Max-Flow Min-Cut

CSE 421 Fall 22
Lecture 20

Announcements

HW6 is out, due on Wednesday the 22nd.

It's more dynamic programming (what we were doing last week).

On Friday we'll 'close the loop' on a few things from Ken's feedback session, but one thing to say today:

If you were hoping for "how do I know which technique to use?"

Go to tomorrow's section!

Our next topic

Max-flow/min-cut

Two (closely related) graph problems.

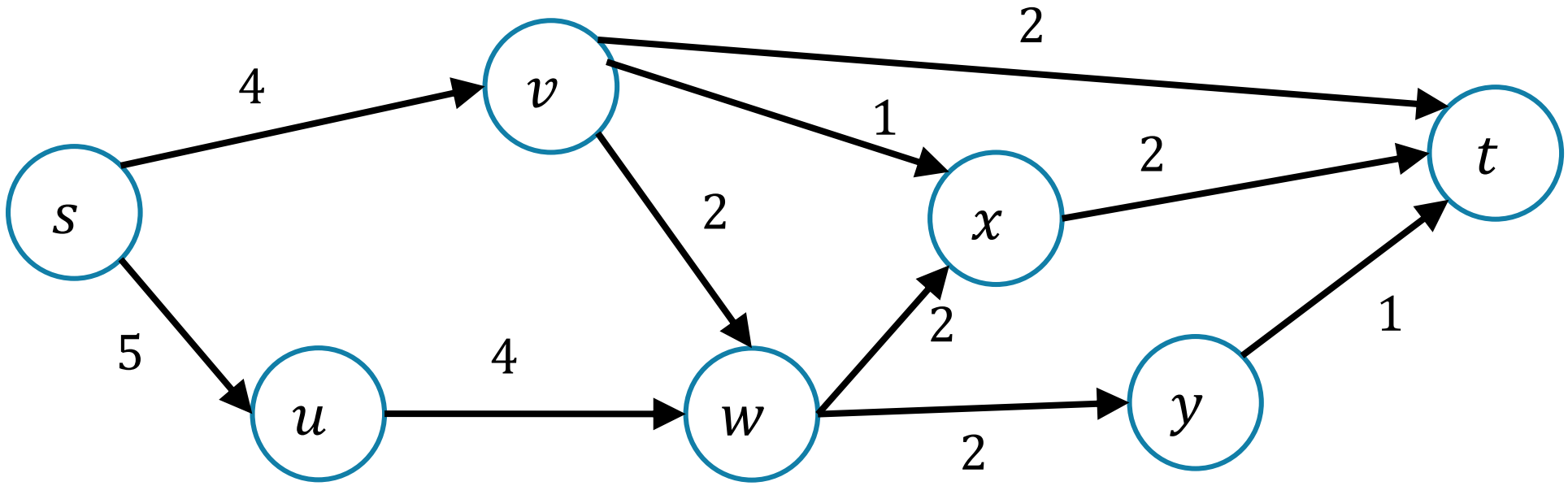
We'll use them for a number of modeling applications.

Max Flow

We have a directed graph G, a source vertex s and a target vertex t.

We have some thing (water or data packets) we have to send from s to t.

Every edge has a capacity, it can only handle so many.

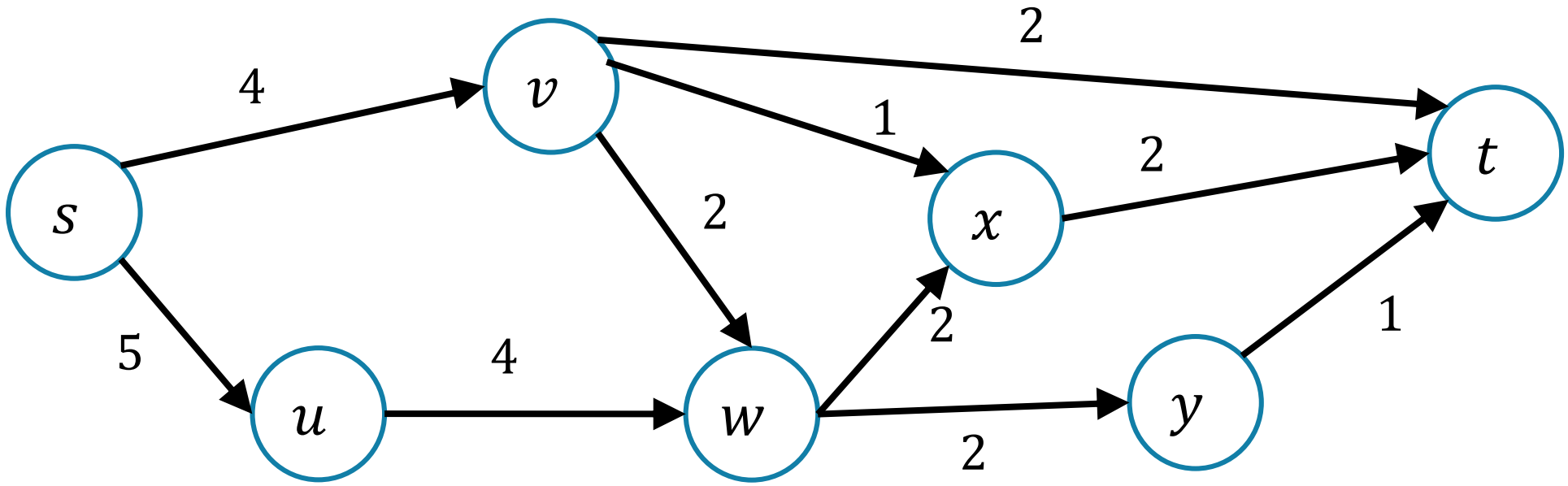


Flows

A **flow** moves units of water from s to t.

Water can only be created at s and only disappear at t.

And you cannot move more water than the capacity on any edge.



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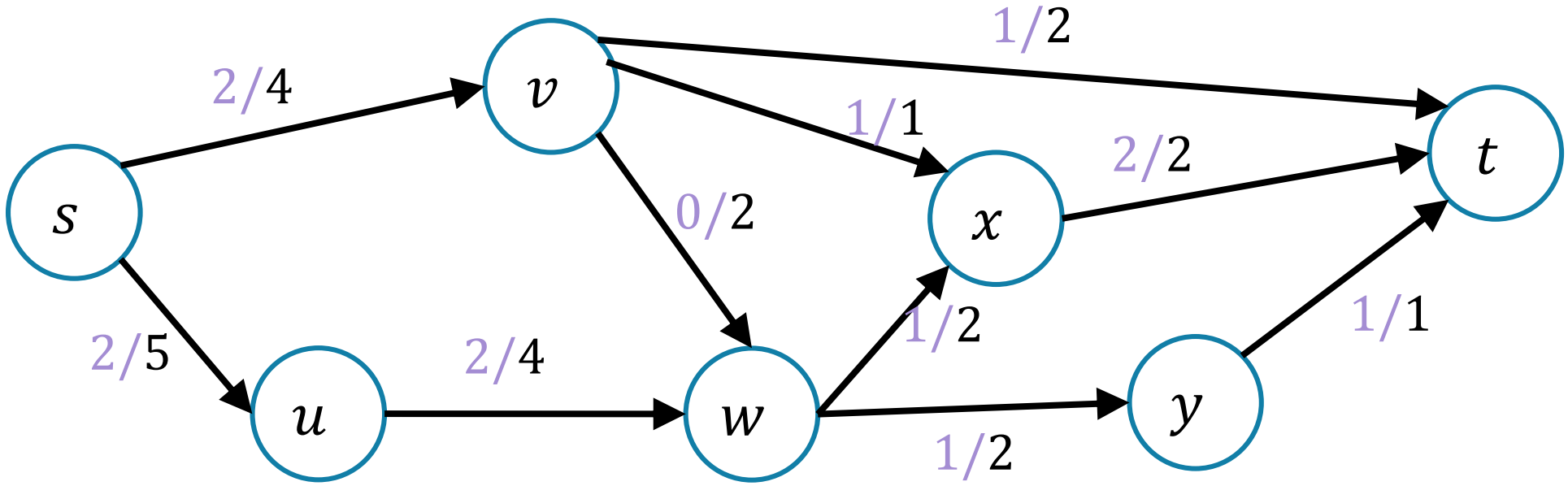


Technical terms

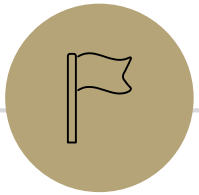
The fancy names for the two requirements

Capacity constraints:

For every edge e, the flow on e is at most the capacity.

Conservation constraints:

For every vertex u (except for s, t), the total flow entering u is equal to the total flow leaving u.



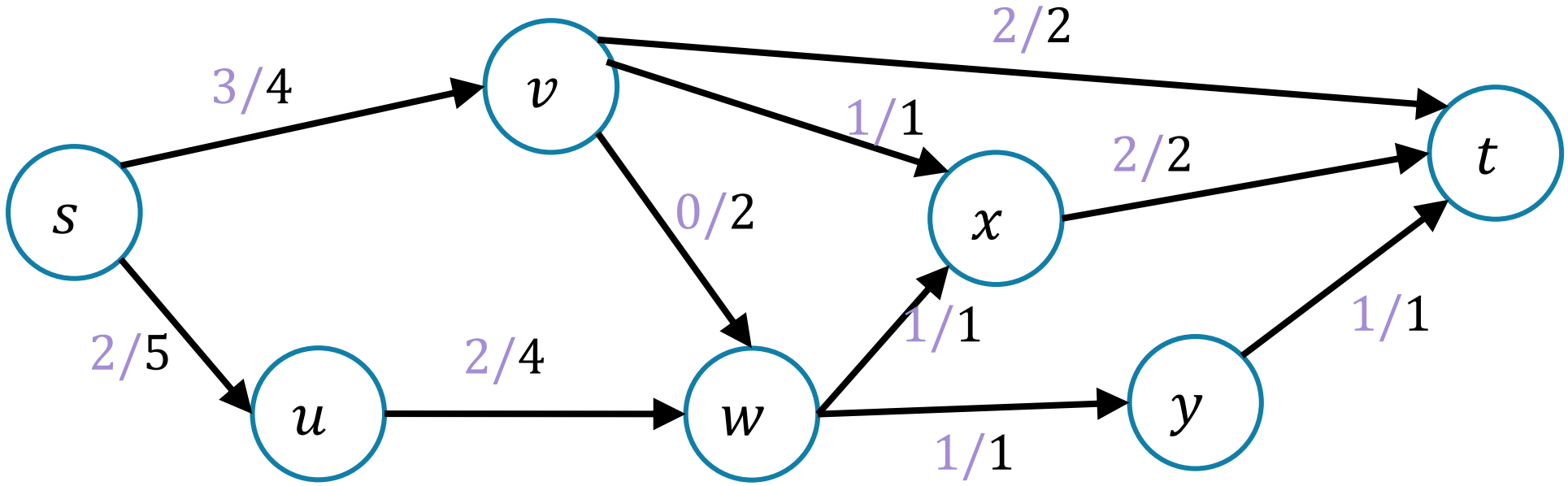
Finding Max Flows

Flows

The **value** or **size** of a flow is the net flow leaving s

Or, equivalently the net flow entering t.

Value of this flow is 5.



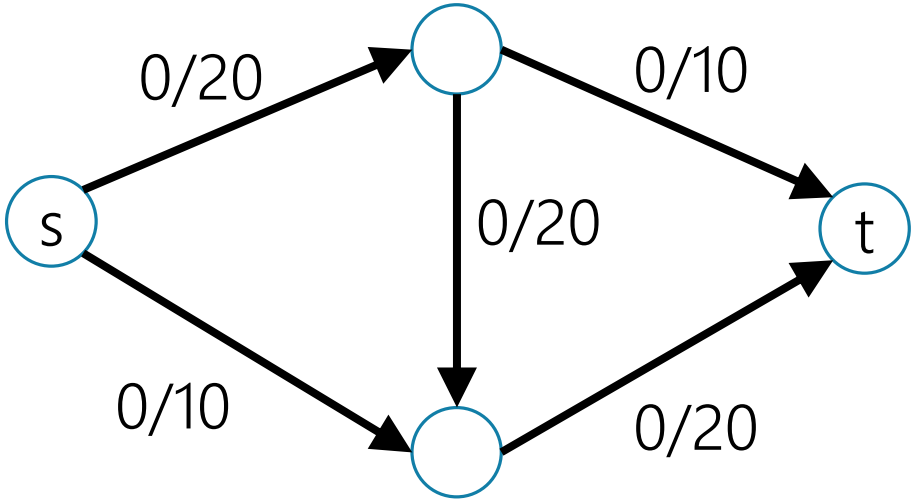
Finding the Max Flow

Idea: find a path that we can push flow along.

Start from s, follow an edge with remaining capacity until you get to t

How much flow can you add? The minimum **remaining** capacity on any of the edges we used.

Greedy: Repeat this as much as you can. Does this work?



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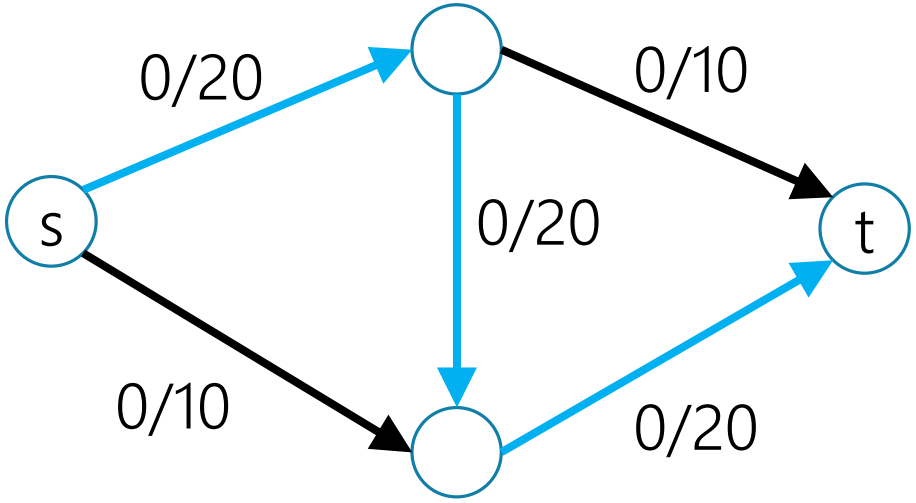

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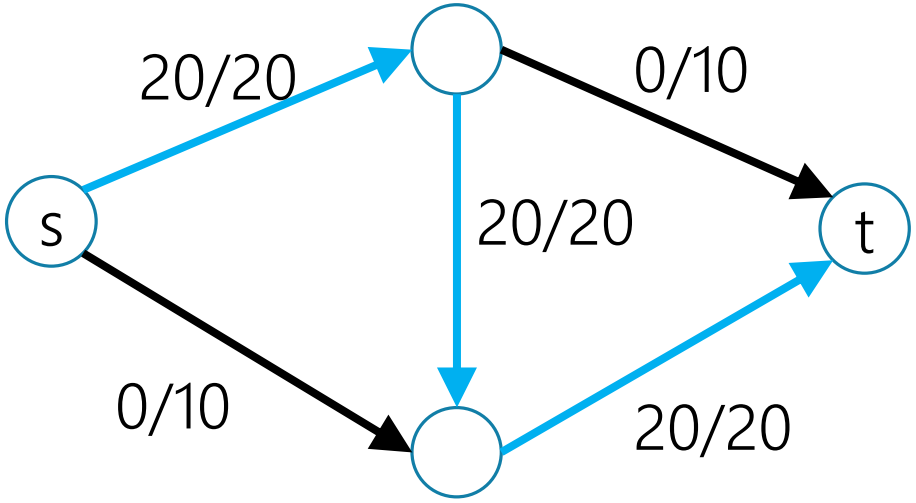

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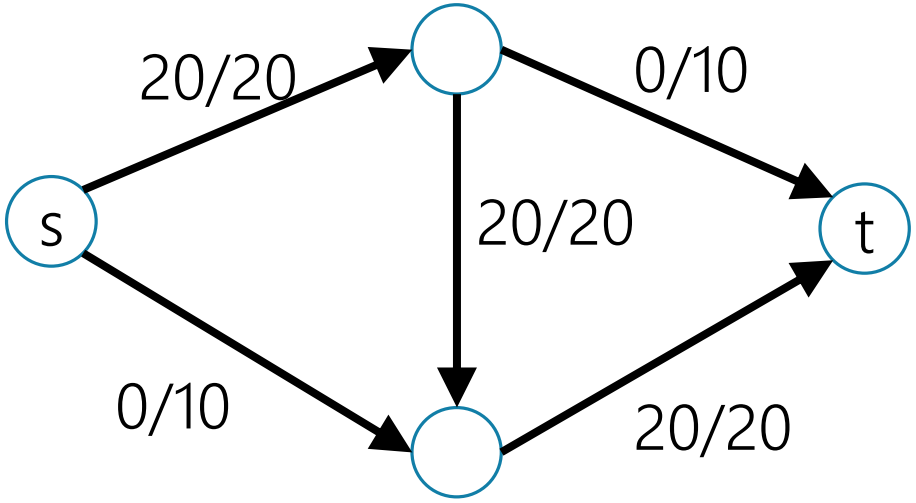
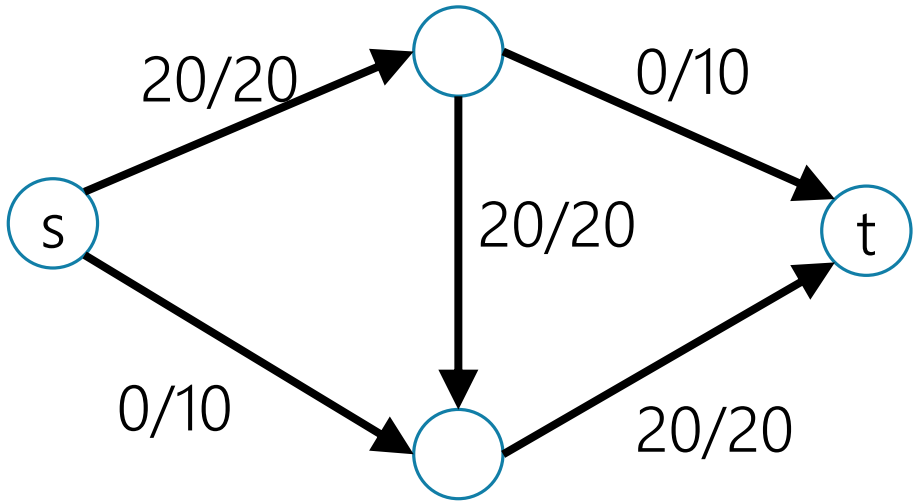


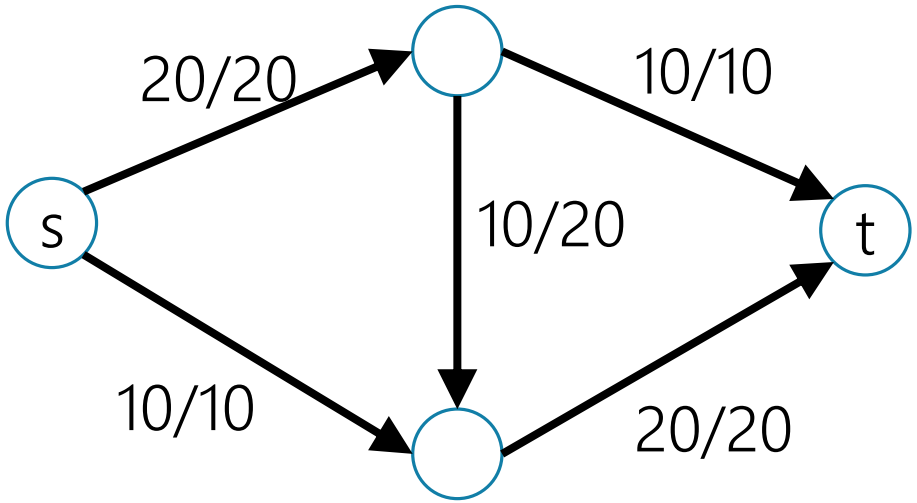
Finding the Max Flow

We find a valid flow…but it might not be the maximum one.

What greedy found



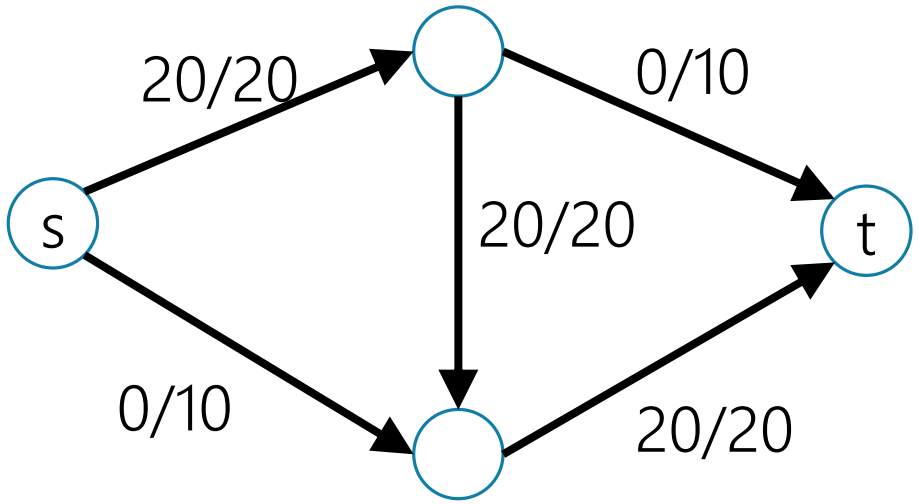
The true optimum



Finding the Max Flow

How would we fix what our first idea found?

What greedy found



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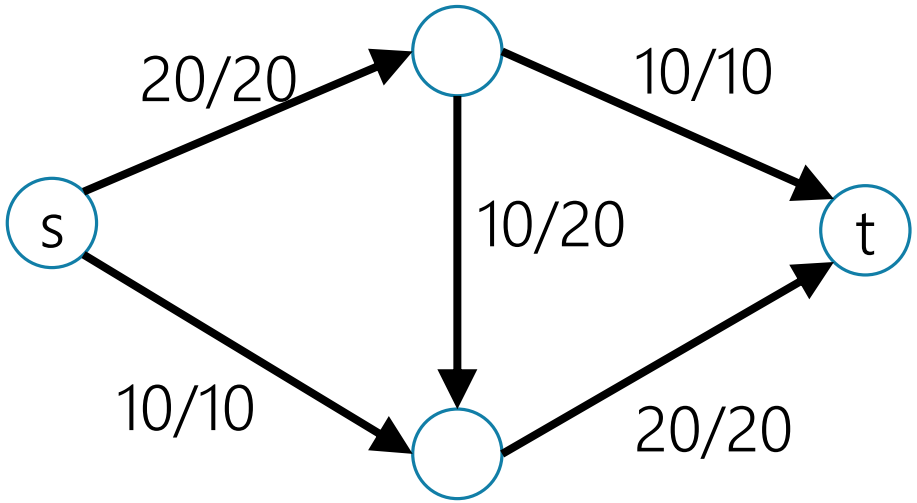
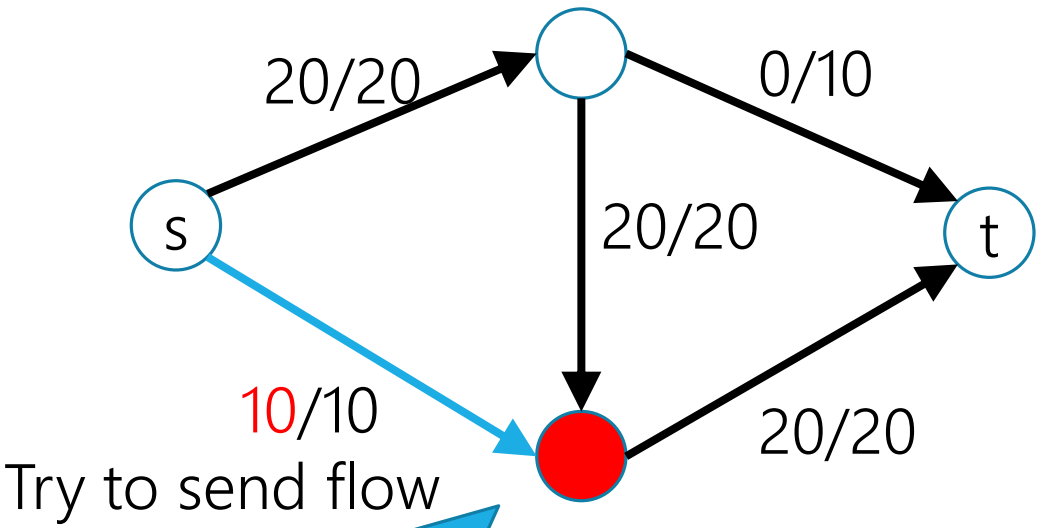


Finding the Max Flow

Try to send more flow

What greedy found



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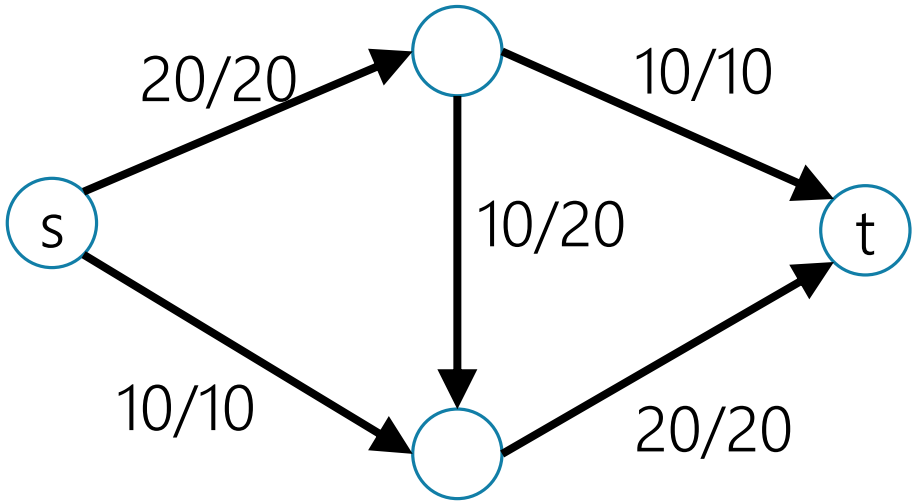
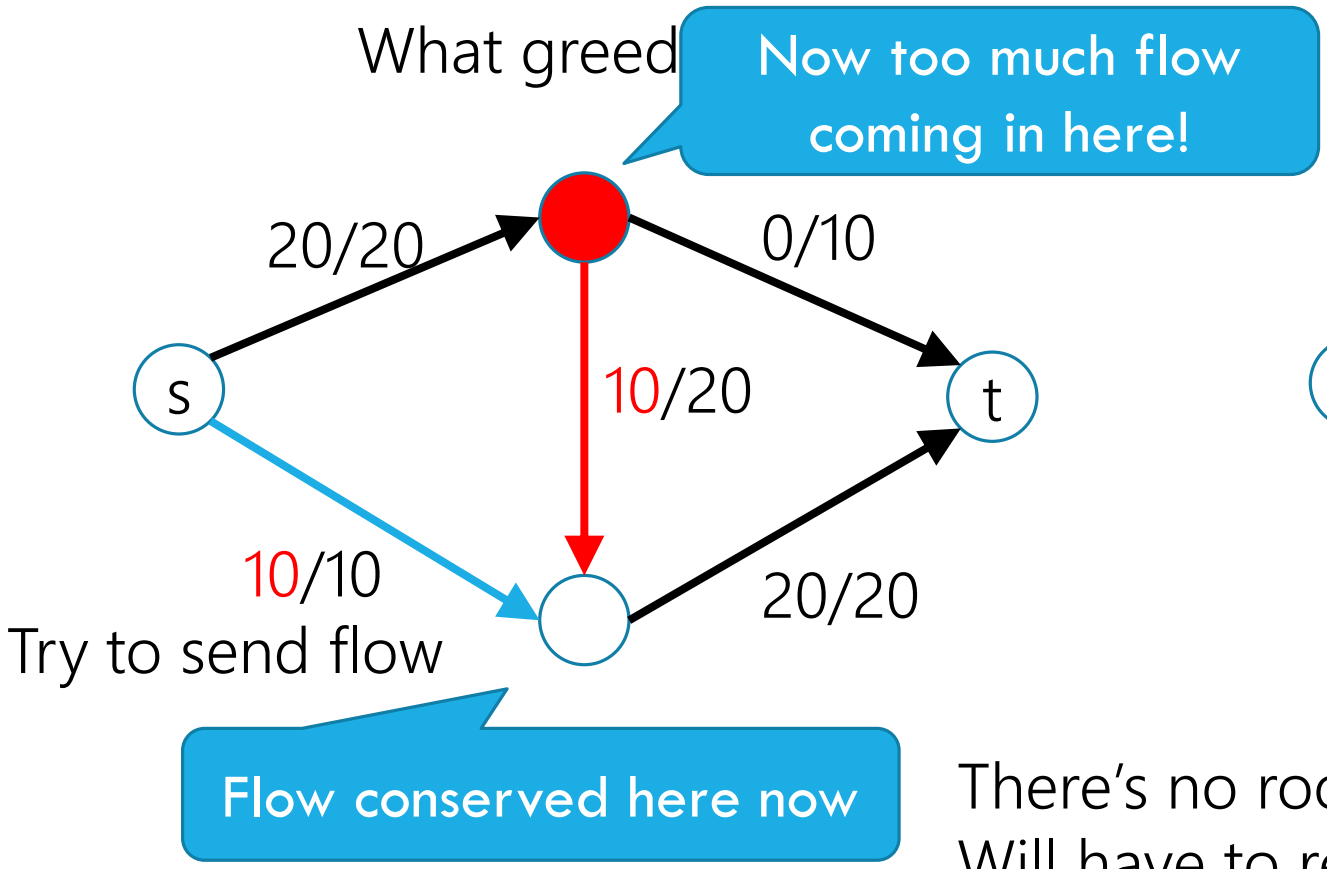


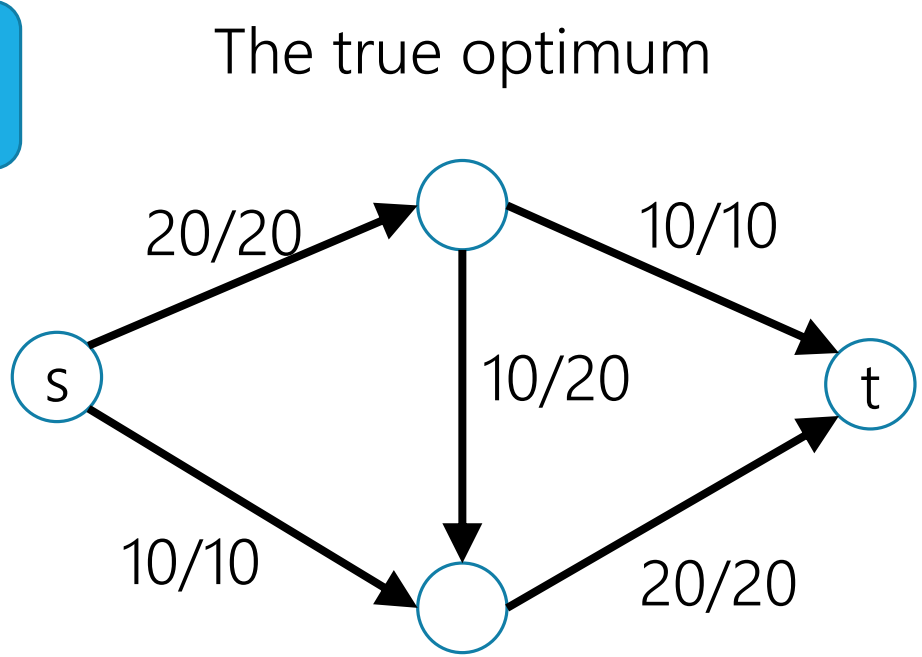
Too much flow going in!
(violates conservation)

There's no room to push the added flow out.
Will have to redirect it.

Finding the Max Flow

Try to send more flow





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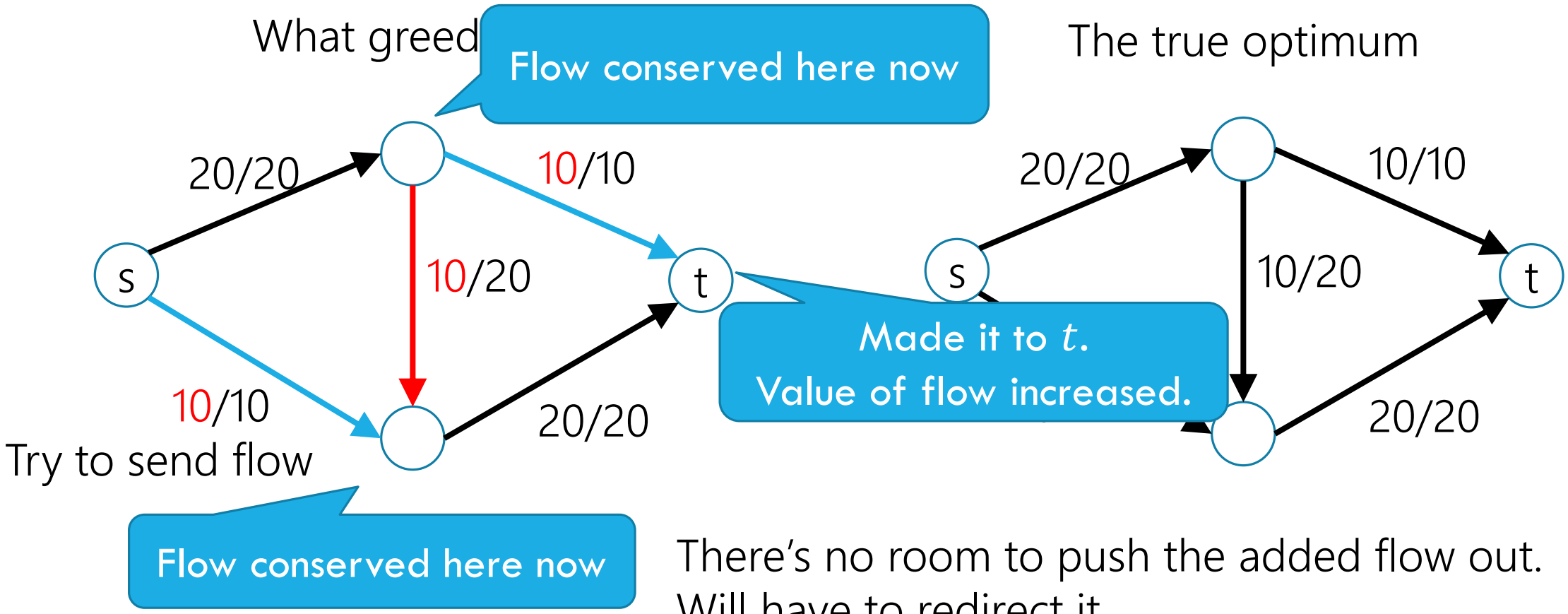


There's no room to push the added flow out.
Will have to redirect it.

Fixing Our First Idea

When can we send flow in a direction?

When there is unused capacity OR

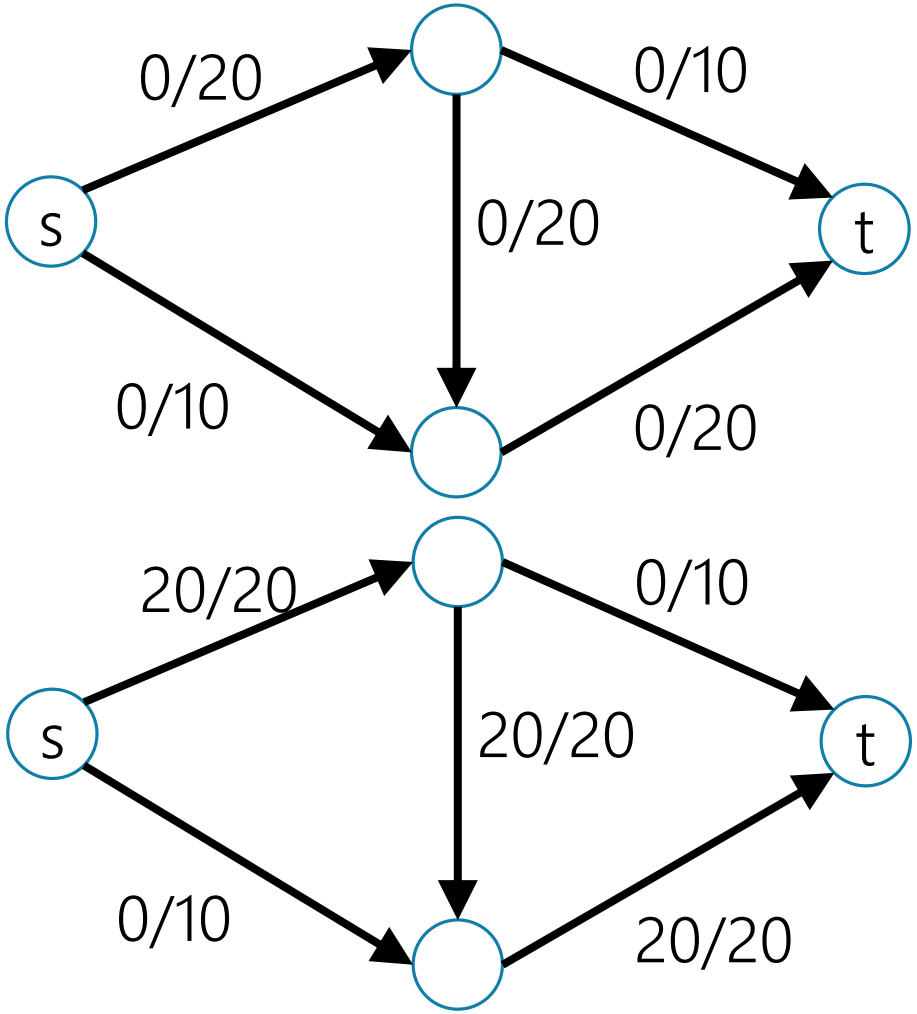
When there is flow going the other direction we can redirect it.

Let's update the graph to take that into account.

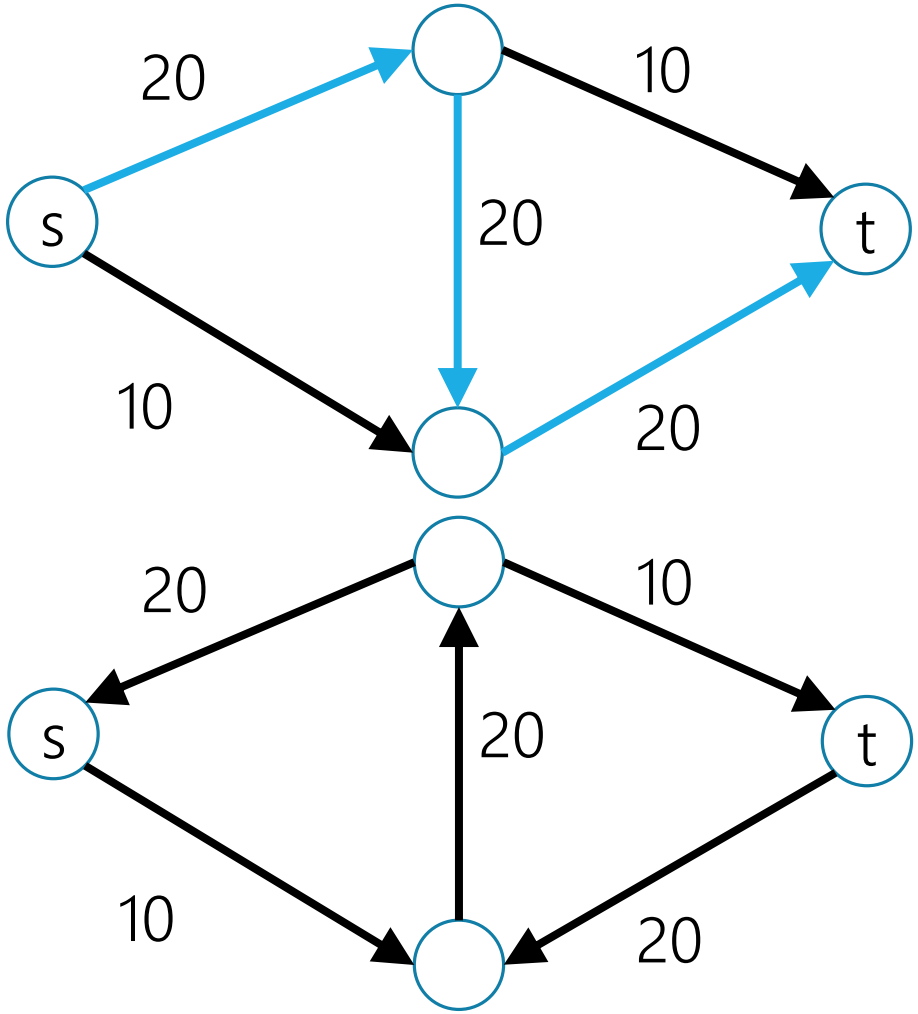
Have an edge weight demonstrate the changeable capacity (the "residual")

An Example

Graph, with flow



"Residual Graph"



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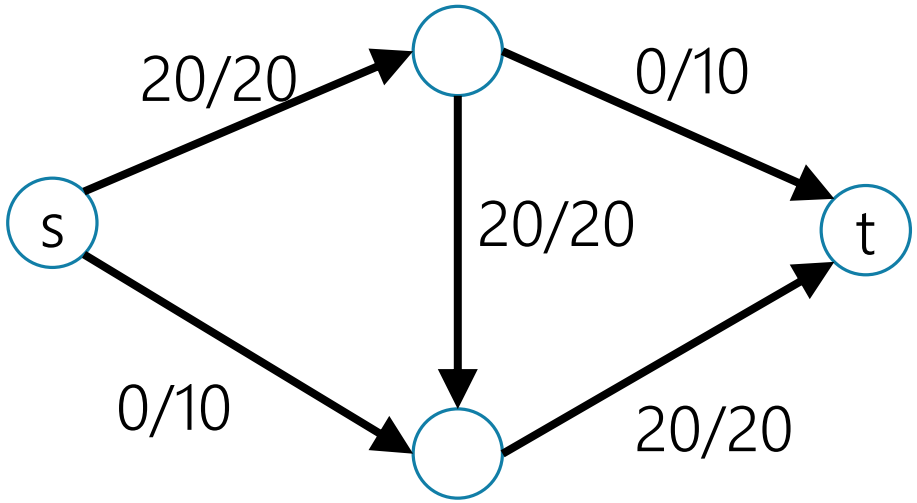

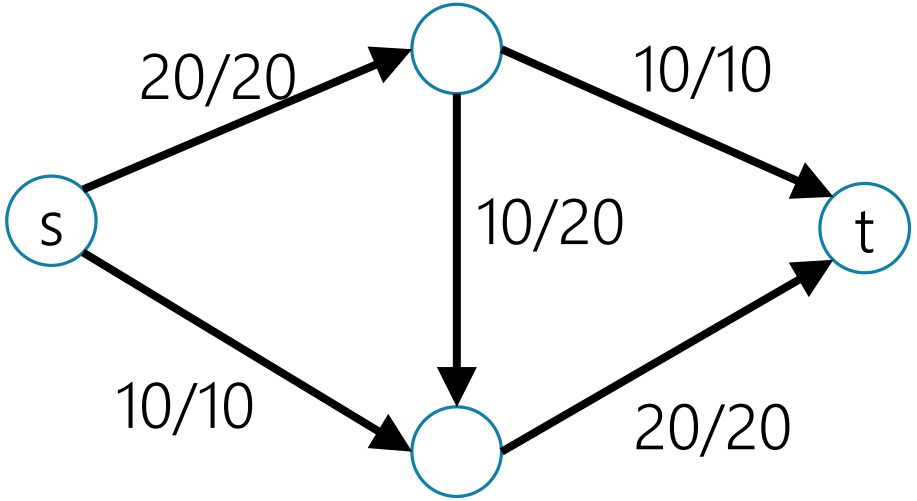

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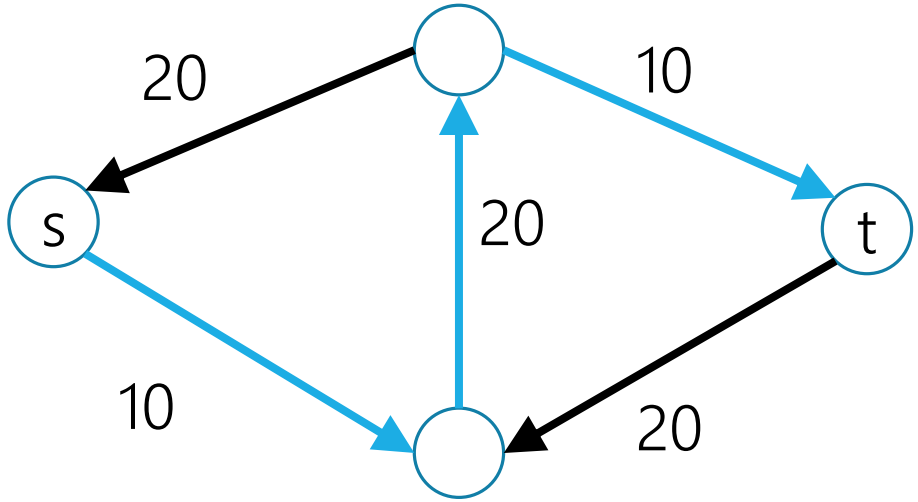
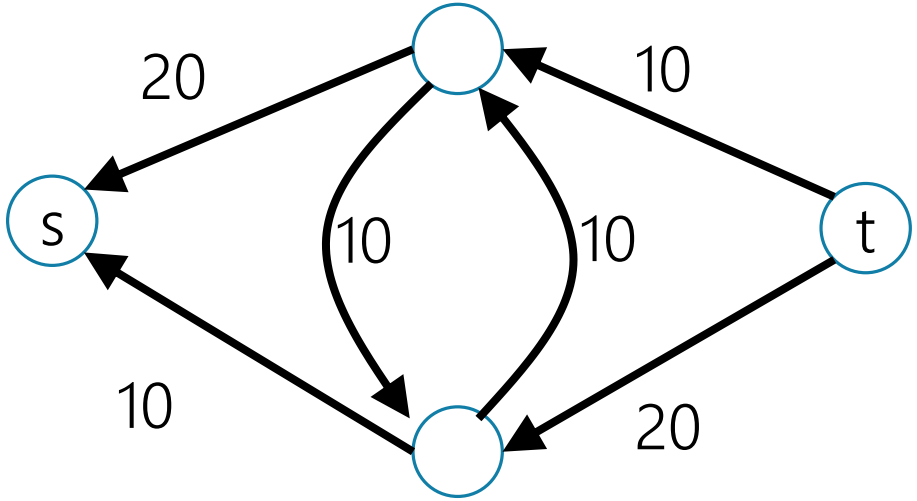




Residual Graph

In general:

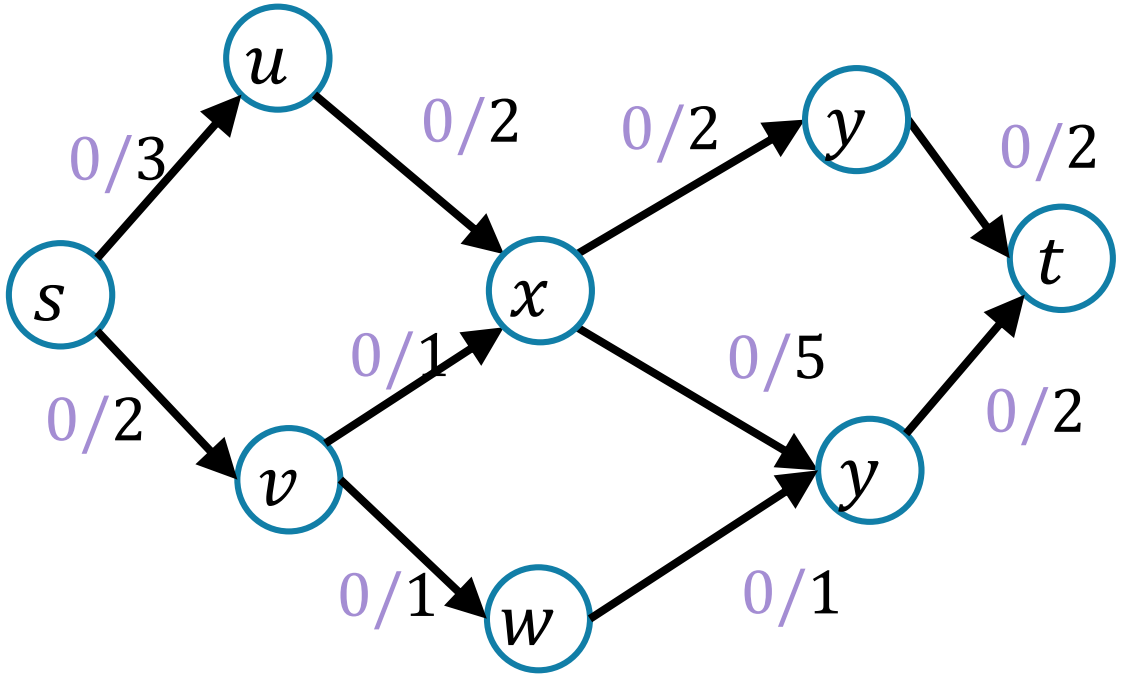
If the original graph has an edge (u, v) of capacity c, and the flow sends $f_{u,v}$ along (u, v):

Include (u, v) in the residual with capacity $c - f_{u,v}$ as long as $c - f_{u,v} > 0$ (if equal to zero, don't include the edge)

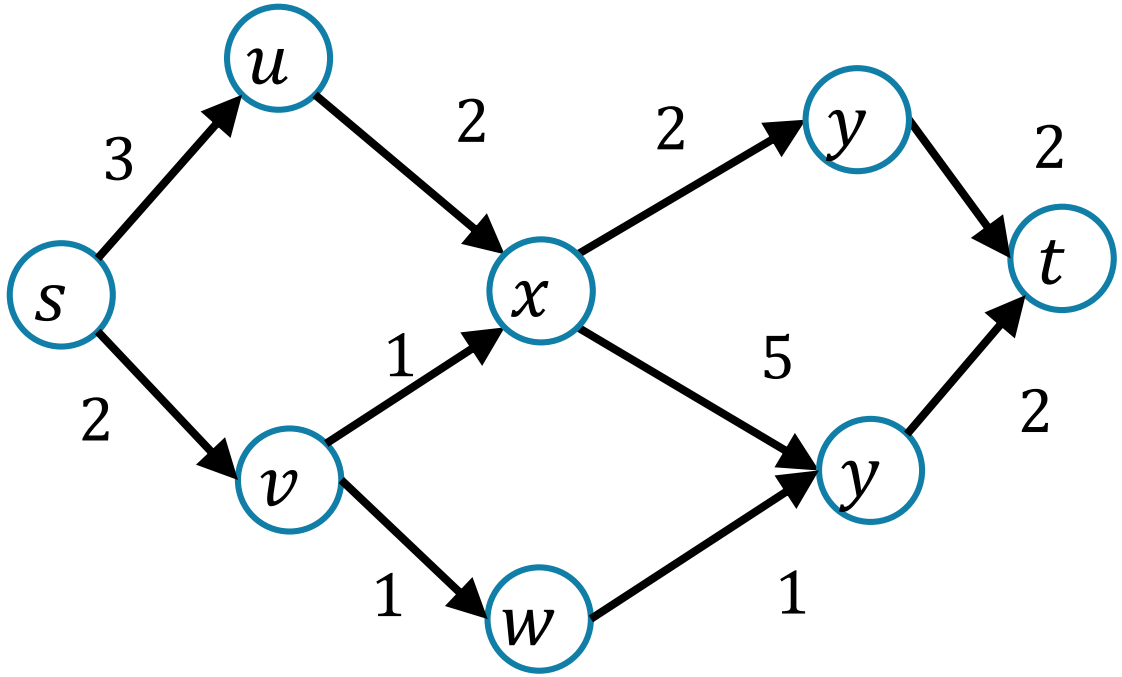
Include (v, u) [the edge going in the reverse direction] with capacity $f_{u,v}$ as long as $f_{u,v} > 0$

Example

Flow graph



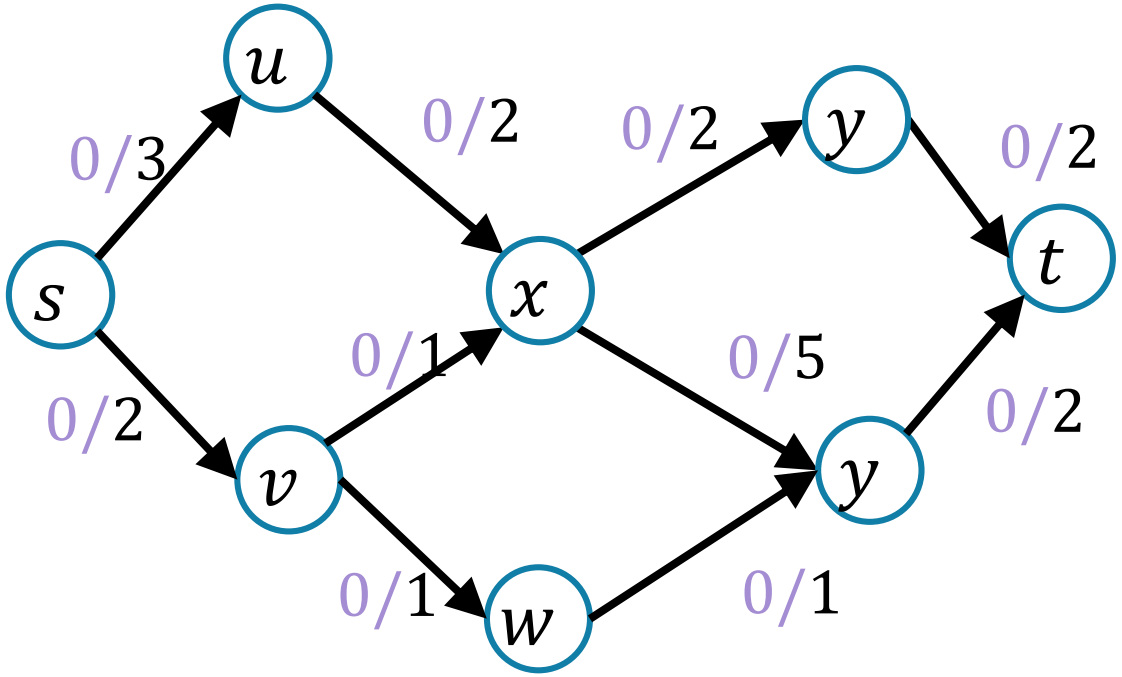
Residual graph



Residual starts with same edges and capacities as the flow graph.

Example

Flow graph



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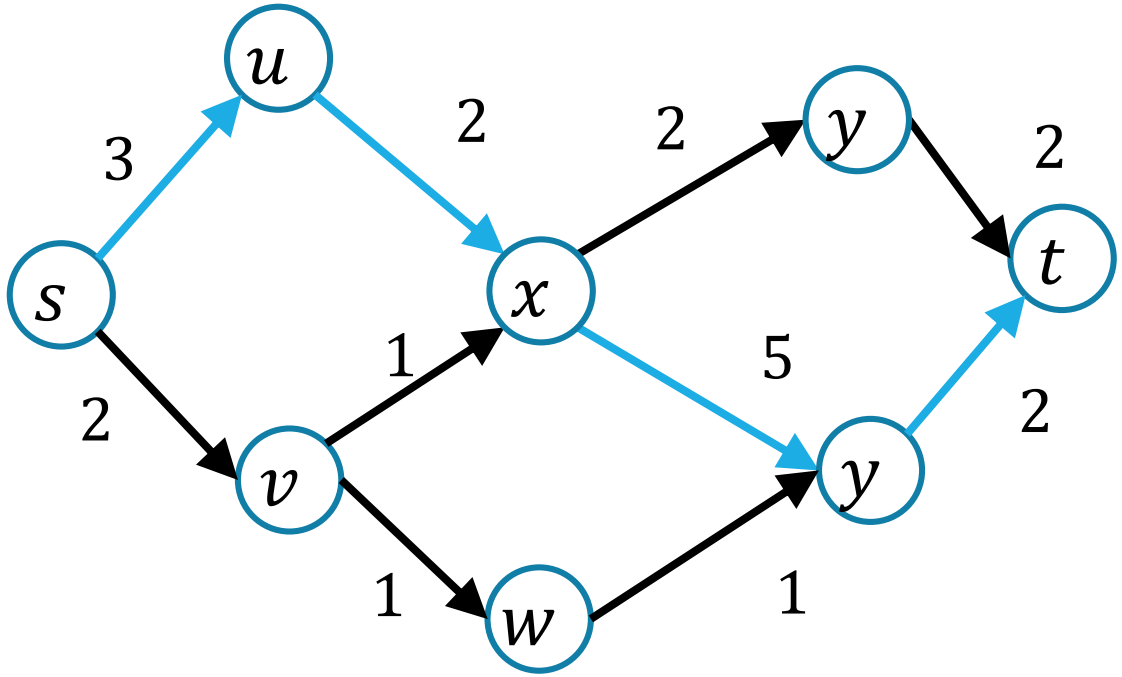
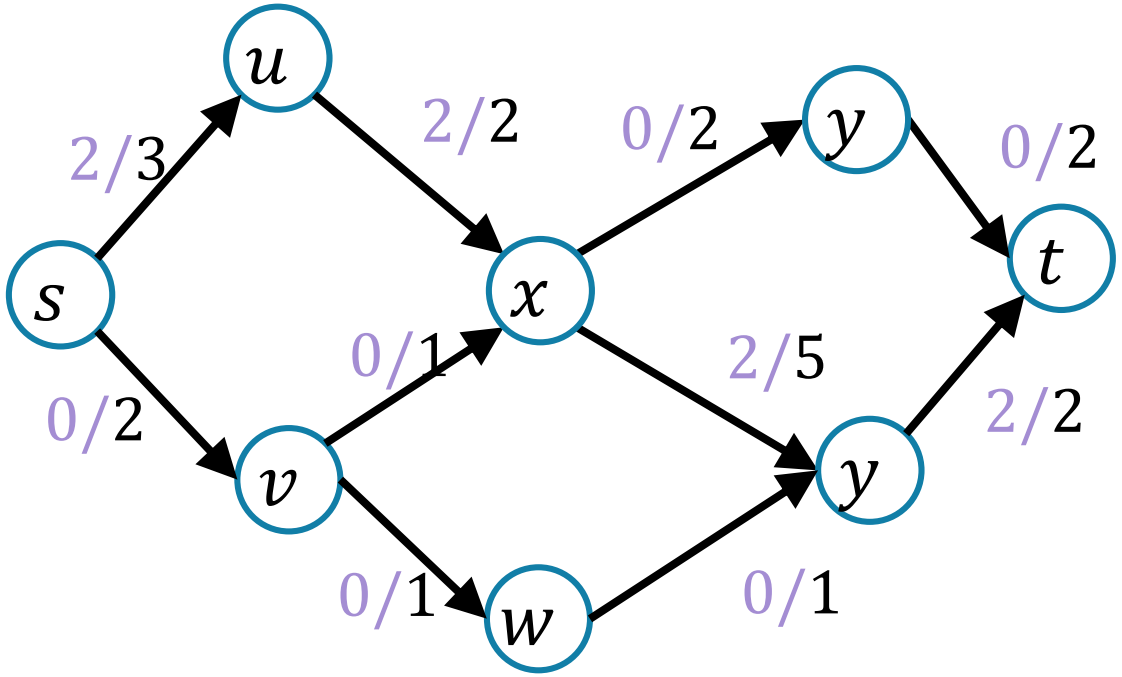


Find an s-t path in the residual.
Min residual capacity is what you augment by (here: 2)

Example

Flow graph



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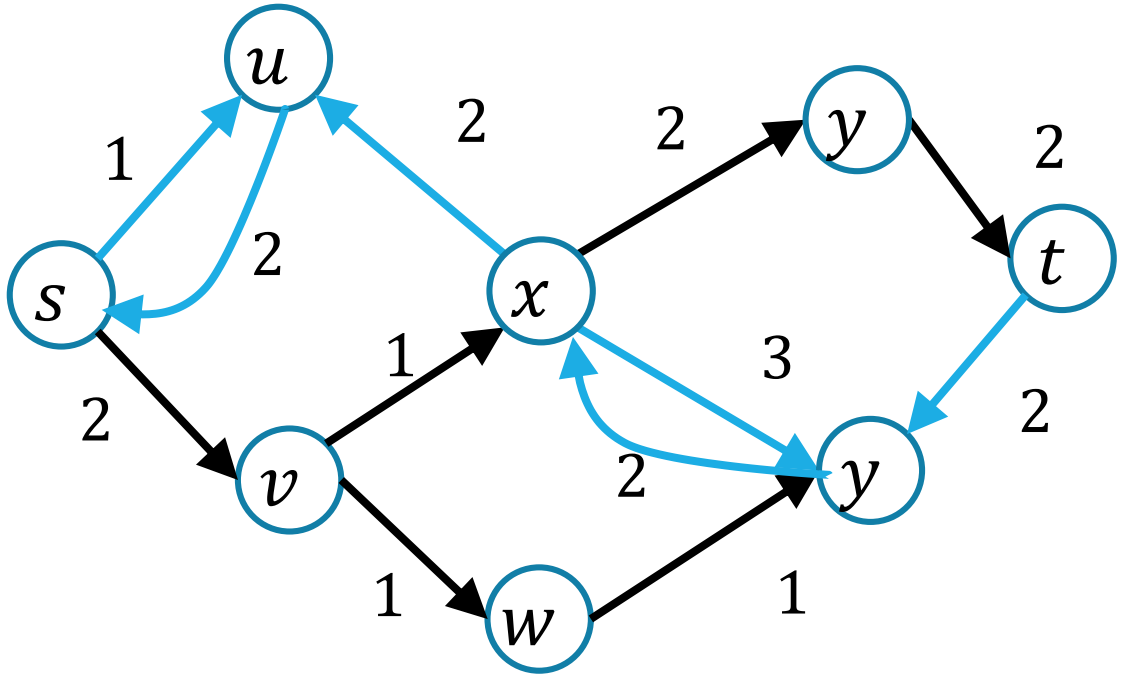
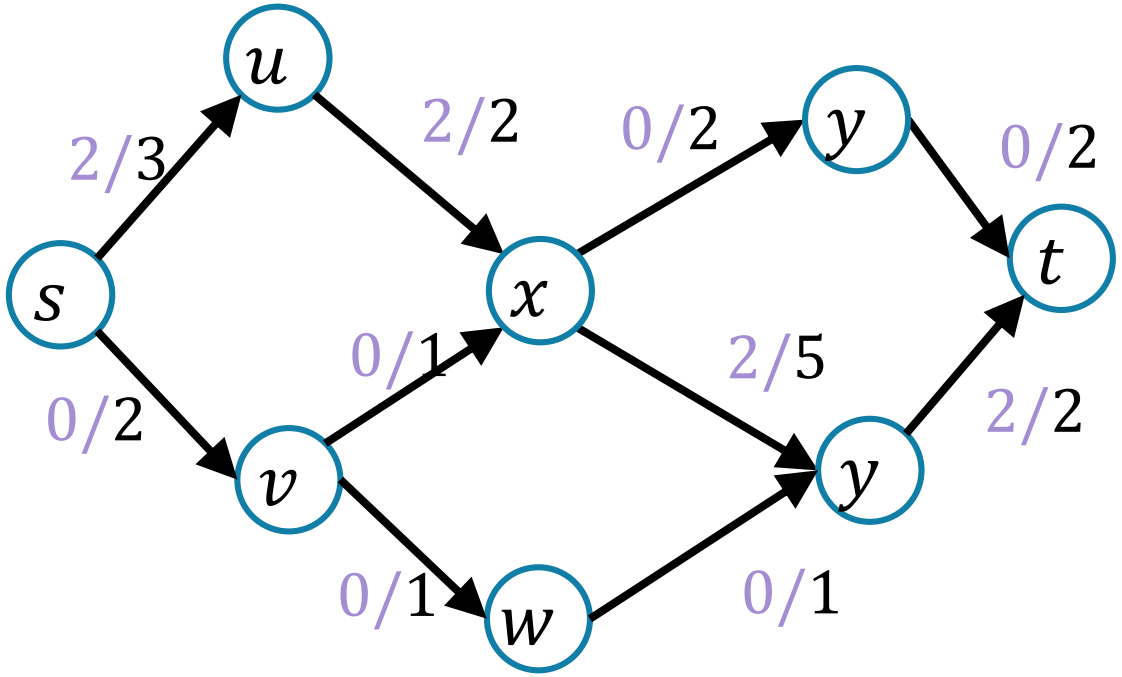


Update flow and residual

Example

Flow graph



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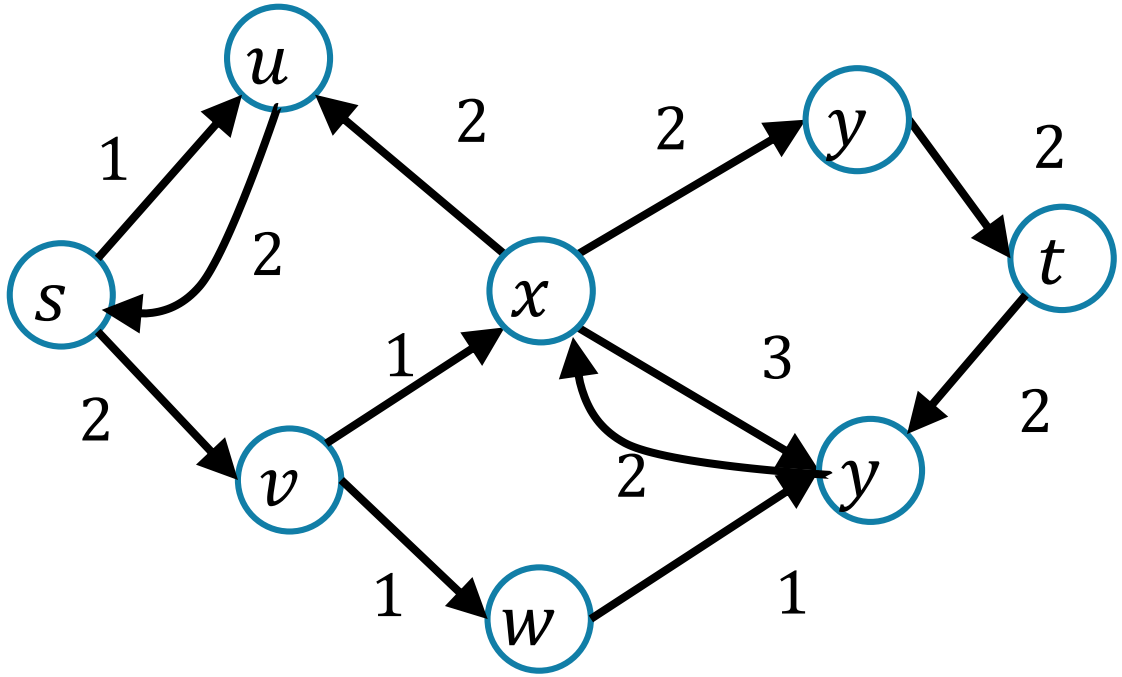
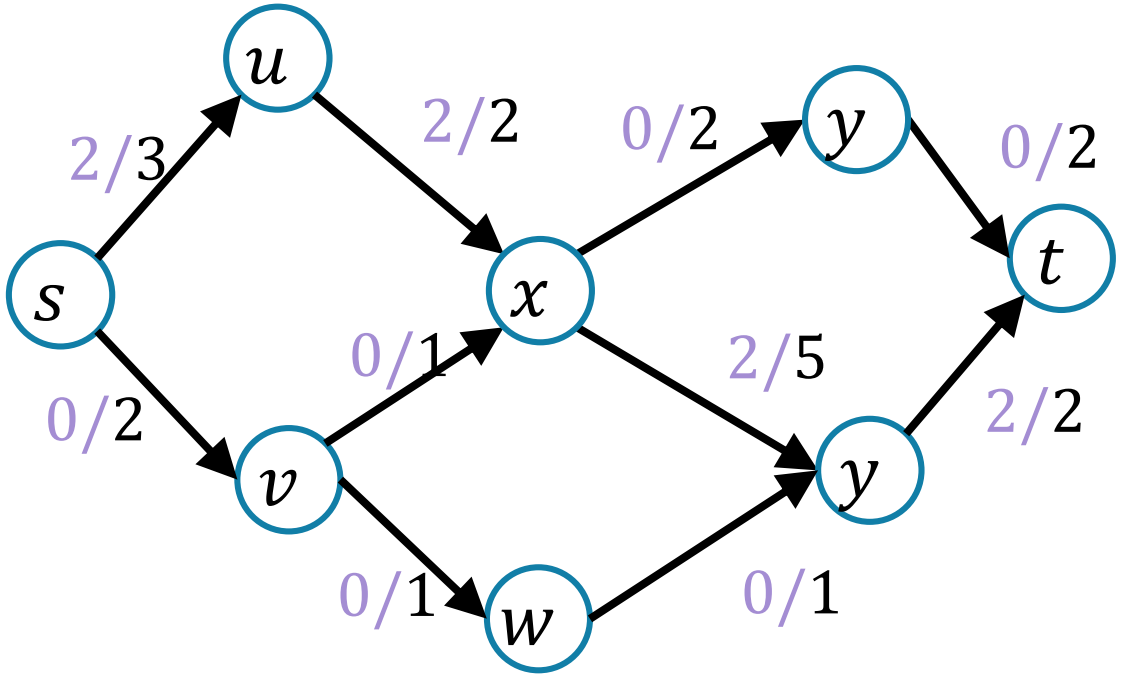


Find another path

Example

Flow graph



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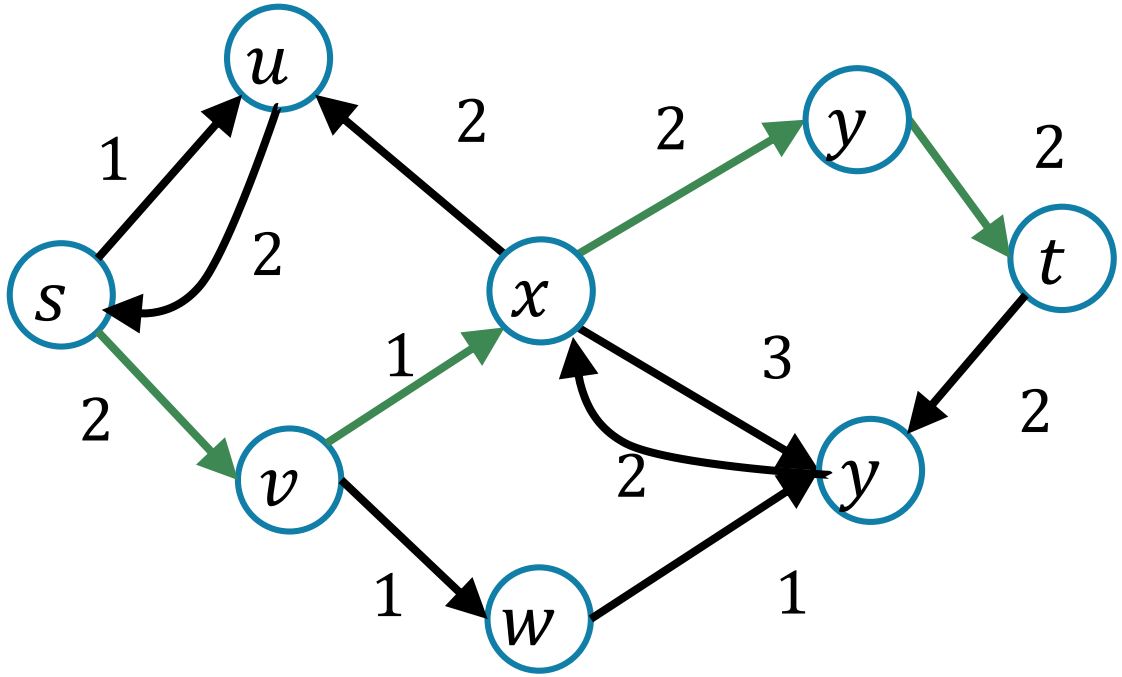
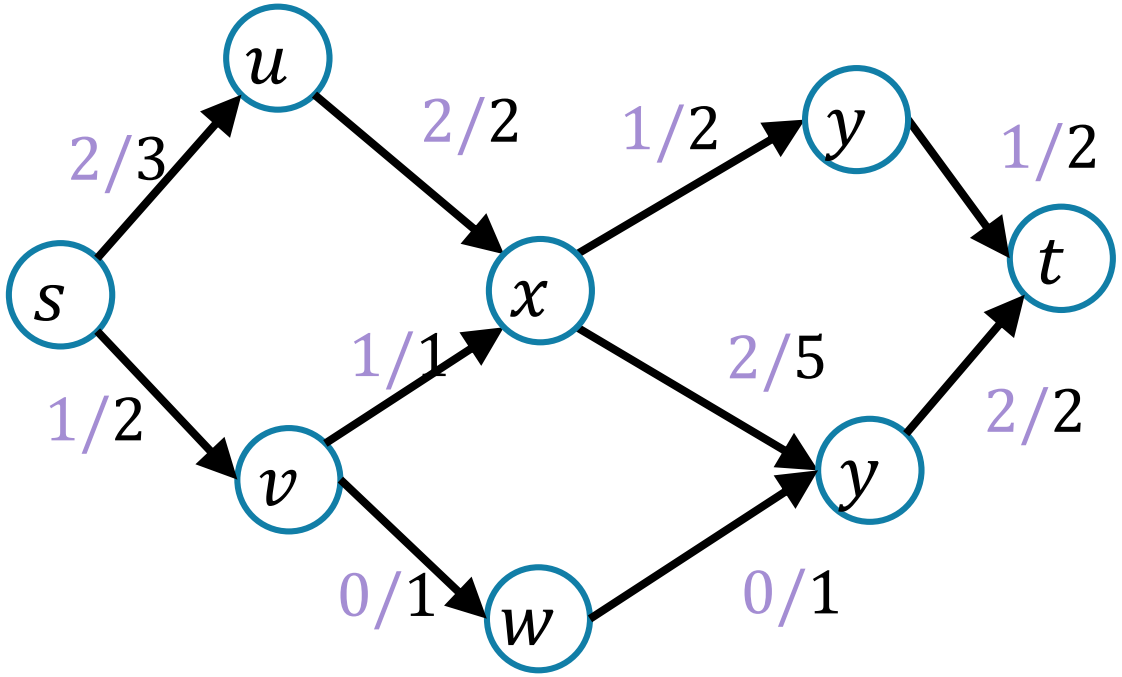


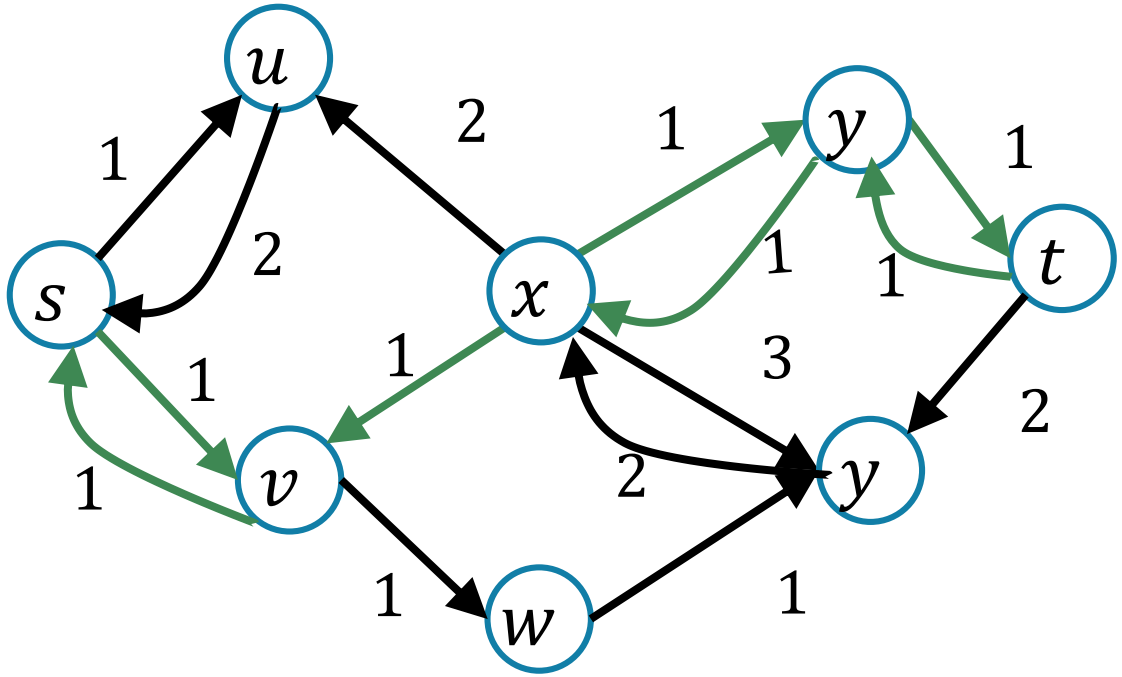
Find another path and update the flow by min (residual) capacity, which is 1

Example

Flow graph



Residual graph



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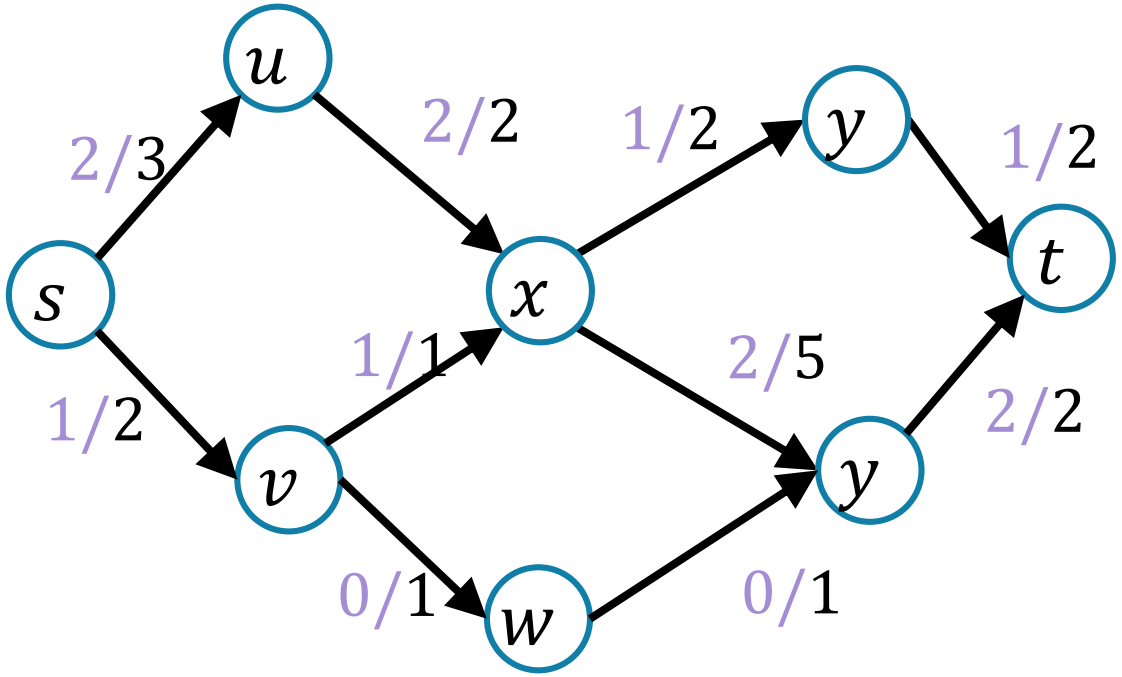

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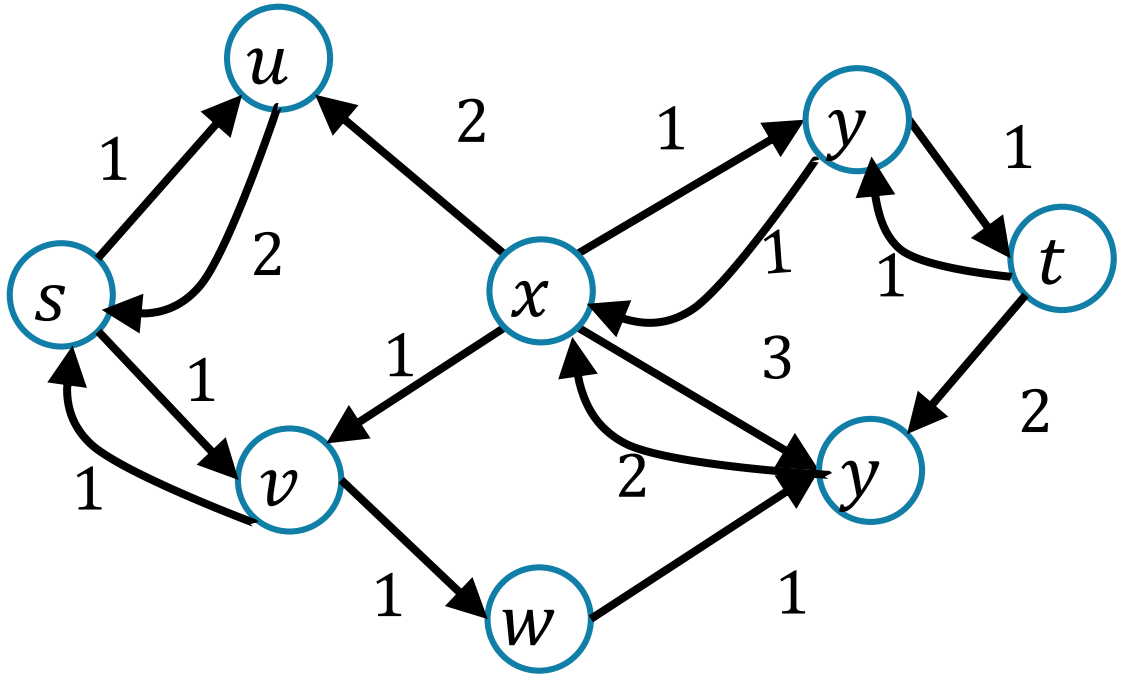
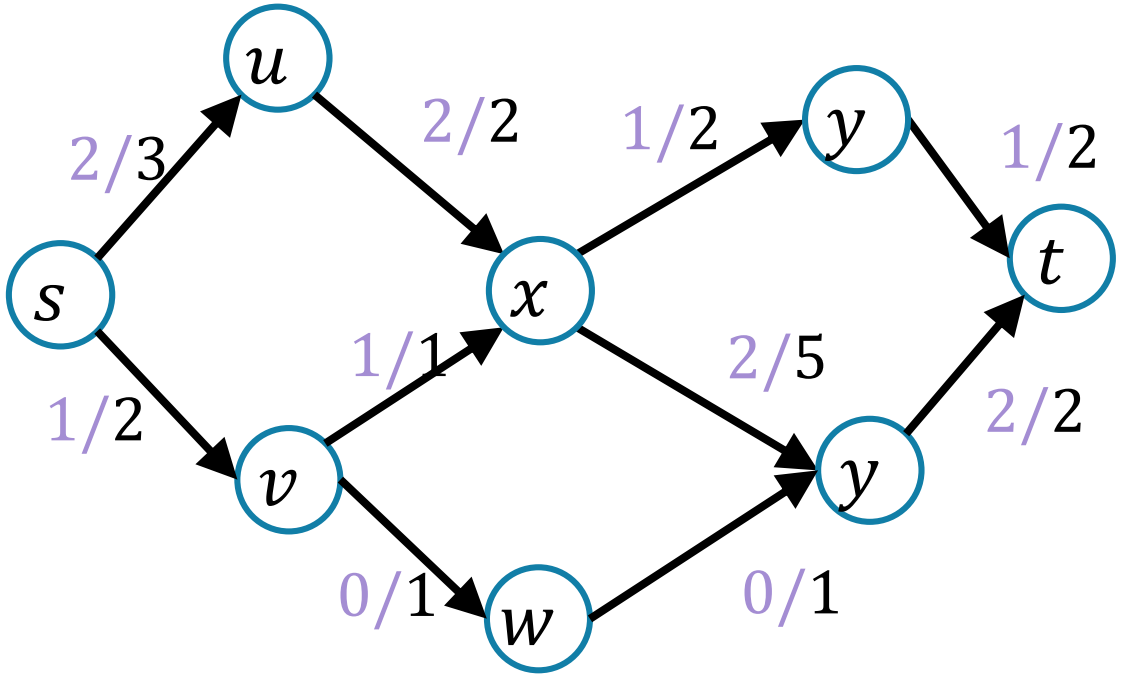


Update completed, let's do another…

Example

Flow graph



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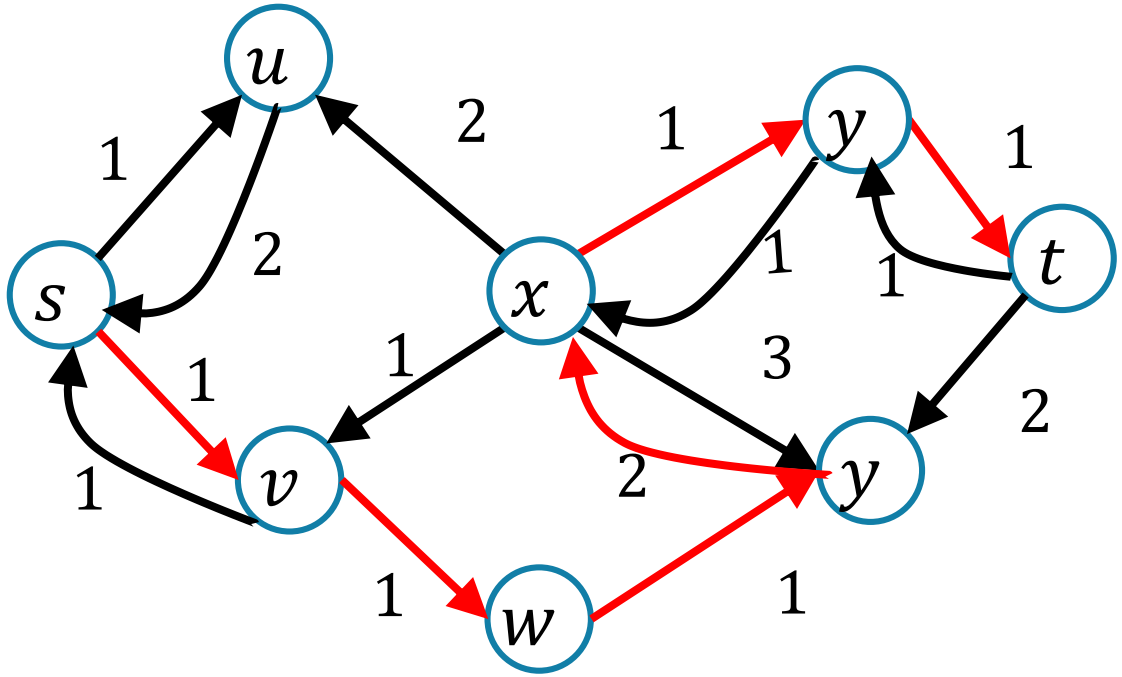
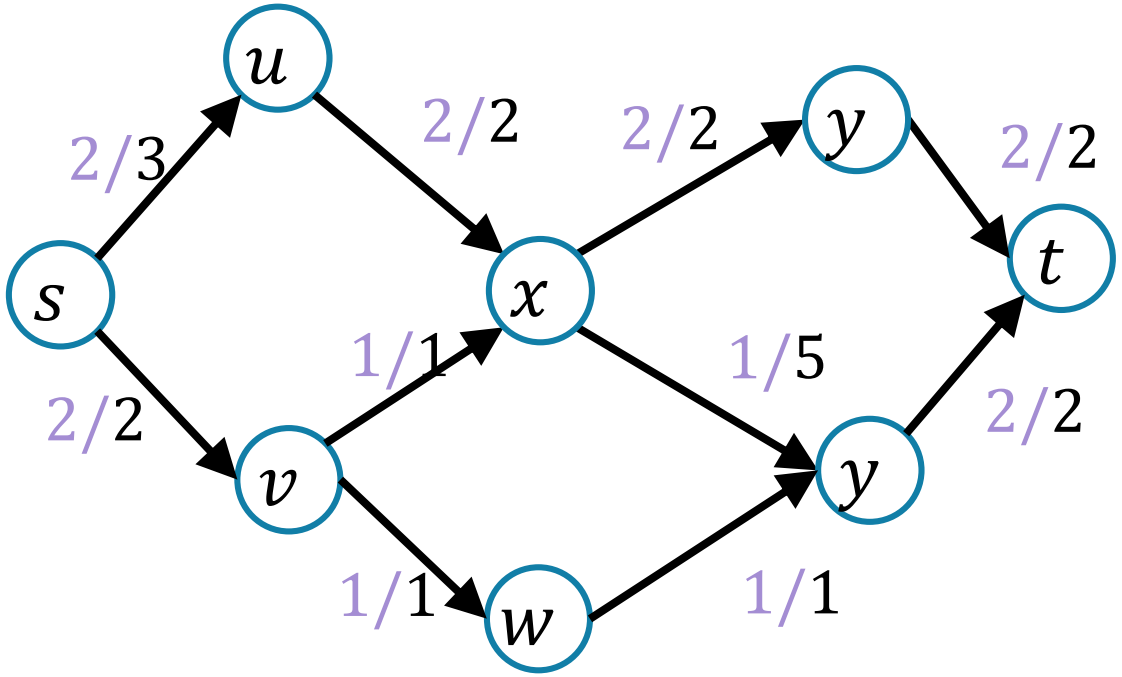


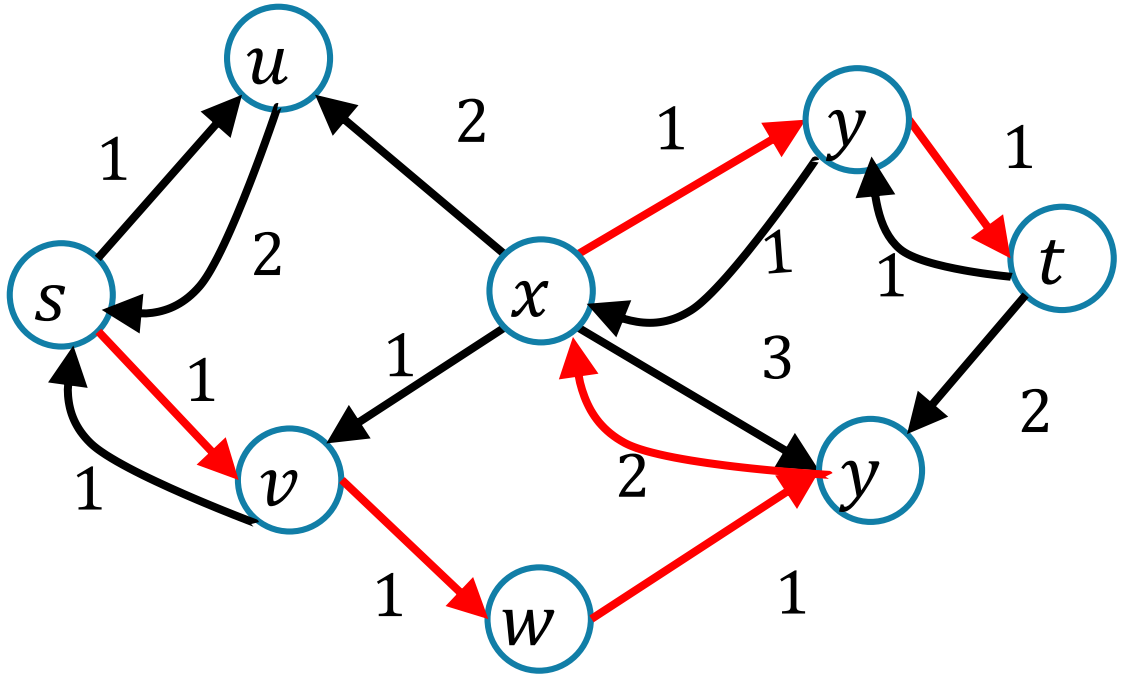
Find another path!

Example

Flow graph



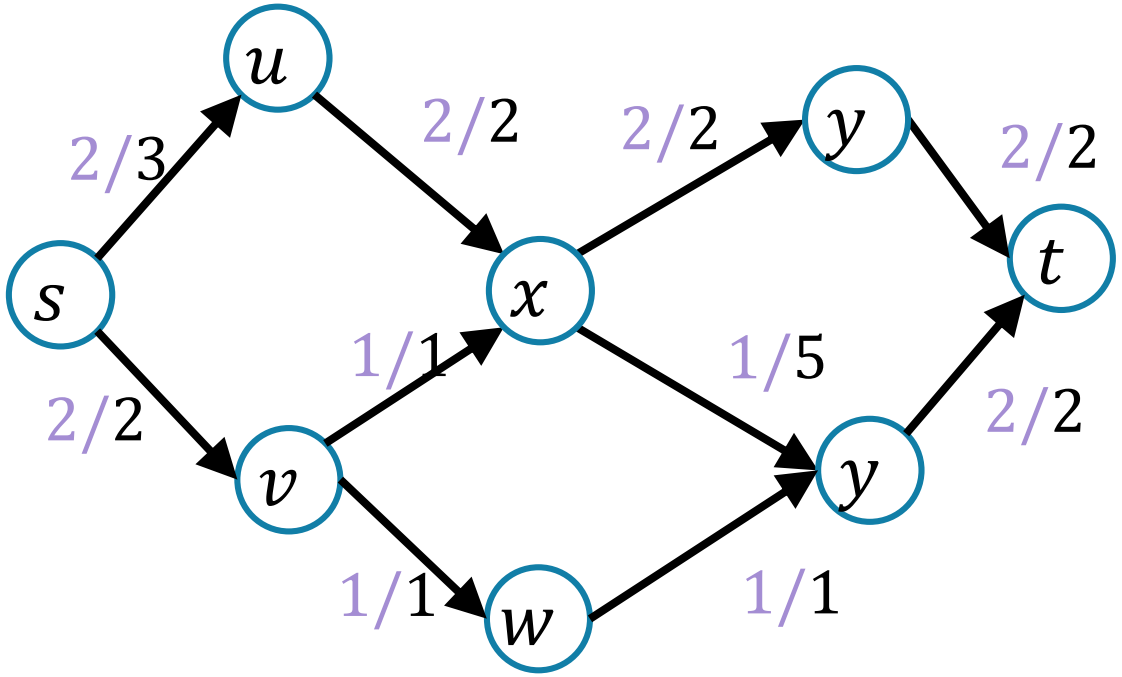
Residual graph



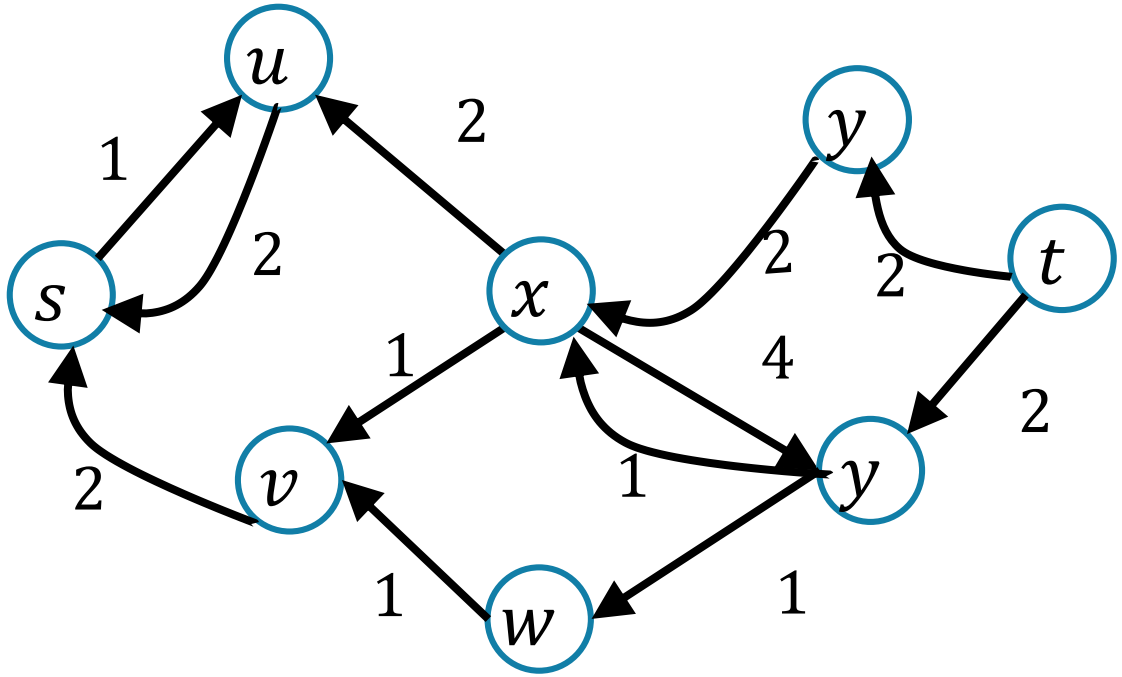
Find another path! Augment by the minimum redisual capacity (1).
When residual edge is “wrong direction” decrease in flow graph.

Example

Flow graph



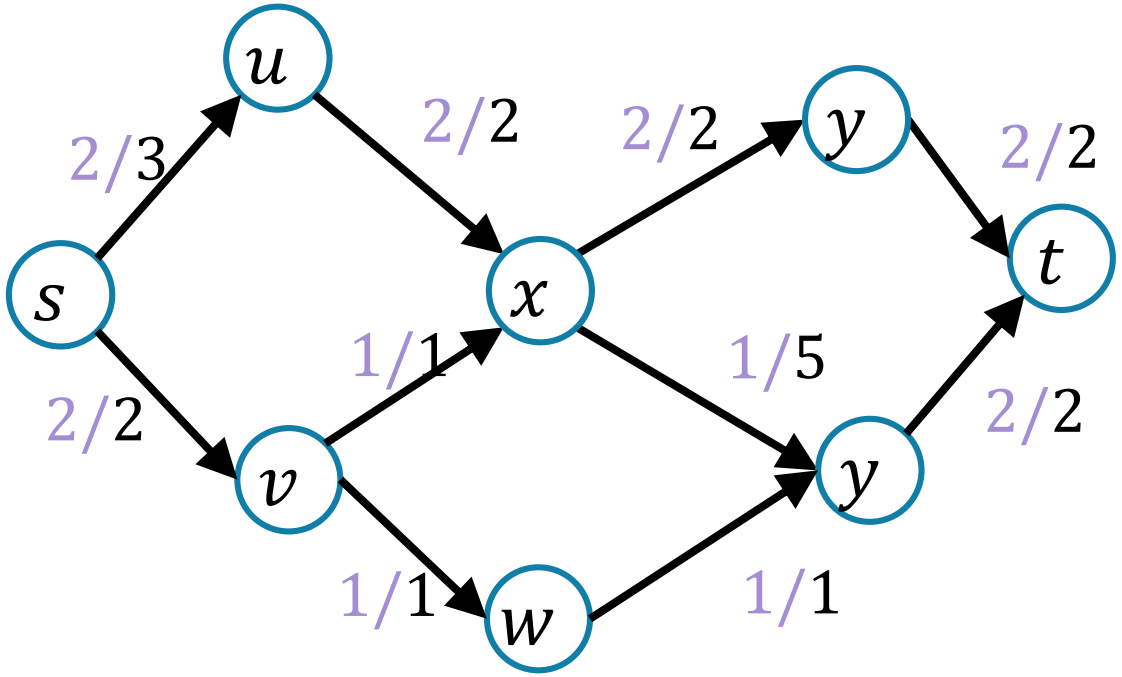
Residual graph



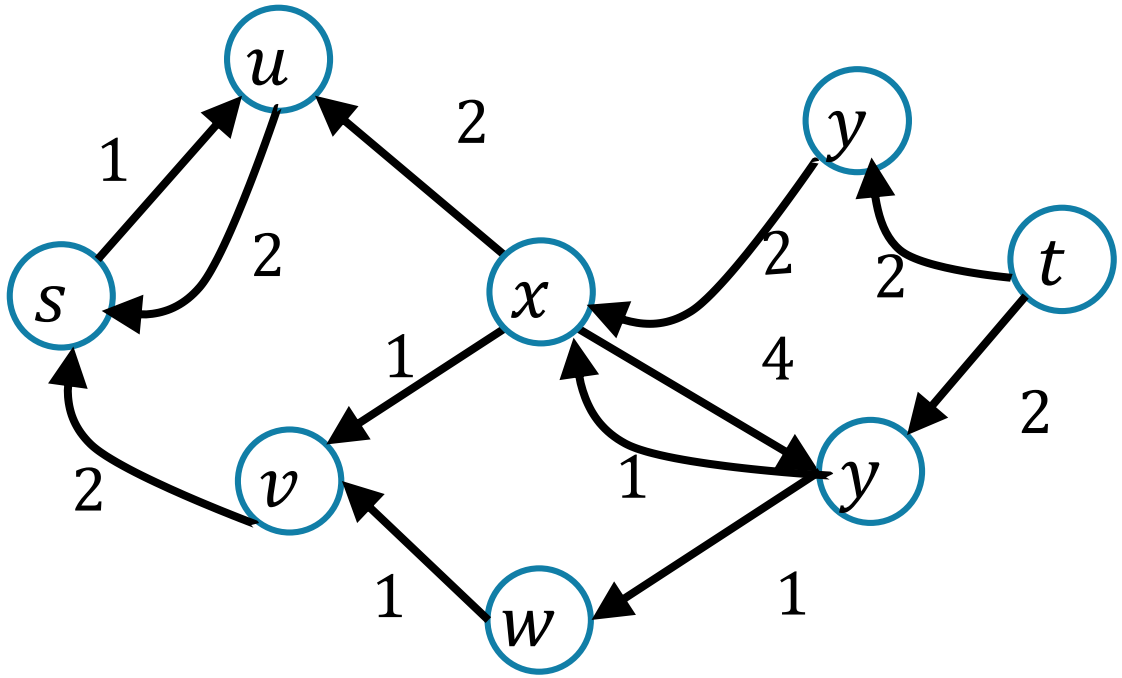
Then update residual

Example

Flow graph



Residual graph



No more s-t path in the residual. We're done!

Ford-Fulkerson Algorithm

While(flow is not maximum)

Run BFS in residual graph starting from s.

Record predecessors to find an s, t-path

Iterate through path, finding c minimum residual capacity on path.

Add c to every edge on path in flow

Update residual graph

Ford-Fulkerson Algorithm

While(true)

Run BFS in residual graph starting from s.

Record predecessors to find an s, t-path

If you don't reach t, break. //otherwise you can still augment

Iterate through path, finding c minimum residual capacity on path.

Add c to every edge on path in flow

Update residual graph

Assuming $E \geq V$ (isolated vertices won't affect the flow, so this is reasonable).

Running Time:? $O(E)$ per iteration. Number of iterations?

Ford-Fulkerson Algorithm

If we have all integer capacities at the start…

The residual graph will always have integer capacities.

Why? The minimum capacity on the first path is an integer.

So we subtract or add integers to the residual graph.

And the result is more integers!

So in every iteration we add at least 1 unit of flow!

Ford-Fulkerson Algorithm

While(true)

Run BFS in residual graph starting from s.

Record predecessors to find an s, t-path

If you don't reach t, break. //otherwise you can still augment

Iterate through path, finding c minimum residual capacity on path.

Add c to every edge on path in flow

Update residual graph

Running Time:? $O(E)$ per iteration. Number of iterations? $O(f)$, where f is the value of the maximum flow. Total $O(Ef)$

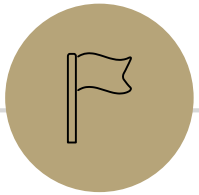
Wait… f?

We haven't seen a running time before that depends on the **answer**

Normally, we want the running time directly in terms of the input.

There are tricks to speed up Ford-Fulkerson so you don't take too long if f is really big.

We'll briefly discuss next time.



Minimum Cut

What's a Cut?

For directed graphs (like we have here)

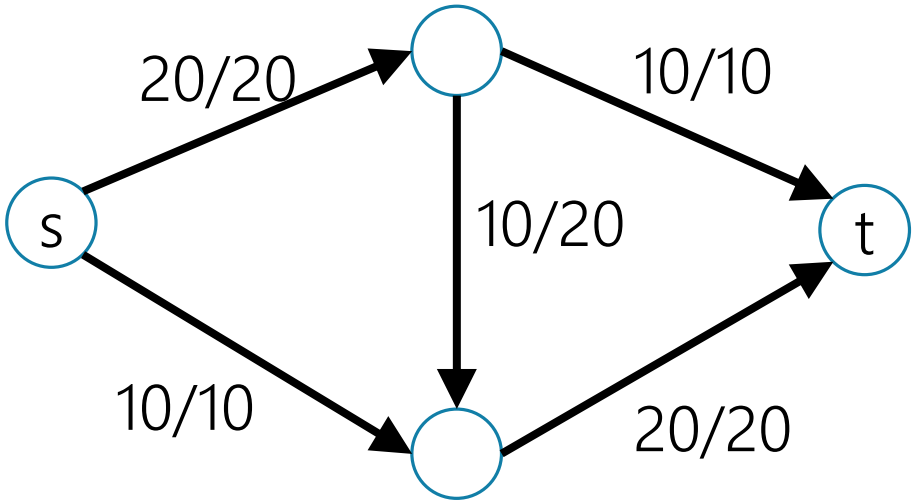
An (s, t)-cut, is a split of the vertices into two sets (S, T)

So that s is in S, t is in T,
and every other vertex is in exactly one of S and T.

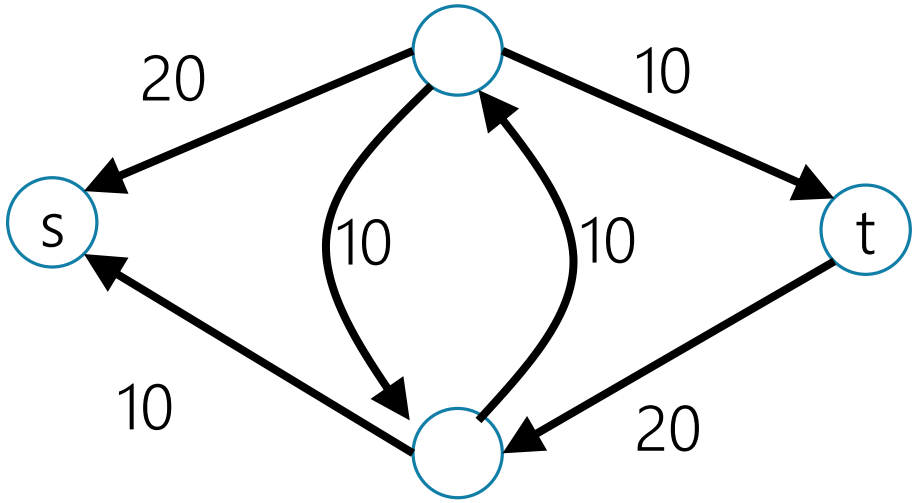
The capacity of a cut (or size of a cut) is the capacity of the edges going from S to T (don't count capacity from T to S).

An Example

Graph, with flow



"Residual Graph"

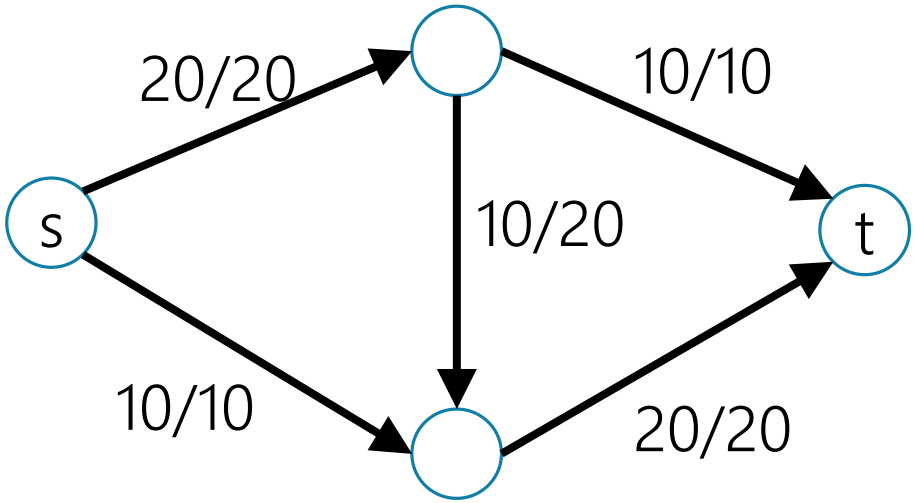


We can't get from s to t anymore in the residual graph.
We're done! That's a maximum flow.

But…why?

An Example

Graph, with flow



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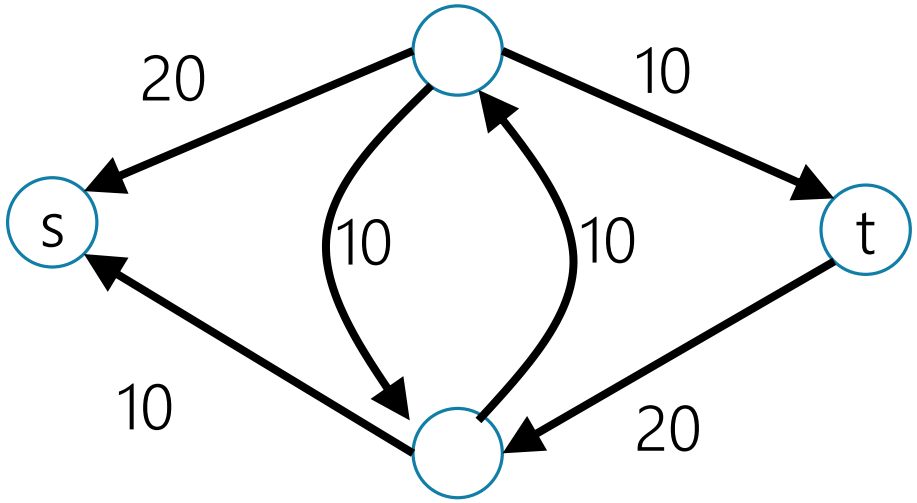


Cut is $(s, V - s)$ (i.e. s on one side, everything else on the other)

Edges from s to everything else? Capacity $\mathbf{30}$
We can't get more than $\mathbf{30}$ units of flow from s to t. Because it all (simultaneously) must cross from one side to the other.

How Do We Know?

How do we know the algorithm is done?

When we can't we get from s to t?

We've **cut** the vertices by looking at the residual graph.
s and all the vertices you can reach from it on one side
and t and all the vertices s can't reach on the other.

Take a look at the edges spanning the cut in the **original** graph.

In our first graph, that capacity was equal to the value of the max flow.

Finding the min-cut

Maintain the residual graph.

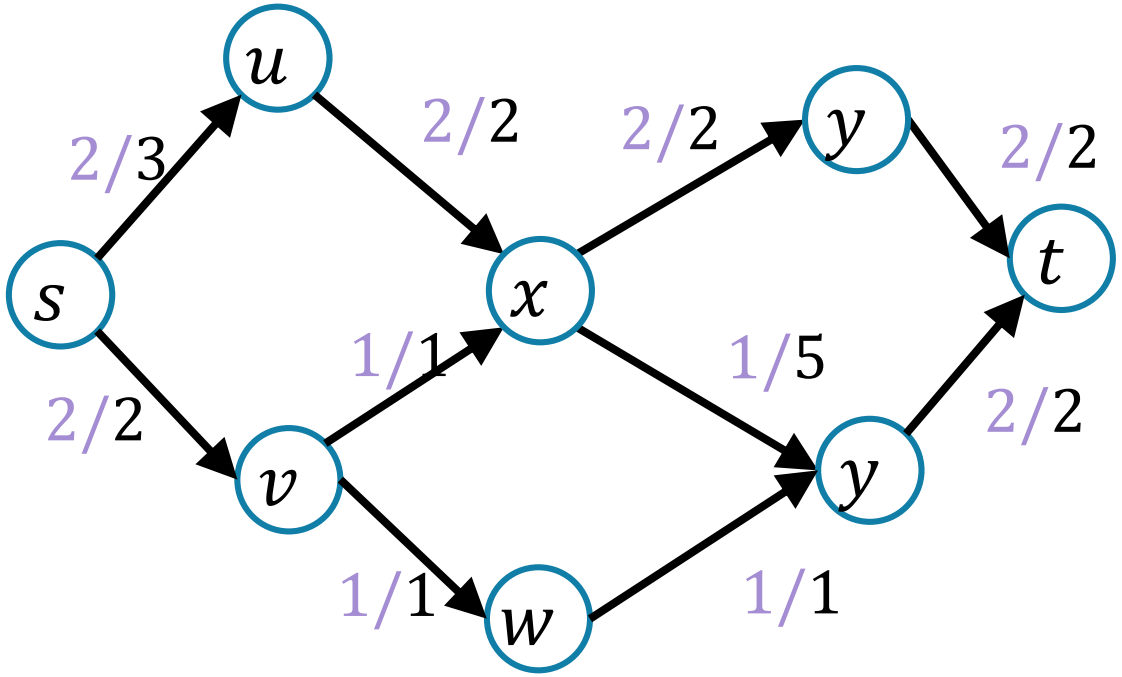
When you search from s and can't get to t:

s and everything you can reach from s is on one side of the cut

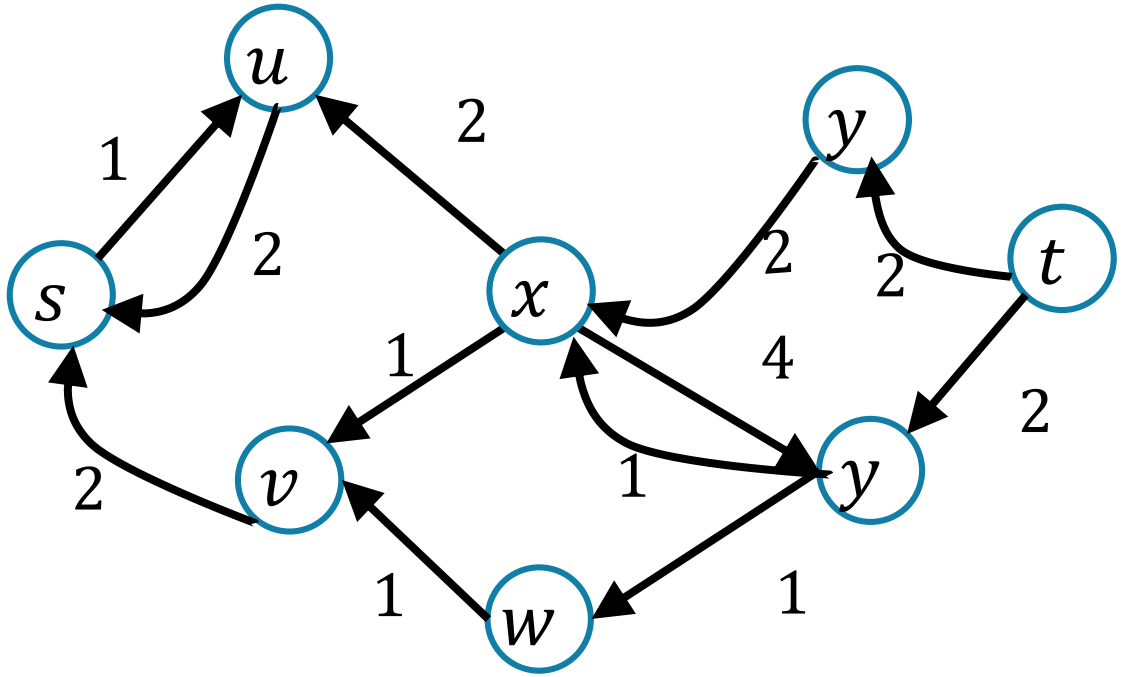
t and everything you can't reach from s is on the other side.

Example

Flow graph



Residual graph

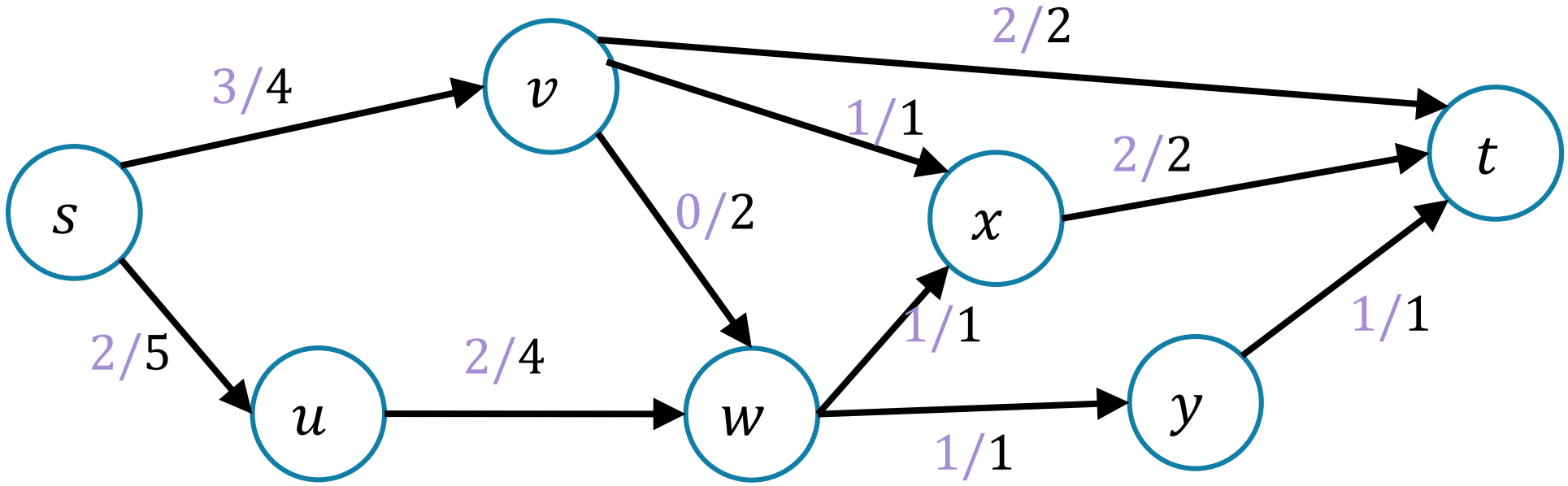


$(\{s, u\}, \{v, x, w, y, z, t\})$ is the cut.
Cut edges are $(u, x), (v, x), (w, x)$. Capacity is $2 + 1 + 1 = 4$.

Another Example

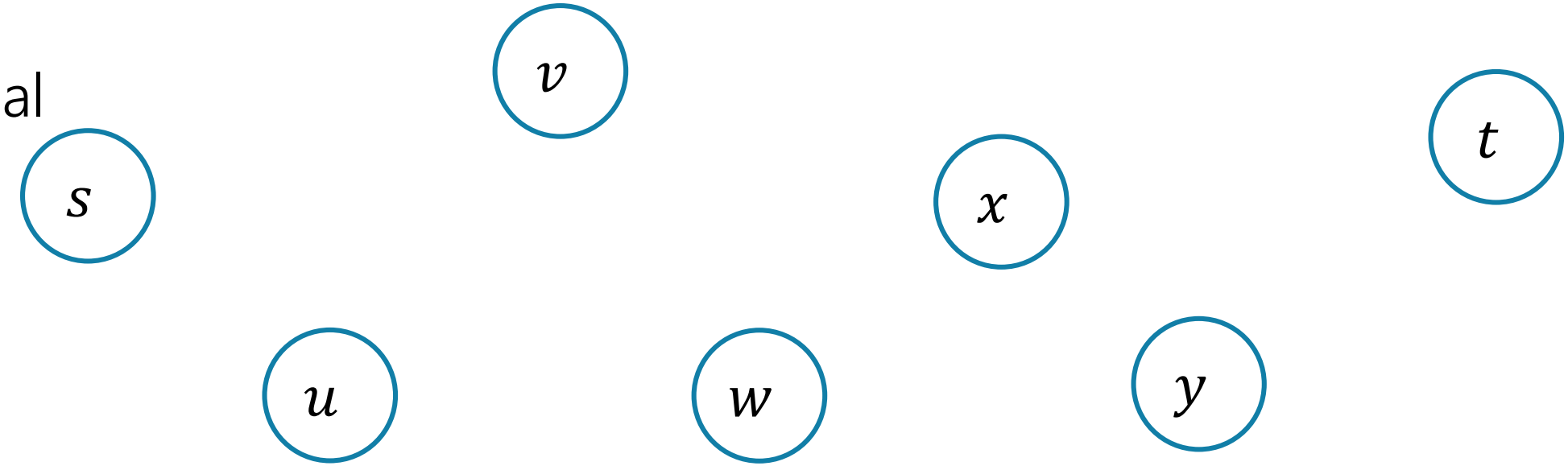
We started lecture with this flow.

What's the residual graph? What's the cut?

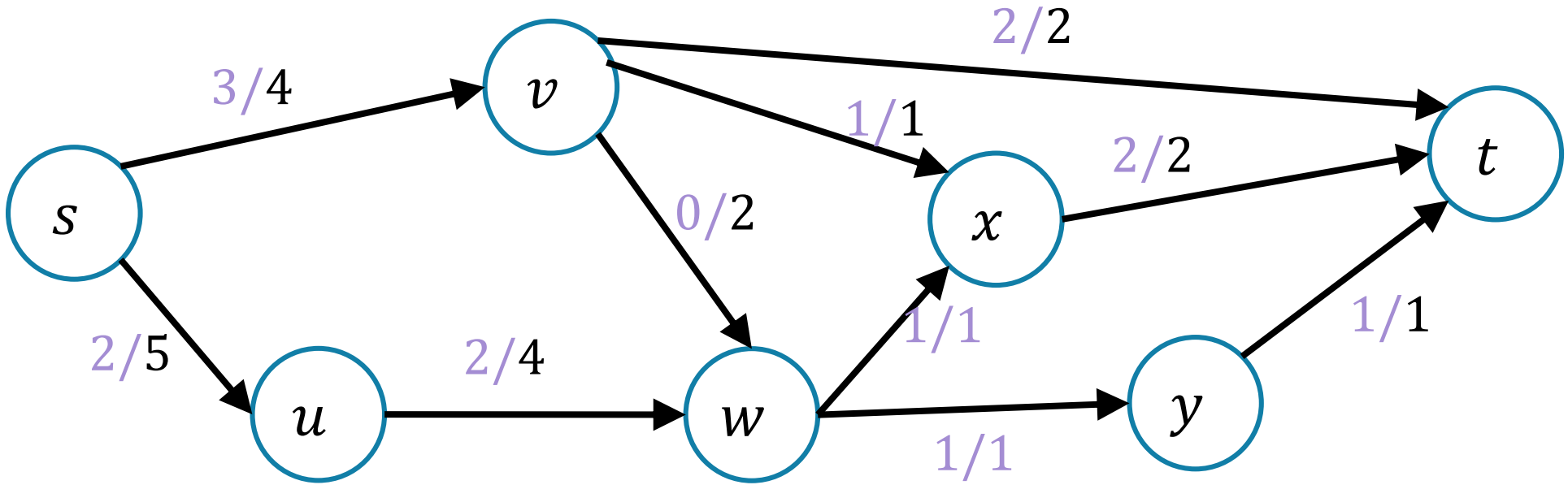


Another Example

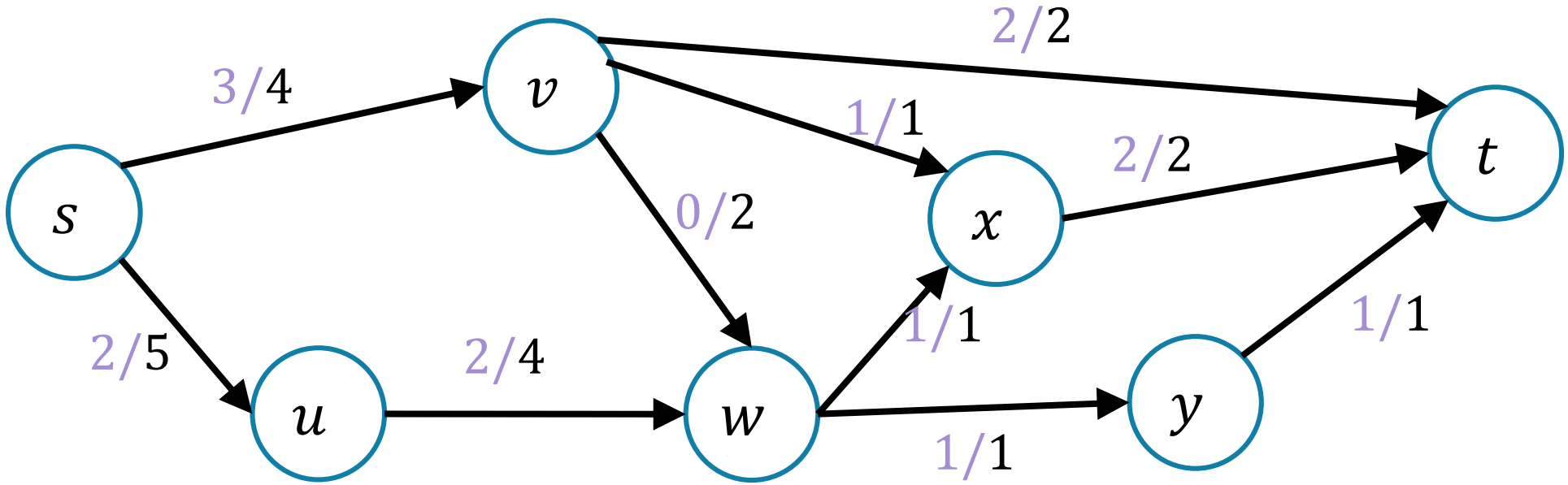
Residual



Flow



Another Example



$(\{s, u, v, w\}, \{x, y, t\})$ is the cut in the residual graph.

What edges span?

$(v, x), (v, t), (w, x), (w, y)$

Total capacity? 5. What's the value of the flow? 5

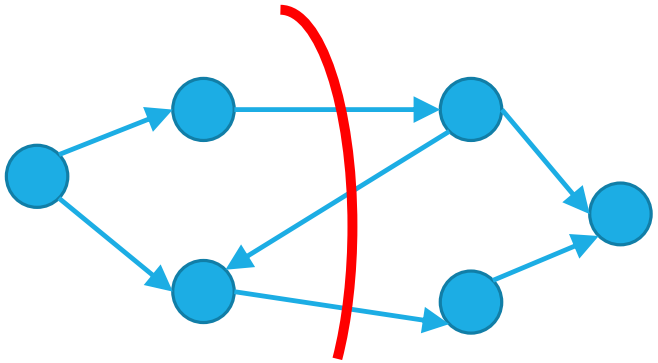
Max Flow-Min Cut Theorem

Max-Flow-Min-Cut Theorem

The value of the maximum flow from s to t is equal to the value of the minimum cut separating s and t.

The full proof is VERY notation heavy.
Focus on the words and intuition. The notation is there to support your intuition; the notation is not the main point.
We're going to skip a few steps for the sake of minimizing notation. See any textbook for all the details.

Some notation (more formally)



Let f be a flow.

For an edge e, $f(e)$ is the flow on e.

$\text{val}(f)$ is the sum of flow leaving s (equivalently entering t).

For a cut (A, B), $\text{cap}(A, B) = \sum_{e:e=(u,v),u\in A,v\in B} c(e)$

i.e., the sum of the capacities on edges going from A to B.

Direction matters!

Notice the capacity of a cut is independent of any particular flow. It's a property of the **original** graph, not the flow or the residual graph.

Step 1: The Flow Goes Somewhere

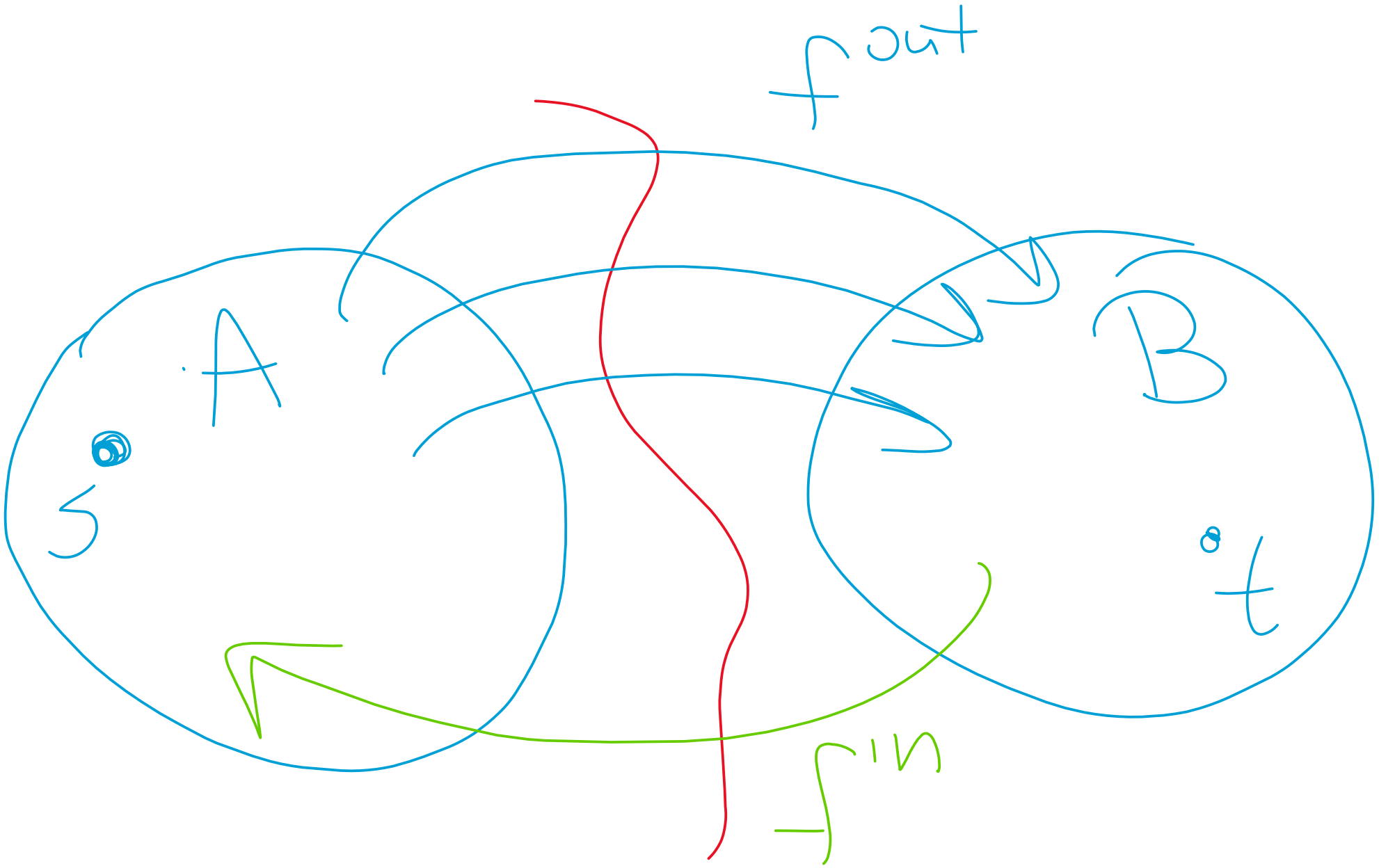
For every s-t cut, (A, B):

$$\mathrm{val}(f) = f^{out}(A) - f^{in}(A) = \sum_{e=(u,v):u\in A,v\in B} f(e) - \sum_{e=(v,u):u\in A,v\in B} f(e)$$

Intuitively, the net-flow for *every* cut is the same as the net flow for the cut $(s, V \setminus \{s\})$.

Why? Well the flow has to go somewhere! It can only disappear at t.

Why care? It's a technical observation we'll need later.


f^{out}
A
B
s
t
f^{in}

Step 2: Cuts limit flows ('weak duality')

Let f be any s-t flow, and (A, B) be any s-t cut.
Then $\text{val}(f) \leq \text{cap}(A, B)$

Cuts limit flows! Intuition: to get the flow to t it has to "all get through" every cut. So you can't have a flow of value more than any given cut.

Proof:

$$\text{val}(f) = f^{out}(A) - f^{in}(A) \leq f^{out}(A) = \sum_{e=(u,v):u\in A,v\in B} f(e)$$

$$\leq \sum_{e=(u,v):u\in A,v\in B} c(e) = \text{cap}(A, B)$$

Step 3: Cuts are the only things that limit flows

Let f^* be an s-t flow such that there is no s-t path in the residual graph.
Then there is a cut (A^*, B^*) such that $\text{val}(f^*) = \text{cap}(A^*, B^*)$

Intuition: going from $A^* \to B^*$, you're saturated; $B^* \to A^*$ is unused.

Sketch:

Let A^* be all the vertices reachable from s in the residual graph, and $B^* = V \setminus A^*$.

Observe that (A^*, B^*) is indeed an s-t cut. The only way to not be a cut is to have $t \in A^*$. But we assumed t was not reachable from s.

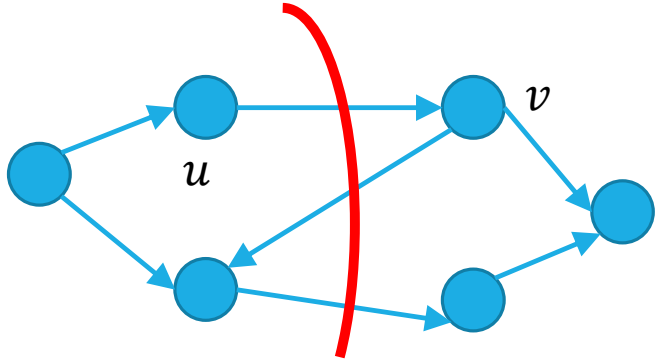
Step 3: Cuts are the only things that limit flows

Let f^* be an s-t flow such that there is no s-t path in the residual graph. Then there is a cut (A^*, B^*) such that $\text{val}(f^*) = \text{cap}(A^*, B^*)$

(sub)-claim: If $e = (u, v)$ such that $u \in A^*$, $v \in B^*$, then $f(e) = c(e)$.

(i.e. edges going from A^* to B^* are saturated).

In the residual graph, we only don't have a copy of e if e is saturated. If we did have the edge e, we would be able to reach v from s, and it would be in A^*, not B^*. So e must be saturated.



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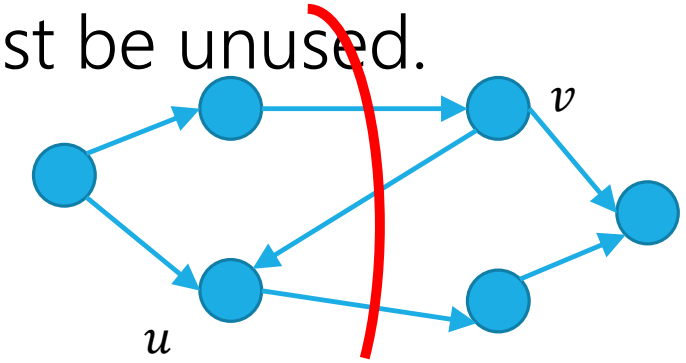
(sub)-claim: If $e = (v, u)$ such that $u \in A^*$, $v \in B^*$, then $f(e) = 0$.

(i.e. edges going from B^* to A^* are unused).

In the residual graph, we add a copy of (u, v) when there is any flow on (v, u). If we did have the edge (u, v), we would be able to reach v from s, and it would be in A^*, not B^*. So $e = (v, u)$ must be unused.



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Let f^* be an s-t flow such that there is no s-t path in the residual graph.
Then there is a cut (A^*, B^*) such that $\text{val}(f^*) = \text{cap}(A^*, B^*)$

Put it together: What's the value of the flow?

$$\text{val}(f^*) = f^{out}(A^*) - f^{in}(A^*)$$

Step 1's lemma

$$= \sum_{e=(u,v):u\in A,v\in B} f(e) - \sum_{e=(v,u):u\in A,v\in B} f(e)$$

Net is flow out minus flow in.

$$= \sum_{e=(u,v):u\in A,v\in B} c(e) - \sum_{e=(v,u):u\in A,v\in B} 0$$

Last 2 slides

$$= \text{cap}(A^*, B^*)$$

Definition of capacity.

Concluding The Theorem

Max-Flow-Min-Cut Theorem

The value of the maximum flow from s to t is equal to the value of the minimum cut separating s and t.

Proof: Run Ford-Fulkerson, you'll get a flow of value f^* such that there is a cut of capacity f^*. There can be no larger flow and no smaller cut, as for all flows f and all cuts (A, B): $\mathrm{val}(f) \leq \mathrm{cap}(A, B)$.

Isn’t This Cool?

Another instance where we prove a big theorem using an algorithm.

The max-flow min-cut theorem doesn’t mention an algorithm, but it can be proved via analyzing Ford-Fulkerson!

So What?

Max-flow and min-cut are each interesting algorithmic problems.

They were first studied in the 1950s

The U.S. military wanted to know how much the Soviets could ship on their rail network.

And also which rail lines they would target to *disrupt* the network.

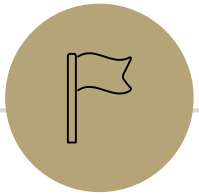
So What?

Great quick check for if you've found the maximum flow (or min-cut).
Check the other and see if the value is the same!

We'll see examples next time of max-flow used for modeling. In those cases the min-cut can be interpreted as a "barrier" to a good assignment.

It's also a nice example of **duality**

In a few weeks we'll see another instance of this phenomenon with LPs.



Applications

Applications of Max-Flow-Min-Cut

Max-Flow and Min-Cut are useful if you work for the water company…

But they're also useful if you don't.

The most common application is assignment problems.

You have jobs and people who can do jobs – who is going to do which?

Big idea:

Let one unit of flow mean "assigning" one job to a person.

Hey Wait…

Isn’t this what stable matching is for?

Stable matching is very versatile, and it lets you encode preferences.

Max-flow assignment is even more versatile on the types of assignments.

But there’s not an easy way to encode preferences.

Example Problem

You and your housemates need to decide who is going to do each of the chores this week.

Some of your housemates are unable to do some chores.

Housemates: 1,2,3

Chores:

Arrange furniture, clean the **B**athroom, **C**ook dinner, do the **D**ishes

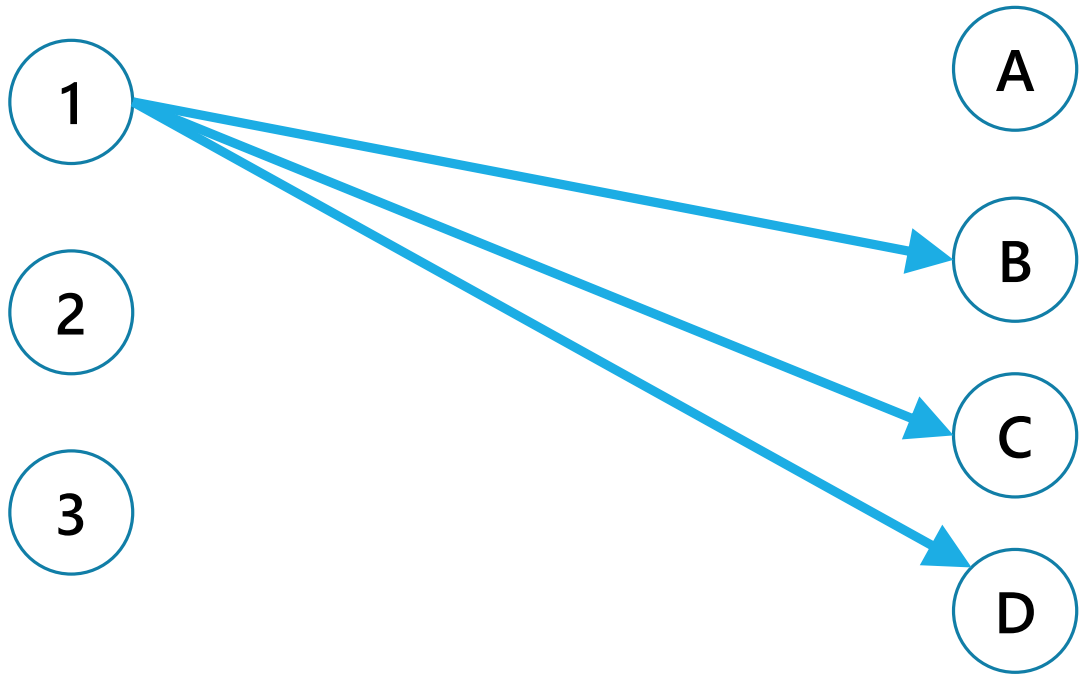
Housemate 1 is unable to arrange furniture, 2 is unable to cook.

Example Problem

Housemate 1 is unable to arrange furniture, 2 is unable to cook.

Vertex for each housemate and chore.

Edge if the housemate **could** do the chore



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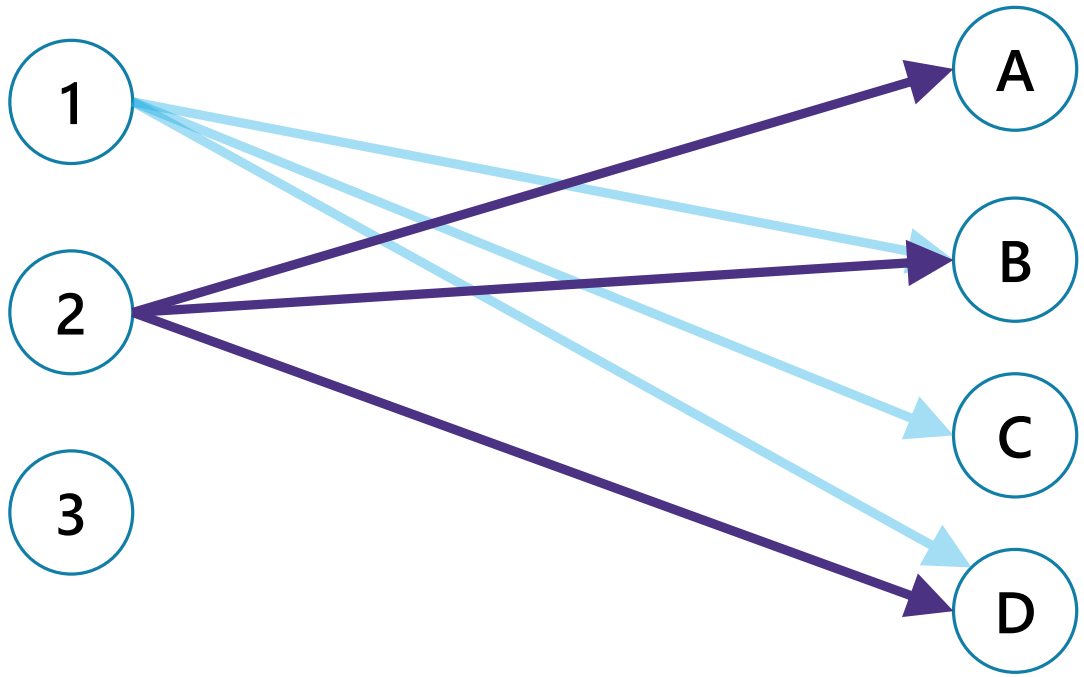

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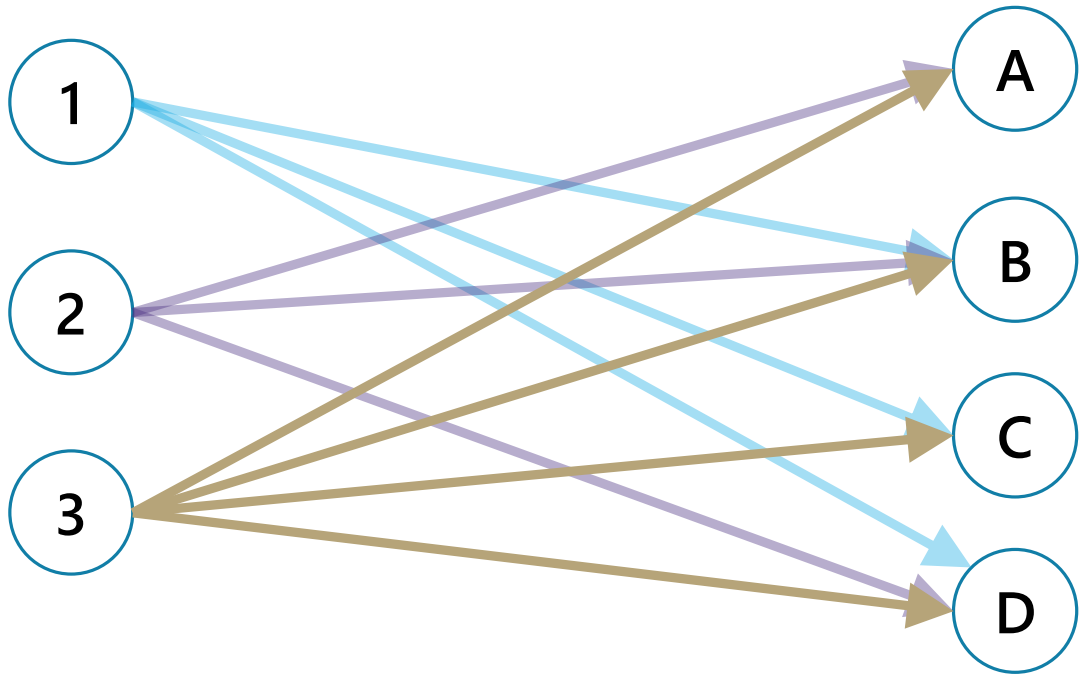

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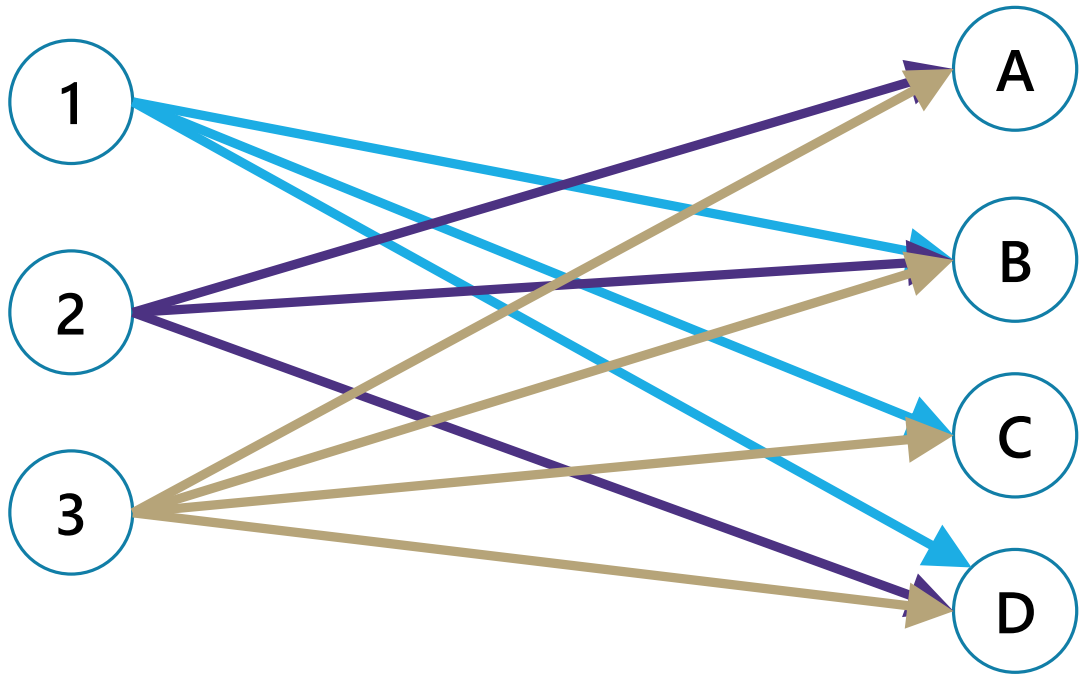
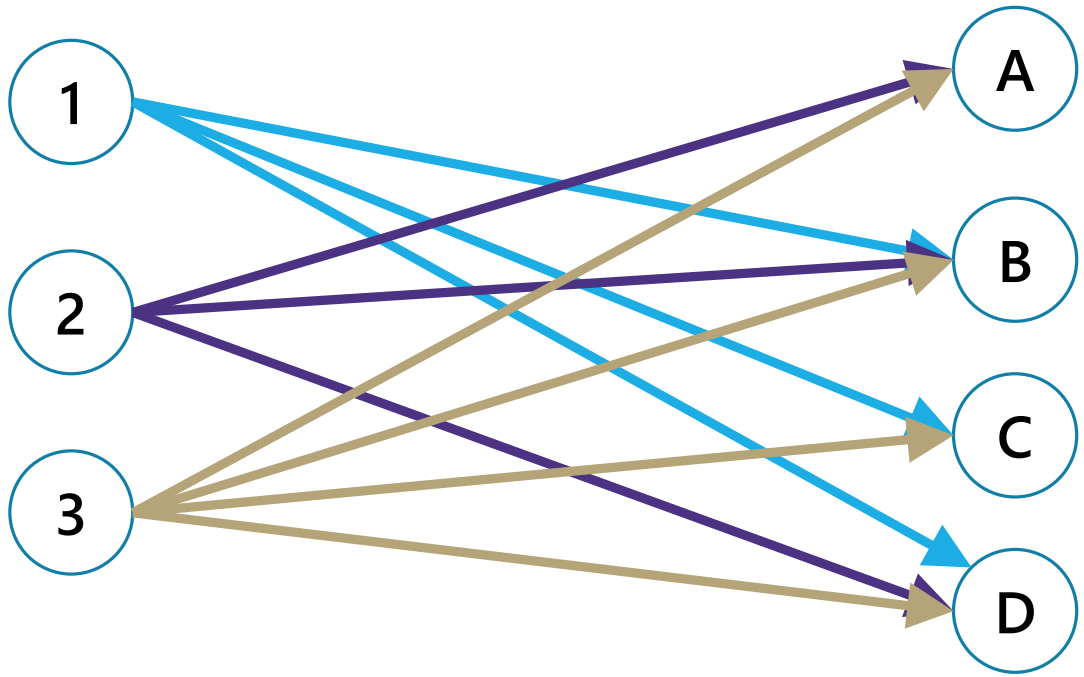


Example Problem

Idea: Flow from 1 to B means "make housemate 1 do chore B."

Every chore needs to be done (by one person).

Every person needs to do at most two chores.



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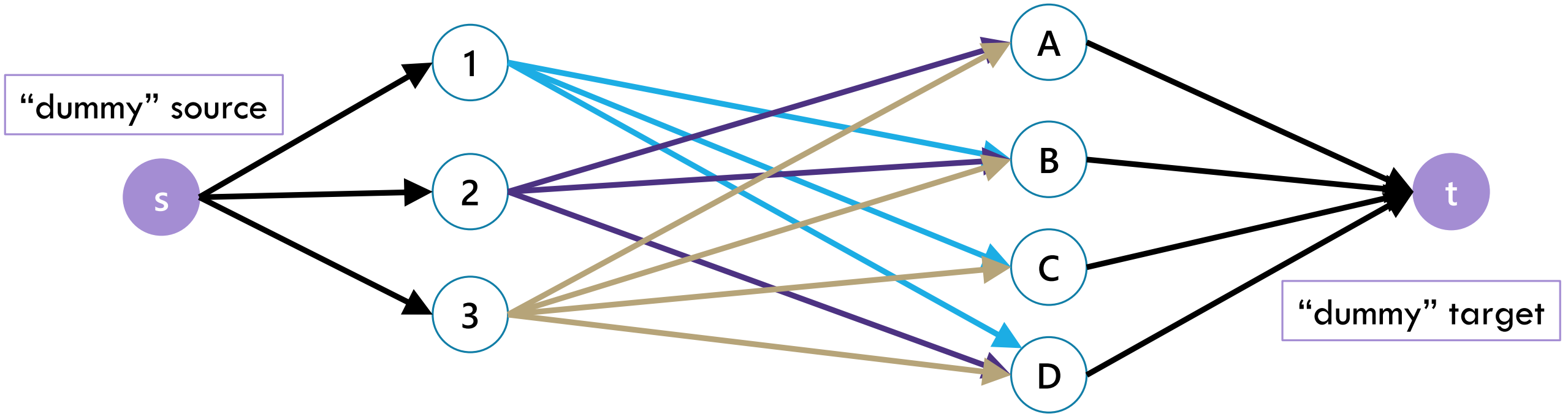

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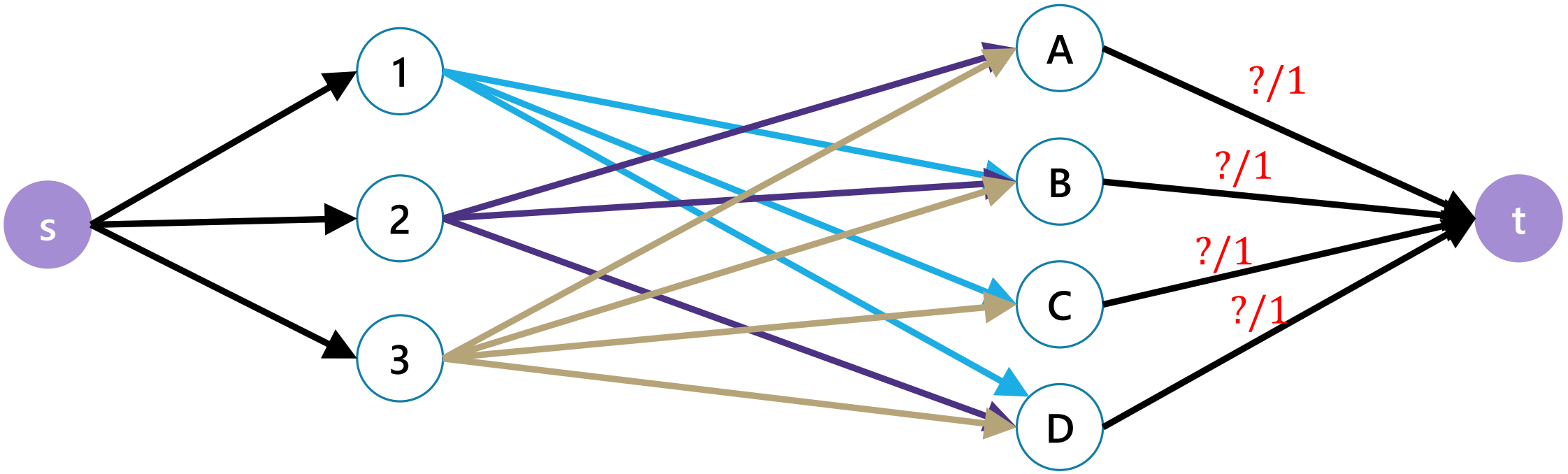

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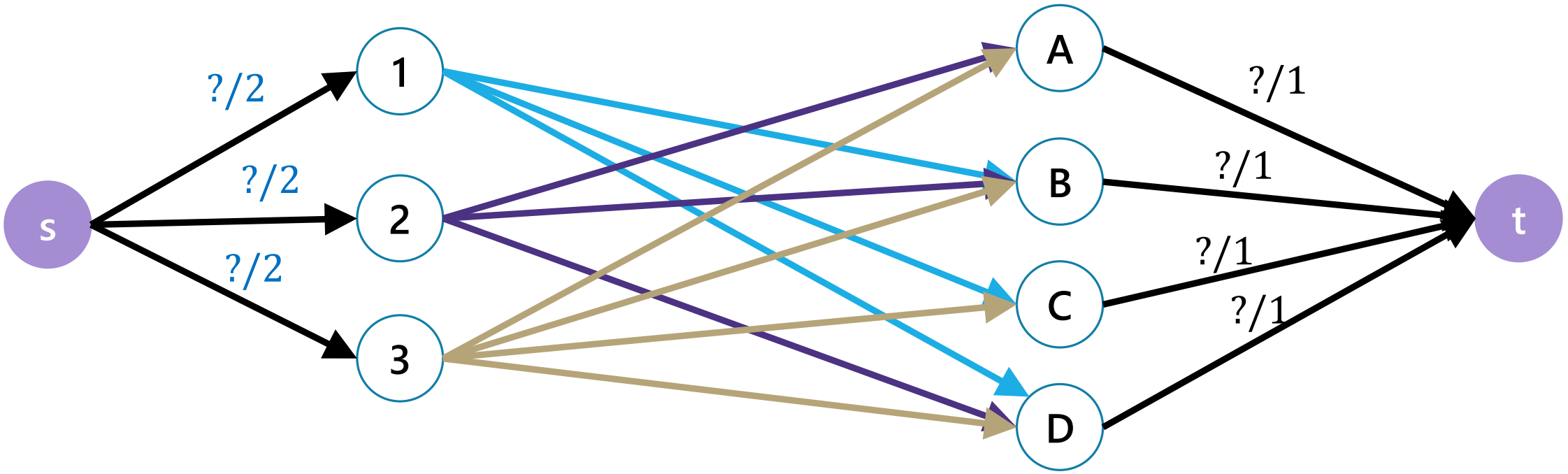
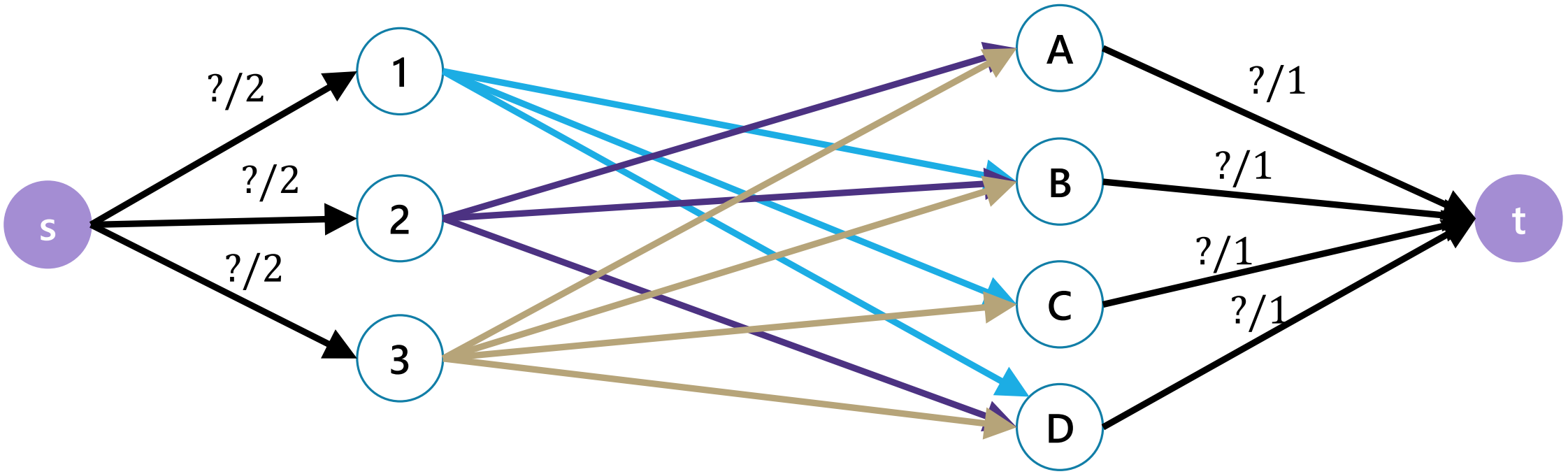


Example Problem

What are the capacities for the middle edges?

Could make them 1 (make sure you don't get "two units of cooking"

All our requirements are already (implicitly) encoded. So could make them ∞ instead.

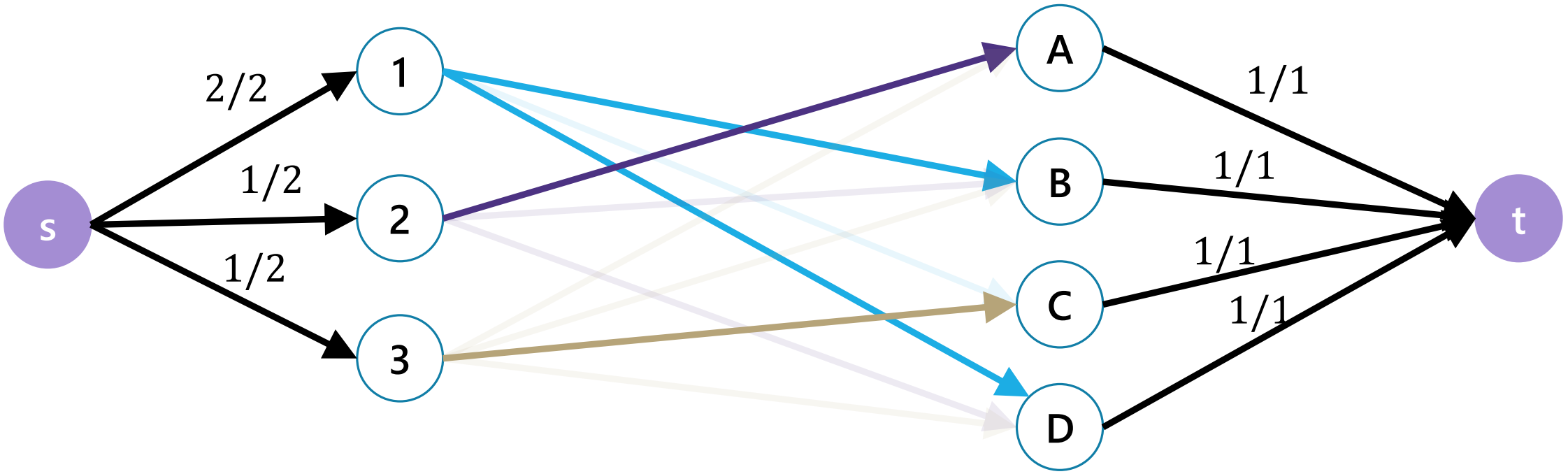


Example Problem

Find a max flow…And read off the assignment!

Full color: 1 unit of flow, faded: 0 units of flow

1 cleans the bathroom and does the dishes, 2 arranges furniture, 3 cooks.



Why are all of our constraints met?

Every chore gets done

No one does more than 2 chores

People only do chores they're capable of

Why are all of our constraints met?

Every chore gets done

A flow of value 4 sends one unit of flow through each of A,B,C,D (because the edges to t are all capacity 1), so a max-flow ensures if possible we'll find an assignment.

No one does more than 2 chores

Only 2 units of flow can go through any person vertex (because edges from s to people are all capacity 2).

People only do chores they're capable of

There is only an edge from a person to a chore if they can do that chore.

One More Requirement…

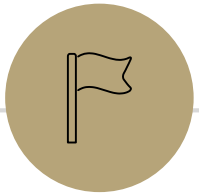
There's another requirement we haven't mentioned:

People only get "whole units" of chores

i.e. you don't have two people each doing half of the cooking.

The max-flow approach guarantees this! As long as our requirements are integers (or ∞) as well.

Same logic as Friday's lecture – Ford-Fulkerson will only add integers to the current flow.



Speeding Up Ford-Fulkerson

Speeding Up Ford-Fulkerson

Ford-Fulkerson is only slow if f is big and we keep doing very small updates.

If instead of finding **any** s, t path you find a particular one (the one with the fewest edges, or the one that will lead to the biggest increase) you can guarantee you'll do fewer iterations.

- Each iteration will take a little longer to find the good path, but the tradeoff is often worth it.
- In particular some algorithms end up with running times independent of f.

In practice, Ford-Fulkerson is usually fine!

The fastest-known (theoretically) is Orlin's algorithm: running time $O(VE)$.