DP on Trees and Graphs

CSE 421 Winter 2023
Lecture 15

Today

Dynamic Programming on Graphs

Getting more complicated: Trees → DAGs → General graphs

We're building up to "Bellman-Ford" and "Floyd-Warshall"

Two very clever algorithms – we won't ask you to be as clever.

Tree and DAG DPs are common; general graph DPs are less common, but these are standard library functions, so good to know.

And deriving them together is good for practicing DP skills.

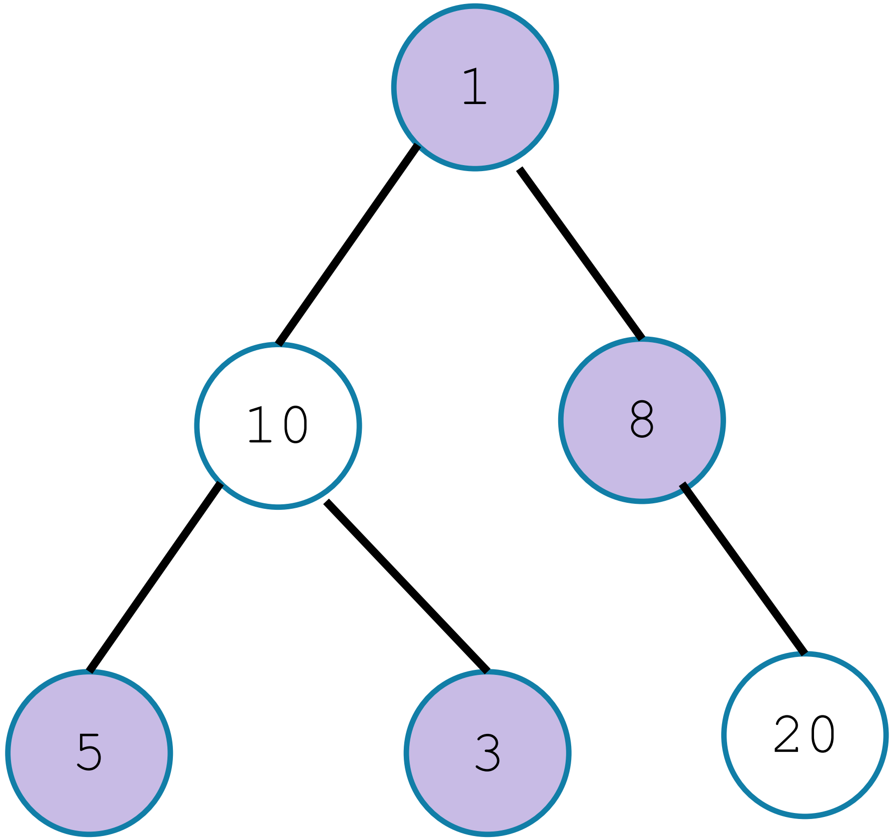
Vertex Cover

Vertex Cover

A set S of vertices is a vertex cover if for every edge (u, v): u is in S, or v is in S, (or both)

Notice, the minimum weight vertex cover might have both endpoints of some edges

Even though only one of 1, 8 is required on the edge between them, they are both required for other edges.



Vertex Cover – Recursively

Let's try to write a recursive algorithm first.

What information do we need to decide if we include u?

If we don't include u then to be a valid vertex cover we need…

If we do include u then to be a valid vertex cover we need…

Vertex Cover – Recursively

Let's try to write a recursive algorithm first.

What information do we need to decide if we include u?

If we don't include u then to be a valid vertex cover we need…

to include **all** of $u's$ children, and vertex covers for each subtree

If we do include u then to be a valid vertex cover we need…

just vertex covers in each subtree (whether children included or not)

Recurrence

Let $OPT(v)$ be the weight of a minimum weight vertex cover for the subtree rooted at v.

Write a recurrence for $OPT()$

Then figure out how to calculate it

Recurrence

$OPT(v)$ – the weight of the minimum weight vertex cover for the tree rooted at v (whether or not v is included).

$INCLUDE(v)$ – the weight of the minimum weight vertex cover for the tree rooted at v where v is included in the vertex cover.

$$OPT(v) = \begin{cases} \min\{\sum_{u:u \text{ is a child of } v} INCLUDE(u), weight(v) + \sum_{u:u \text{ is a child of } v} OPT(u)\} \text{ if } v \text{ is not a leaf} \\ 0 \text{ if } v \text{ is a leaf} \end{cases}$$

$$INCLUDE(v) = weight(v) + \sum_{u:u \text{ is a child of } v} OPT(u)$$

Vertex Cover Dynamic Program

What memoization structure should we use?

What code should we write?

What's the running time?

Vertex Cover Dynamic Program

What memoization structure should we use?

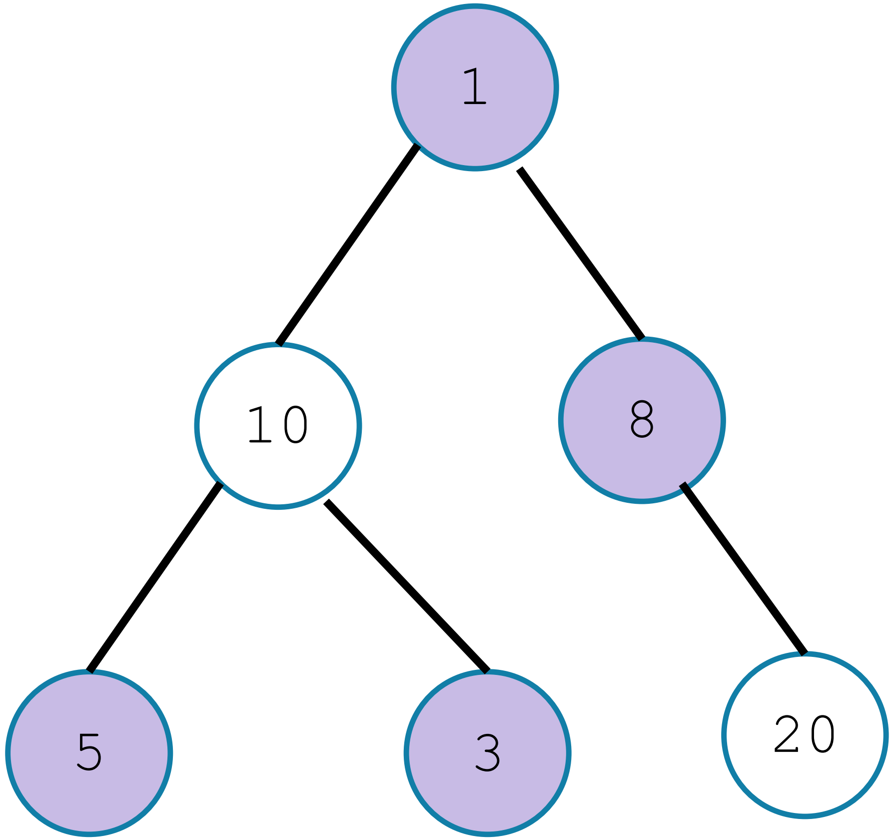
the tree itself!

What code should we write?

What's the running time?

Vertex Cover

What order do we do the calculation?



Vertex Cover Dynamic Program

What memoization structure should we use?

the tree itself!

What code should we write?

A post-order traversal (make recursive calls, then look up values in children to do calculations)

What's the running time?

$\Theta(n)$

Shortest Paths

Shortest Path Problem
Given: A directed graph and a vertex s
Find: The length of the shortest path from s to t.

The length of a path is the sum of the edge weights.

Baseline: Dijkstra's Algorithm

Dijkstra's Algorithm

```
Dijkstra(Graph G, Vertex source)
    initialize distances to ∞
    mark source as distance 0
    mark all vertices unprocessed
    while(there are unprocessed vertices){
        let u be the closest unprocessed vertex
        foreach(edge (u,v) leaving u){
            if(u.dist+weight(u,v) < v.dist){
                v.dist = u.dist+weight(u,v)
                v.predecessor = u
            }
        }
    mark u as processed
    }
```

In 332, we said the running time was $O(m \log n + n \log n)$

Can be sped up to $O(m + n \log n)$ by inserting a different heap implementation.

A recurrence

Suppose you have a directed acyclic graph G.

How could you find distances from s?

What's one step in this problem?

A recurrence

Suppose you have a directed acyclic graph G.

How could you find distances from s?

What's one step in this problem?

Choosing the predecessor, i.e. "the last edge" on a path.

A recurrence

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min\limits_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$

Our memoization structure can be the graph itself.

What's an evaluation order? (Remember we're in a DAG!)

A recurrence

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$

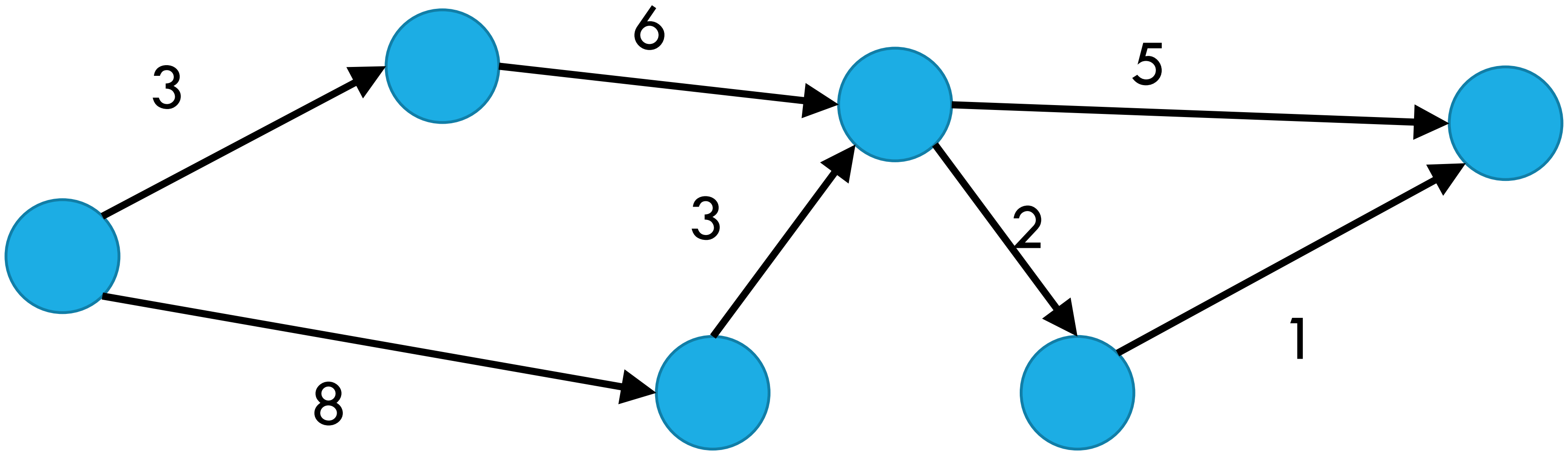
Our memoization structure can be the graph itself.

What's an evaluation order? (Remember we're in a DAG!)

A topological sort! – we need to have distances for all incoming edges calculated.

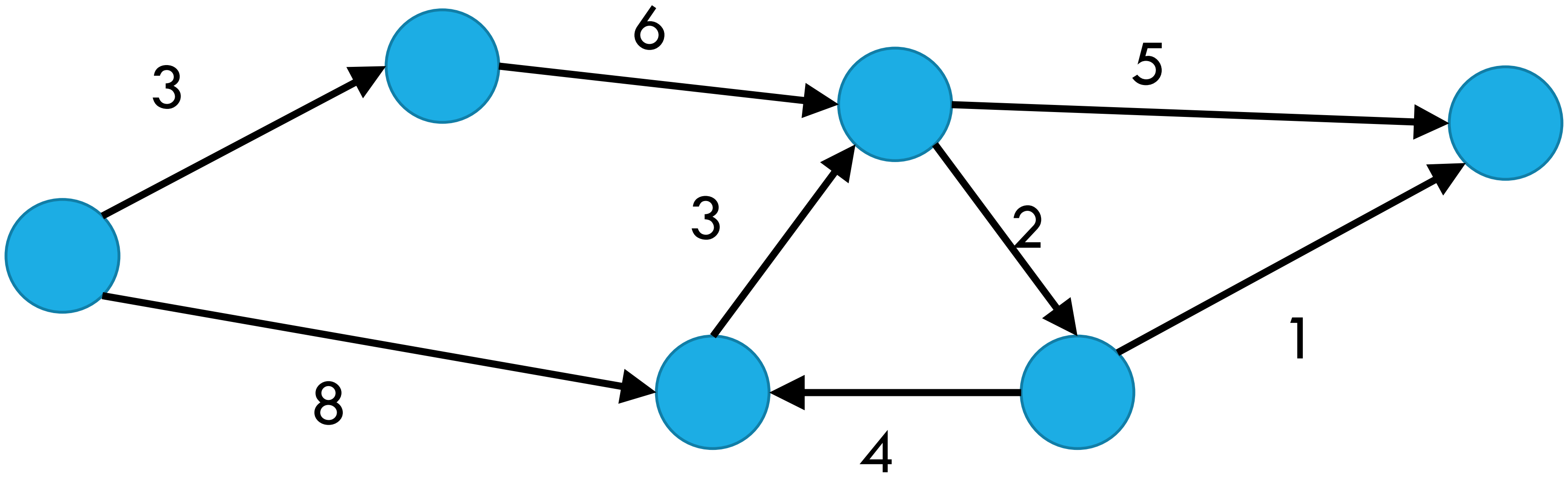
In a DAG

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$



What about cycles?

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$



Cycles

We need some way to “order” the paths.

I.e. we need to be sure we always have **something** to look up.

It doesn’t have to be the perfect distance necessarily…

As long as we’ll realize it and update later

And as long as we can fix it to the true distance eventually.

Ordering

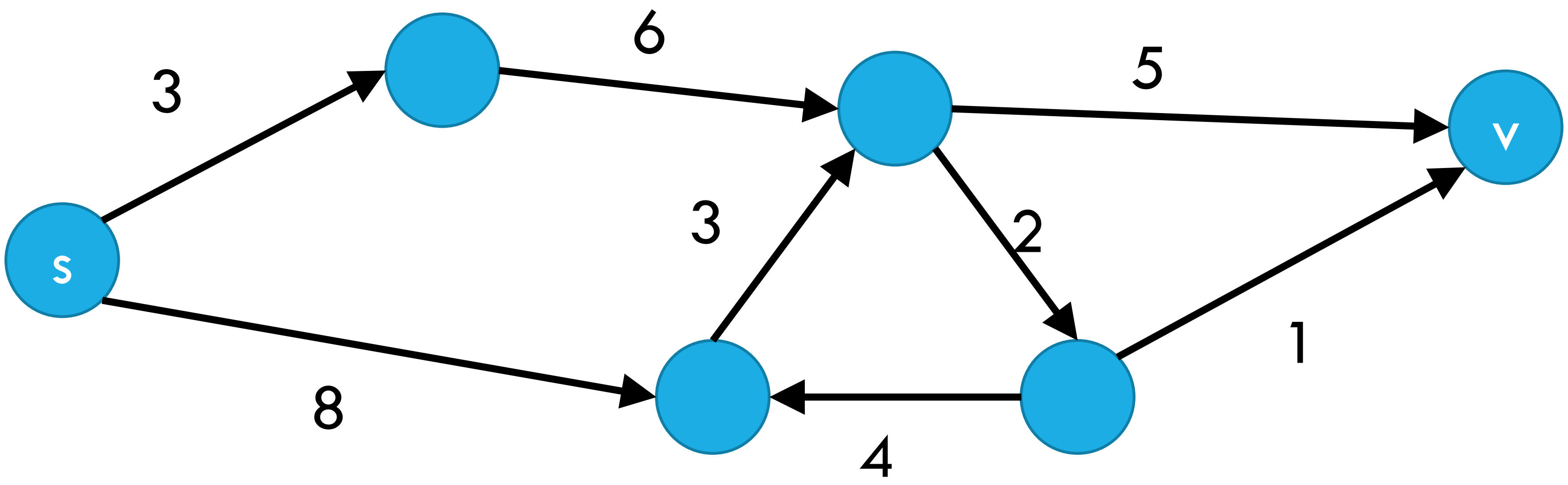
Instead of $dist(v)$, (the true distance) right from the start, we'll let

$dist(v, i)$ to be the length of the shortest path from the source to v that uses at most i edges.

That breaks ties – counting the number of edges required!

$dist(v, i) =$

Distances



$dist(v, 2) = \infty$ (can't get there in 2 hops)

$dist(v, 3) = 14$

$dist(v, 4) = 12$

Ordering

Instead of $dist(v)$, (the true distance) right from the start, we'll let

$dist(v, i)$ to be the length of the shortest path from the source to v that uses at most i edges.

That breaks ties – counting the number of edges required!

$dist(v, i) =$

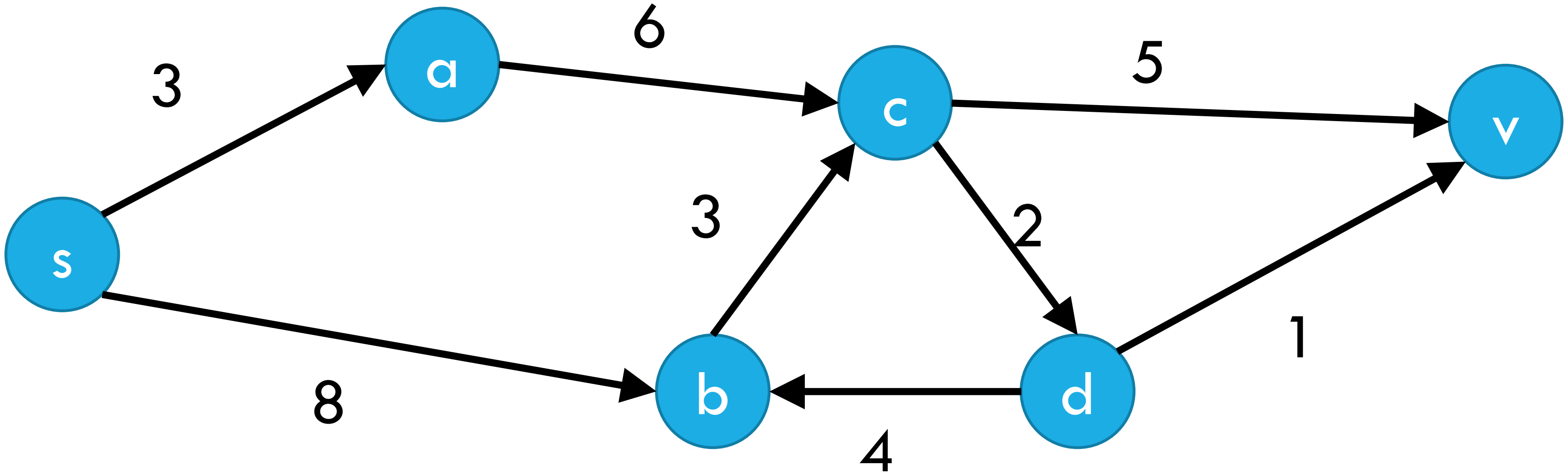
Ordering

Instead of $dist(v)$, we want

$dist(v,i)$ to be the length of the shortest path from the source to v that uses at most i edges.

$$dist(v,i) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v \text{ is the source} \\ \infty & \text{if } i = 0 \text{ and } v \text{ is not the source} \\ \min\left\{ \min_{u:(u,v)\in E} \{dist(u,i-1)\} + w(u,v), dist(v,i-1)\right\} & \text{o/w} \end{cases}$$

Sample calculation



Vertex\\i	0	1	2	3	4	5
S	0	0	0	0	0	0
A	∞	3	3	3	3	3
B	∞	8	8	8	8	8
C	∞	∞	9	9	9	9
D	∞	∞	∞	11	11	11
V	∞	∞	∞	14	12	12

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to ??)
    for(every vertex v) //what order?
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

$$dist(v,i) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v \text{ is the source} \\ \infty & \text{if } i = 0 \text{ and } v \text{ is not the source} \\ \min\left\{ \min_{u:(u,v)\in E} \{dist(u,i-1)\} + w(u,v), dist(v,i-1) \right\} & \end{cases}$$

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
    for(every vertex
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

The shortest path will never need more than $n-1$ edges
(more than that and you've got a cycle)

Pseudocode

Only ever need values from the previous iteration

Order doesn't matter!!

```
Initialize sour
for(i from 1 to
    for(every vertex v) //what order?
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
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```

Pseudocode

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Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
    for(every vertex v) //any order
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

Graphs don't usually have easy access to their incoming edges (just the outgoing ones)

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
    for(every vertex v) //any order
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

But the order doesn't matter – as long as we check every edge, the processing order is irrelevant.
So if we only have access to outgoing edges…

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
    set u.dist[i] to u.dist[i-1] for every u
    for(every vertex u) //any order
        for(each outgoing edge (u,v))//better!
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

Pseudocode

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Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
   set u.dist[i] to u.dist[i-1] for every u
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      for(each outgoing edge (u,v))//better!
         if(u.dist[i-1]+weight(u,v)<v.dist[i])
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         endIf
      endFor
   endFor
endFor
```

We don’t really need all the different values…
Just the most recent value.

Pseudocode

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Initialize source.dist=0, u.dist=∞ for others
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         endIf
      endFor
   endFor
endFor
```

We don't really need all the different values…
Just the most recent value.

A Caution

We did change the code when we got rid of the indexing

You might have a mix of `dist[i]`,`dist[i+1]`,`dist[i+2]`,... at the same time.

That's ok!

You'll only "overwrite" a value with a better one.

And you'll eventually get to $dist(u, n-1)$

After iteration i, u stores $dist(u, k)$ for some $k \geq i$.

Exit early

If you made it through an entire iteration of the outermost loop and don't update any $dist()$

Then you won't do any more updates in the next iteration either. You can exit early.

More ideas to save constant factors on Wikipedia (or a textbook)

Laundry List of shortest pairs (so far)

Algorithm	Running Time	Special Case	Negative edges?
BFS	$O(m+n)$	ONLY unweighted graphs	X
Simple DP	$O(m+n)$	ONLY for DAGs	X
Dijkstra’s	$O(m+n\log n)$		X
Bellman-Ford	$O(mn)$		???

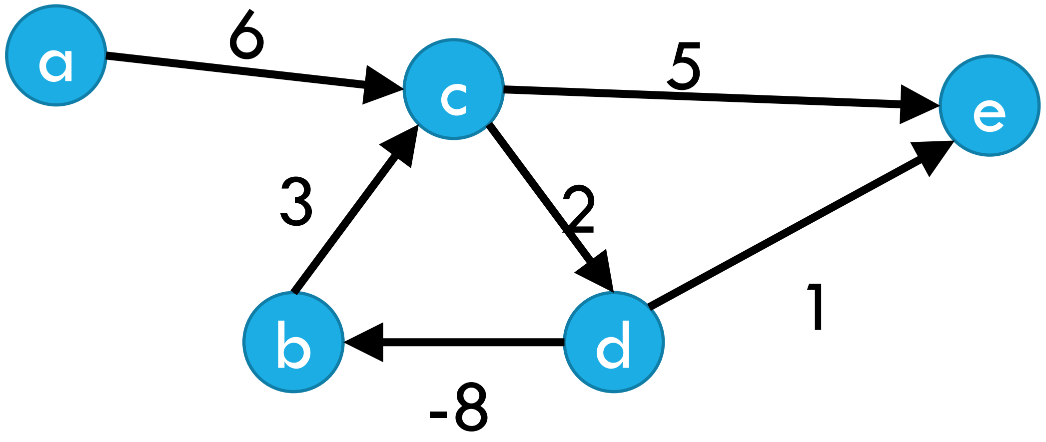
Pseudocode

```
Initialize source.dist=0, u.dist=∞ for others
for(i from 1 to n-1)
   for(every vertex u) //any order
      for(each outgoing edge (u,v))//better!
         if(u.dist+weight(u,v)<v.dist)
            v.dist=u.dist+weight(u,v)
         endIf
      endFor
   endFor
endFor
```

What happens if there's a negative cycle?

Negative Edges

Negative Cycles

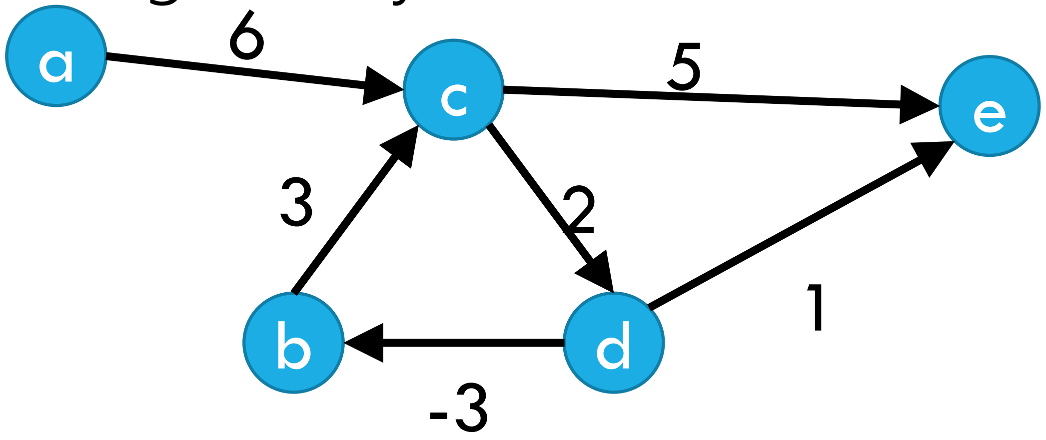


The fastest way from a to e

(i.e. least-weight walk) isn't defined!

No valid answer $(-\infty)$

Negative edges, but only non-negative cycles



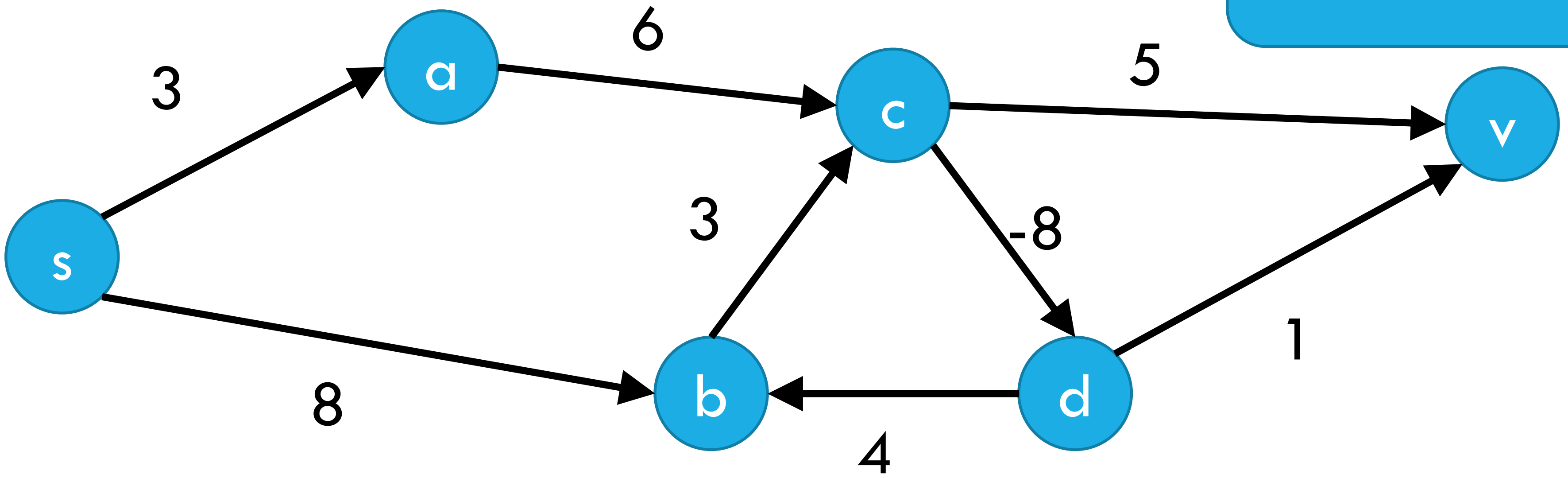
Dijkstra's might fail

But the shortest path IS defined.

There is an answer

Negative Cycle

Pollev.com/Robbie



Vertex\\i	0	1	2	3	4	5	6
S	0	0	0	0	0		
A	∞	3	3	3	3		
B	∞	8	8	8	5		
C	∞	∞	9	9	9		
D	∞	∞	∞	1	1		
V	∞	∞	∞	14	2		

Negative Cycles

If you have a negative length edge: Dijkstra's might or might not give you the right answer.

And it can't even tell you if there's a negative cycle (i.e. whether some of the answers are supposed to be negative infinity)

For Bellman-Ford:

Run one extra iteration of the main loop– if any value changes, you have a negative length cycle. Some of the values you calculated are wrong.

Run a BFS from the vertex that just changed. Anything you can find should have $-\infty$ as the distance. (anything else has the correct [finite] value).

If the extra iteration doesn't change values, no negative length cycle.

Laundry List of shortest pairs (so far)

Algorithm	Running Time	Special Case only	Negative edges?
BFS	$O(m+n)$	ONLY unweighted graphs	X
Simple DP	$O(m+n)$	ONLY for DAGs	X
Dijkstra's	$O(m+n\log n)$		X
Bellman-Ford	$O(mn)$		Yes!

All Pairs Shortest Paths

All Pairs

For Dijkstra's or Bellman-Ford we got the distances from the source to every vertex.

What if we want the distances from every vertex to every other vertex?

All Pairs

For Dijkstra’s or Bellman-Ford we got the distances from the source to every vertex.

What if we want the distances from every vertex to every other vertex?

Why? Most commonly pre-computation.

Imagine you’re google maps – you could run Dijkstra’s every time anyone anywhere asks for directions…

Or store how to get between transit hubs and only use Dijkstra’s locally.

Another Recurrence

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min\limits_{u:(u,v)\in E}\{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$

Another clever way to order paths.

Put the vertices in some (arbitrary) order $1,2,\ldots,n$

Let $dist(u,v,i)$ be the distance from u to v where the only **intermediate** nodes are $1,2,\ldots,i$

Another Recurrence

Put the vertices in some (arbitrary) order $1,2,\ldots,n$

Let $dist(u,v,i)$ be the distance from u to v where the only **intermediate** nodes are $1,2,\ldots,i$

$$\text{dist}(u,v,i)=\begin{cases} weight(u,v) & \text{if } i=0, (u,v) \text{ exists} \\ 0 & \text{if } i=0, u=v \\ \infty & \text{if } i=0,\ \text{ no edge } (u,v) \\ \min\{\text{dist}(u,i,i-1)+\text{dist}(i,v,i-1), \text{dist}(u,v,i-1)\} & \text{otherwise} \end{cases}$$

Pseudocode

```
dist[][] = new int[n-1][n-1]
for(int i=0; i<n; i++)
   for(int j=0; j<n; j++)
      dist[i][j] = edge(i,j) ? weight(i,j) : ∞
for(int i=0; i<n; i++)
    dist[i][i] = 0
for every vertex r
   for every vertex u
      for every vertex v
         if(dist[u][r] + dist[r][v] < dist[u][v])
              dist[u][v]=dist[u][r] + dist[r][v]
```

"standard" form of the "Floyd-Warshall" algorithm. Similar to Bellman-Ford, you can get rid of the last entry of the recurrence (only need 2D array, not 3D array).

Running Time

$O(n^3)$

How does that compare to Dijkstra's?

Running Time

If you really want all-pairs…

Could run Dijkstra's n times…

$O(mn \log n + n^2 \log n)$

If $m \approx n^2$ then Floyd-Warshall is faster!

Floyd-Warshall also handles negative weight edges.

If $dist(u, u) < 0$ then there's a negative weight cycle.

Takeaways

Some clever dynamic programming on graphs.

Which library to use (at least asymptotically)?

Need just one source?

Dijkstra's if no negative edge weights.

Bellman-Ford if negative edges.

Need all sources?

Flord-Warshall if negative edges or $m \approx n^2$

Repeated Dijkstra's otherwise

These are all asymptotics! For any "real-world" problem prefer running actual code to see which is faster.