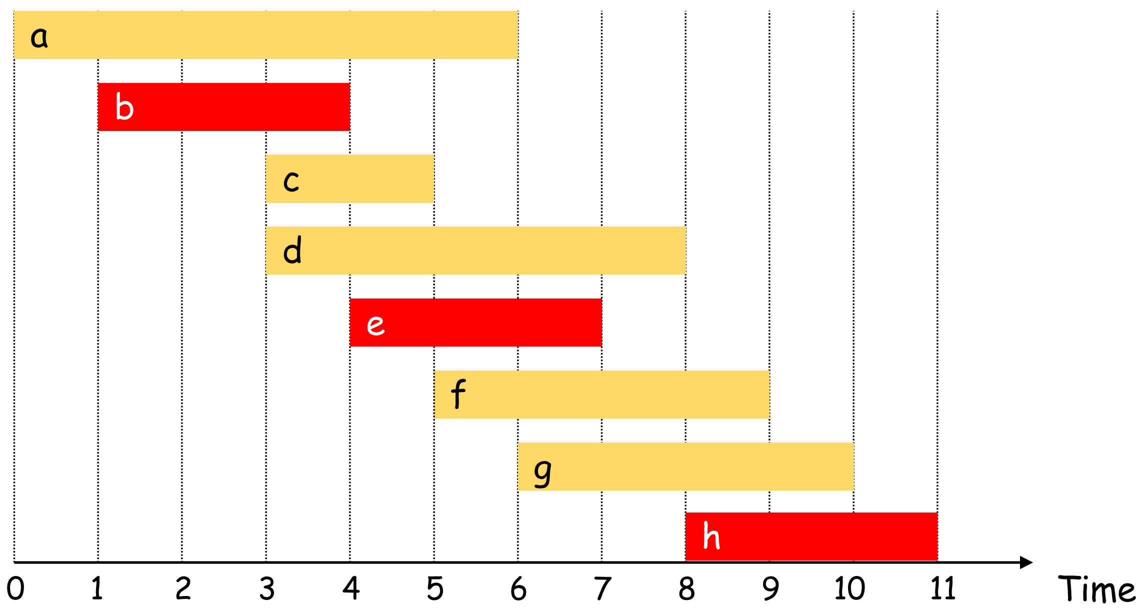


Interval Scheduling

- An optimal solution can be found using a “greedy algorithm”
  - Myopic kind of algorithm that seems to have no look-ahead
  - Greedy algorithms only work when the problem has a special kind of structure
  - When they do work they are typically very efficient

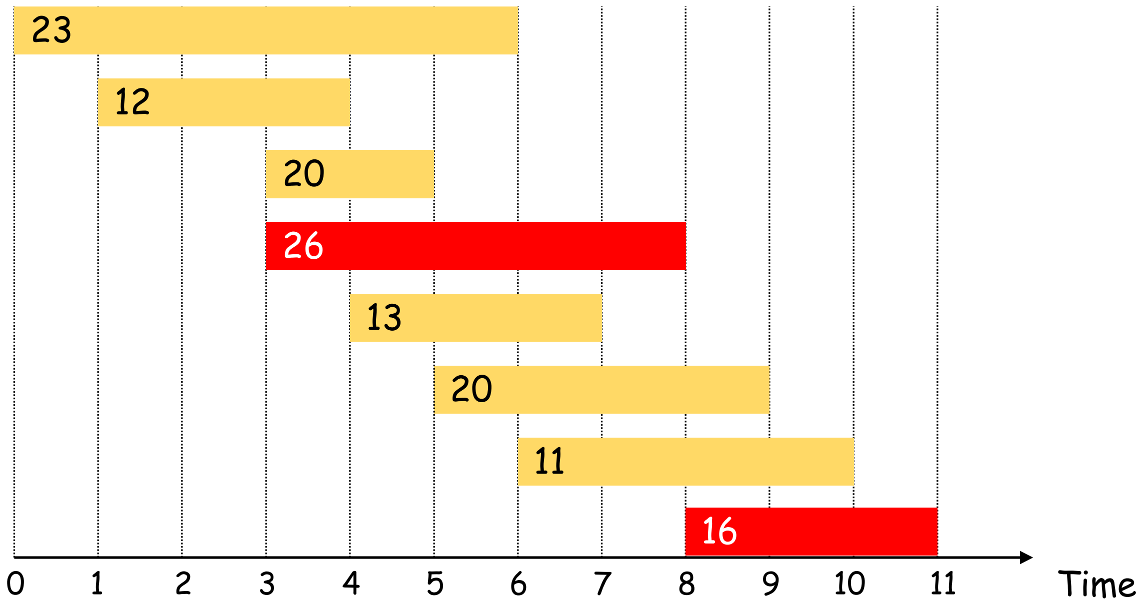
Weighted Interval Scheduling

- Same problem as interval scheduling except that each request i also has an associated value or weight w_i
  - w_i might be
    - amount of money we get from renting out the resource for that time period
    - amount of time the resource is being used

Weighted Interval Scheduling

Input: Set of jobs with start times, finish times, and weights

Goal: Find maximum weight subset of mutually compatible jobs.



Weighted Interval Scheduling

Ordinary interval scheduling is a special case of this problem

- Take all weights $\boldsymbol{w_i = 1}$

Problem is quite different though

- E.g. one weight might dwarf all others

“Greedy algorithms” don’t work

Solution: “Dynamic Programming”

- builds up optimal solutions from a table of solutions to smaller problems

Bipartite Matching

A graph $\boldsymbol{G = (V, E)}$ is bipartite iff

- Set $\boldsymbol{V}$ of vertices has two disjoint parts $\boldsymbol{X}$ and $\boldsymbol{Y}$
- Every edge in $\boldsymbol{E}$ joins a vertex from $\boldsymbol{X}$ and a vertex from $\boldsymbol{Y}$

Set $\boldsymbol{M \subseteq E}$ is a matching in $\boldsymbol{G}$ iff no two edges in $\boldsymbol{M}$ share a vertex

Goal: Find a matching $\boldsymbol{M}$ in $\boldsymbol{G}$ of maximum size.

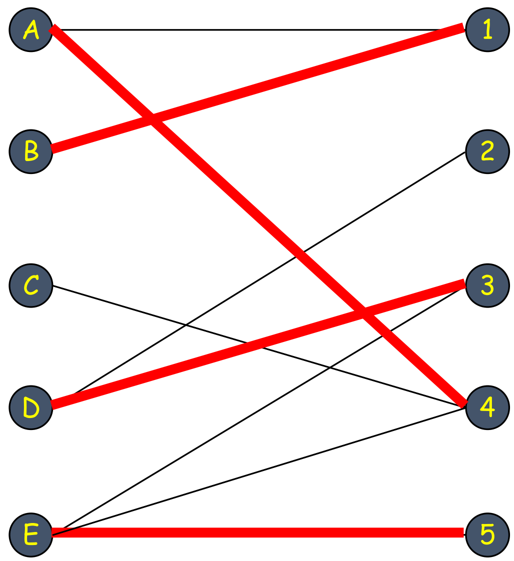
Differences from stable matching

- limited set of possible partners for each vertex
- sides may not be the same size
- no notion of stability; matching everything may be impossible.

Bipartite Matching

Input: Bipartite graph

Goal: Find maximum size matching.



Bipartite Matching

- Models assignment problems
  - $\boldsymbol{X}$ represents customers, $\boldsymbol{Y}$ represents salespeople
  - $\boldsymbol{X}$ represents professors, $\boldsymbol{Y}$ represents courses

- If $|\boldsymbol{X}| = |\boldsymbol{Y}| = \boldsymbol{n}$
  - $\boldsymbol{G}$ has perfect matching iff maximum matching has size $\boldsymbol{n}$

Solution: polynomial-time algorithm using “augmentation” technique

- Also used for solving more general class of network flow problems

Independent Set

Defn: For graph $G = (V, E)$ a set $I \subseteq V$ is independent iff no two nodes in I are joined by an edge

Input: Graph $G = (V, E)$

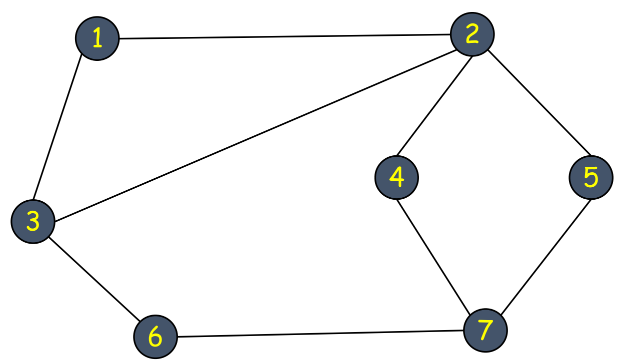
Goal: Find an independent set I in V of maximum possible size

- Models conflicts and mutual exclusion

Independent Set

Input: Graph.

Goal: Find a maximum size independent set.



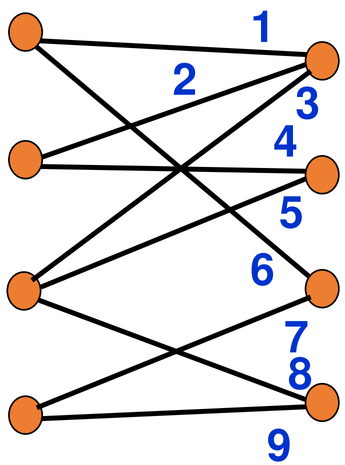
Independent Set

Generalizes

- **Interval Scheduling**
  - Vertices in the graph are the requests
  - Vertices are joined by an edge if they are **not** compatible

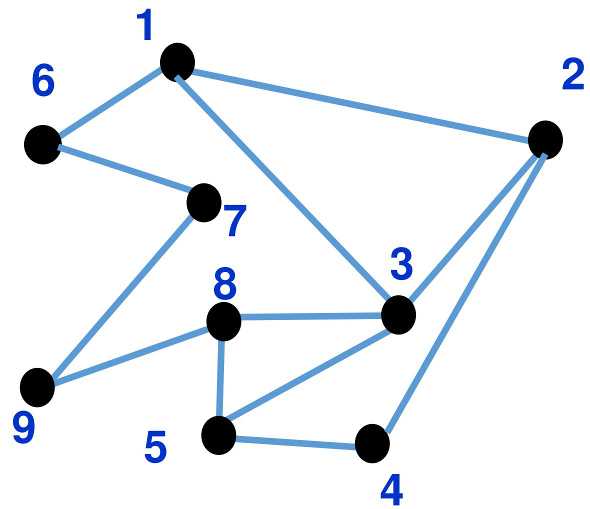
- **Bipartite Matching**
  - Given bipartite graph $G = (V, E)$ create new graph $G' = (V', E')$ (sometimes called the line-graph of G) where
    - $V' = E$
    - Two elements of V' (which are edges in G) are joined iff they touch
  - Independent set I in V' $\Rightarrow$ no edges in I touch $\Rightarrow$ I is matching in G

Bipartite Matching



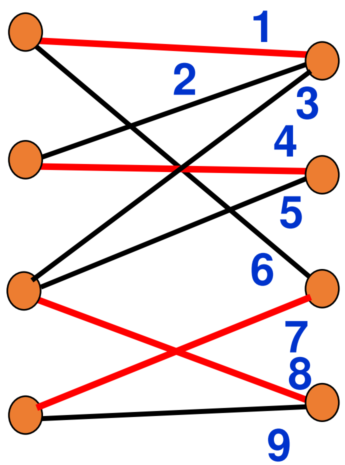
$G = (V, E)$

Independent Set



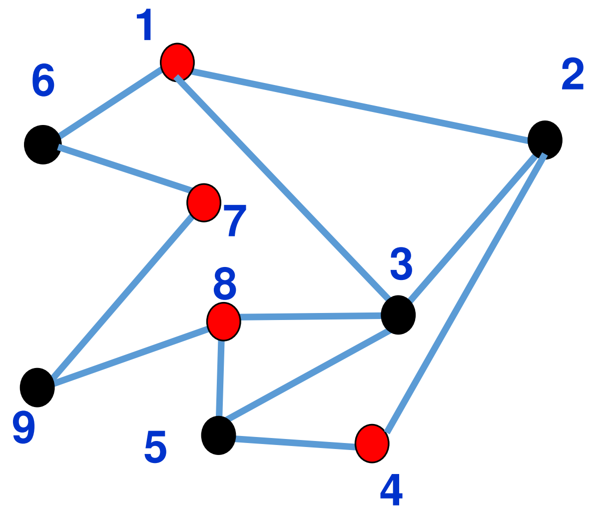
$G' = (V', E')$

Bipartite Matching



$G = (V, E)$

Independent Set



$G' = (V', E')$

Independent Set

No polynomial-time algorithm is known

- But to convince someone that there is a large independent set all you'd only need to tell them what the set is
  - they can easily convince themselves that the set is large enough and independent
- Convincing someone that there isn't such a set seems much harder

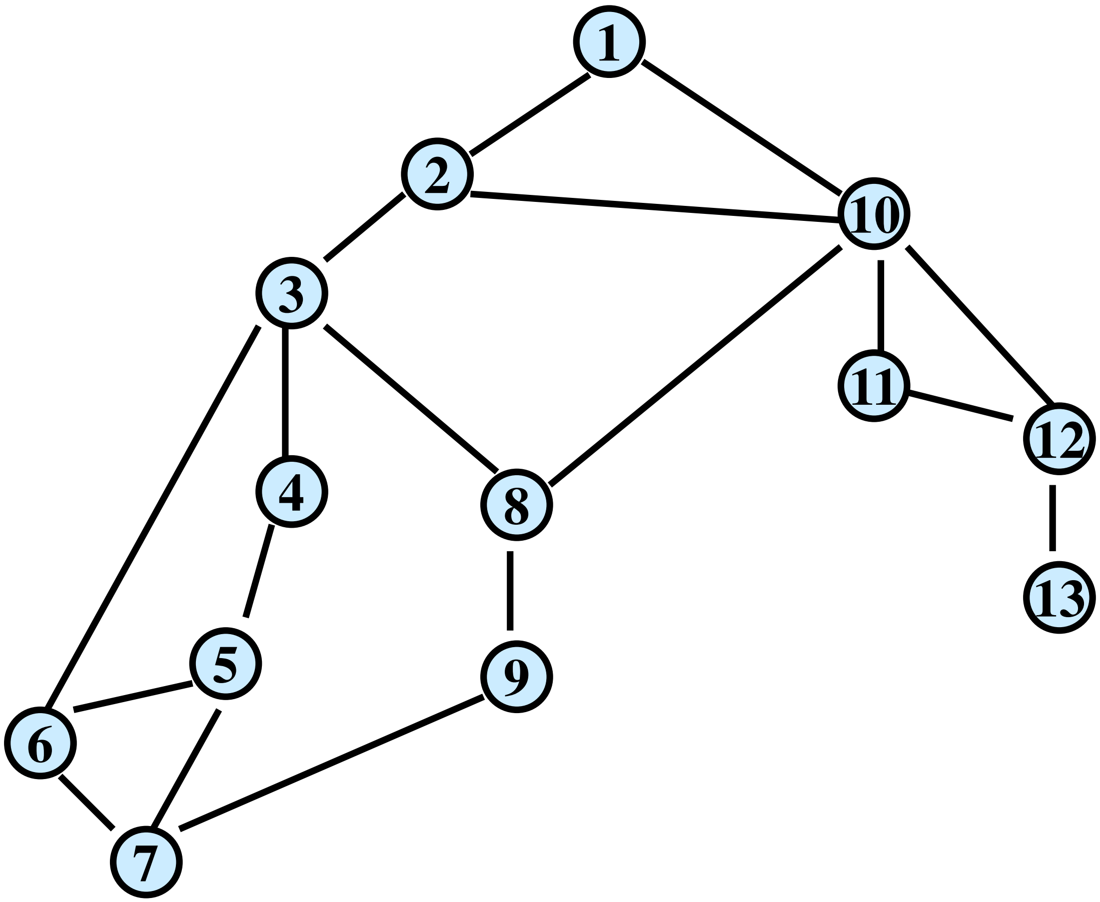
We will show that **Independent Set** is NP-complete

- Class of all the hardest problems that have the property above

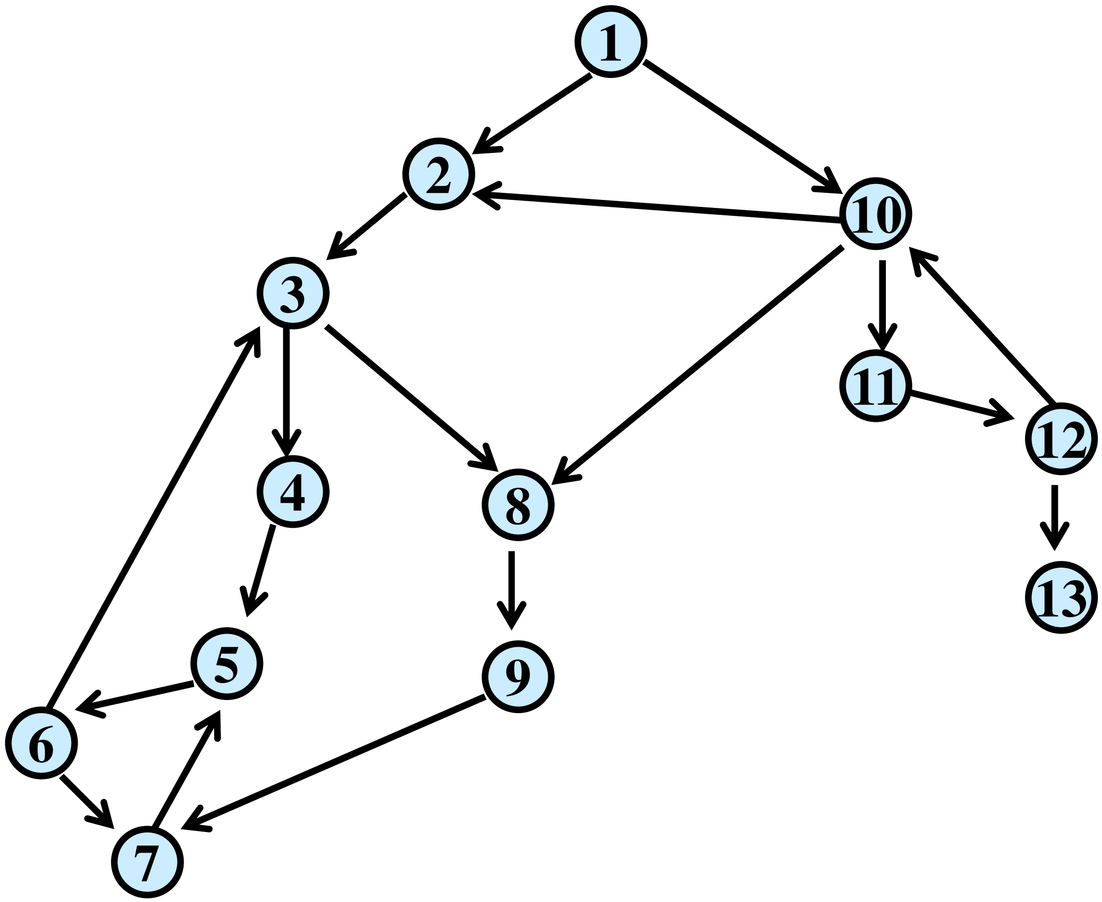
Introduction to Algorithms

- Graph Search/Traversal

Undirected Graph G = (V,E)



Directed Graph G = (V,E)



Graph Traversal

Learn the basic structure of a graph

Walk from a fixed starting vertex s to find all vertices reachable from s

Generic Graph Traversal Algorithm

Given: Graph graph $G = (V, E)$ vertex $s \in V$

Find: set R of vertices reachable from $s \in V$

Reachable(s):

$R \leftarrow \{s\}$

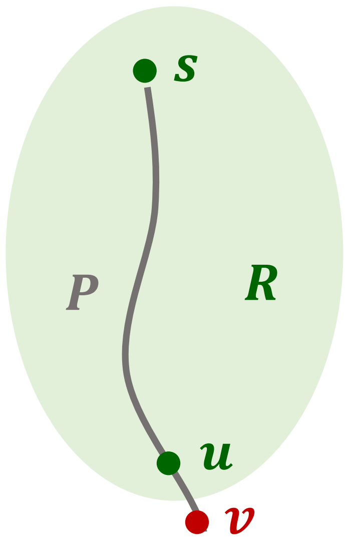
while there is a $(u, v) \in E$ where $u \in R$ and $v \notin R$

Add v to R

return R

Generic Traversal Always Works

Claim: At termination, $\boldsymbol{R}$ is the set of nodes reachable from $\boldsymbol{s}$



Proof

$\subseteq$: For every node $\boldsymbol{v} \in \boldsymbol{R}$ there is a path from $\boldsymbol{s}$ to $\boldsymbol{v}$

- Easy induction based on edges found.

$\supseteq$: Suppose there is a node $\boldsymbol{w} \notin \boldsymbol{R}$ reachable from $\boldsymbol{s}$ via a path $\boldsymbol{P}$

- Take first node $\boldsymbol{v}$ on $\boldsymbol{P}$ such that $\boldsymbol{v} \notin \boldsymbol{R}$
- Predecessor $\boldsymbol{u}$ of $\boldsymbol{v}$ in $\boldsymbol{P}$ satisfies
  - $\boldsymbol{u} \in \boldsymbol{R}$
  - $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}$
- But this contradicts the fact that the algorithm exited the while loop. ■

Graph Traversal

Learn the basic structure of a graph

Walk from a fixed starting vertex $\boldsymbol{s}$ to find all vertices reachable from $\boldsymbol{s}$

Three states of vertices

- **unvisited**
- **visited/discovered** (in $\boldsymbol{R}$)
- **fully-explored** (in $\boldsymbol{R}$ and all neighbors have been visited)

Breadth-First Search

Completely explore the vertices in order of their distance from s

Naturally implemented using a queue

BFS(s)

Global initialization: mark all vertices "unvisited"

BFS(s)

mark s "visited"; $R \leftarrow \{s\}$; layer $L_0 \leftarrow \{s\}$; $i \leftarrow 0$

while L_i not empty

$L_{i+1} \leftarrow \varnothing$

for each $u \in L_i$

for each edge (u, v)

if (v is "unvisited")

mark v "visited"

Add v to set R and to layer L_{i+1}

mark u "fully-explored"

$i \leftarrow i + 1$

Properties of BFS

BFS(s) visits x iff there is a path in G from s to x.

Edges followed to undiscovered vertices define a
breadth first spanning tree of G

Layer i in this tree:

L_i = set of vertices u with shortest path in G from root s of length i.

Properties of BFS

Claim: For undirected graphs:

All edges join vertices on the same or adjacent layers of BFS tree

Proof: Suppose not...

Then there would be vertices (x, y) s.t. $x \in L_i$ and $y \in L_j$ and $j > i + 1$.

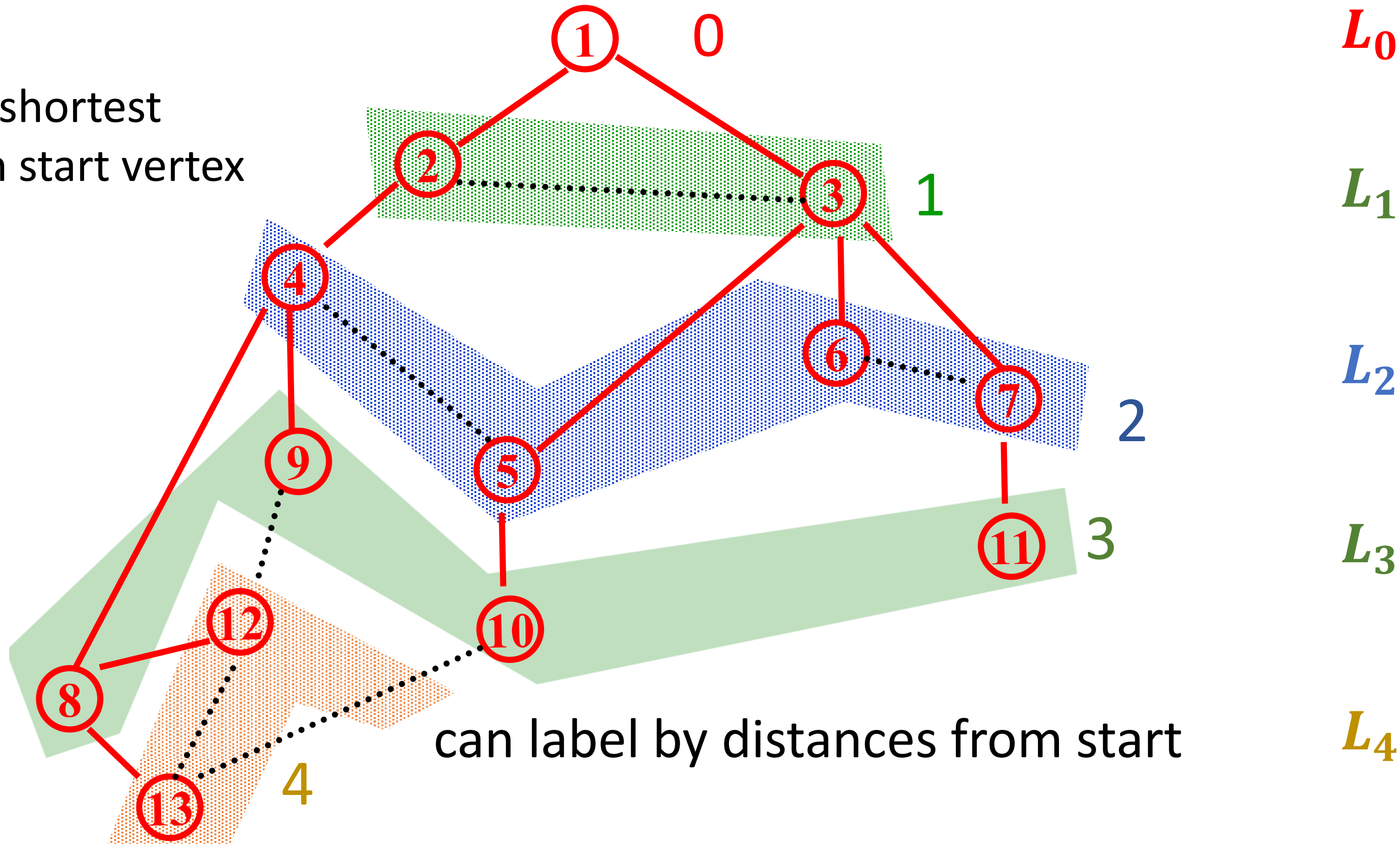
Then, when vertices adjacent to x are considered in BFS,

y would be added to L_{i+1} and not to L_j.

Contradiction. ■

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start