

# Section 8: Linear Programming + What Tool?

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This section has two parts. The first involves getting a bit of practice with thinking about linear programming. You have only had one lecture on it so far but this should help you get used to it.

Also, so far this quarter, every week we've told you what technique to use. That's not how things work in the real world! In the second part of this section, we're going to practice "when you first read through a problem, how do you decide which technique to use?" Often the answer is "try a bunch of different options and see what happens", but in a time-constrained setting (say your final exam, or an interview) it's nice if your first or second choice turns out to work.

## Linear Programming

### 1. Diet happiness

You need to get a certain amount of nutrition: protein, calories, fat, vitamins, but you also shouldn't get too much of certain things. Each quantity of food you could choose has certain amounts of each but you are also happier eating amounts of some foods versus others. The general problem here is to figure out how to choose a combination of amounts of food in your diet to maximize your happiness while meeting your nutritional needs.

Suppose that someone has no dietary restrictions but has food options per 100 grams:

Bread: 7g protein, fat 2 g, calories 260, happiness 5

Cheese: 27g protein, fat 17g, calories 250, happiness 4

Meat: 30g protein, fat 9g, calories 200, happiness 2

Lentils: 10g protein, fat 1g, calories 120, happiness 3

Daily values: At least 50g protein, 80g fat, 2000 calories. No more than 100g fat, no more than 2300 calories.

Write this optimization problem as a linear program in standard form.

### 2. A more varied diet

You are given a list of foods indexed  $1, \dots, n$ , as well as the calories  $c_i$ , sugars (g)  $s_i$ , and vitamin D (mcg)  $d_i$  in each food. You're trying to maintain a healthy diet by eating exactly 2000 calories per day. You also heard that the American Heart Association recommends at most 30 grams of sugar per day. And because you just moved to Seattle from LA this year, it's your first winter and you need to eat at least 15 mcg of vitamin D to avoid SAD.

Along with the nutrition information, you also know that eating each food will

give you happiness  $h_i$ . Find a way to compute a healthy diet that makes you happiest.

### 3. Technique Toolbox

#### 3.1. Step 1: Read the problem carefully

And answer the usual quick-check questions

- Are there any technical terms in the problem? Any words that look like normal words but really are technical terms?
- What is the input type?
- What is the output type?

If you can't answer these, there's no way to figure out what technique to use—you don't even know what problem you're solving!

#### 3.2. Step 2: Make some example input/outputs

You might stumble upon which technique to use here; for example, if you try to visualize an example input, and you start drawing a graph.

#### 3.3. Some questions to ask

At this point, you should ask “is this really a graph modeling problem?” Some signs to look for

- The problem mentions a graph or something graph-sounding (like “routes” or “maps”).
- There are “direct connections” between elements that could be edges.
- When you try to visualize an input example you end up drawing a graph.

If it **doesn't** feel like graph modeling, the next step to ask is probably “could I solve this problem recursively?” Try asking all of these

- Is there a natural way to split things “in half” (or thirds, or...)?
- Could I make the problem a little bit smaller?
- What's “one step” toward the solution?

These might start leading you to either a divide and conquer (with the first bullet) or dynamic programming solution. In all cases, be sure you can see “how the recursion is helping”.

Finally

- Did an idea immediately jump to mind?
- Did you start a sentence with something like “Well, couldn’t I *just...*”?

Then **maybe** it’s time for a greedy algorithm. But really it’s time for you to generate like 3 more examples and try them against your proposed algorithm. It’s not fun to write a bunch of code only to realize it doesn’t work! And greedy algorithms very often fail. Do some more checks before you jump into code writing.

### 3.4. None of the ideas worked

Take a deep breath, it’s going to be ok.

#### 3.4.1. Get a baseline algorithm

Figure out what “brute force” or any other baseline would be, and jot it down quickly on paper. And when you’re scared, look to your baseline like that hang-in-there-cat motivational poster. Worst-case, you’re going to use that one. And if you figured out that one, you can probably find another one. Hang in there.

#### 3.4.2. Write a few more examples

And solve them. How are **you**, as a person, solving them? You’re doing *some* process. See if you can reflect on it and realize what it looks like. That might inspire you toward an algorithm.

#### 3.4.3. Ask yourself more questions

- Does this remind you of any of the problems you’ve seen before? If so, a similar approach **might** work.
- Can you solve a simpler version of the problem? If there are two variables, make one of them a constant (a **small** constant, like 1) and ask “now what would I do?” Maybe you can generalize from there.
- Can you sort the input? Assume a graph is connected or topologically sorted? What if it’s a tree? What if the array contains only positive elements? Any of these might give you inspiration for the general case.

## 4. Try it yourself!

For each of these problems, get far enough that you’re able to guess what technique you might want to use, and write down a sentence or two about what in the problem lead you toward that technique.

- (a) There are a total of  $n$  courses you have to take, labeled from 1 to  $n$ . You are given a list `prerequisites` where `prerequisites[i] = (ai, bi)` indicates that you must take course  $b_i$  first if you want to take  $a_i$ .

Return `true` if you can finish all courses. Otherwise return `false`.

- (b) You are given a list of integers `coins` representing coins of different denominations and an integer `amount` representing a total amount of money. Return the fewest number of coins you need to make up that amount. If that amount of money cannot be made up by any combination of coins, return -1.

- (c) You are given an integer array `prices` where `prices[i]` is the price of a given stock on the  $i^{th}$  day. On each day, you may decide to buy and/or sell the stock. You can hold at most one share of the stock at any time. However, you can buy it and then immediately sell it on the same day. Find and return the maximum profit you can achieve.

- (d) You are given an array of  $k$  linked-lists, each linked-list is sorted in ascending order. Merge all the linked-lists into one sorted linked-list and return it.

- (e) Given an array of distinct integers `nums` and a target integer `target`, return the number of possible combinations that add up to `target`.

- (f) There are  $n$  cities. Some of them are connected, while some are not. If city  $a$  is connected directly with city  $b$  and city  $b$  is connected directly with city  $c$ , then city  $a$  is connected indirectly with city  $c$ . A province is a group of directly or indirectly connected cities and no other cities outside of the group. You are given an  $n \times n$  matrix `isConnected` where `isConnected[i][j] = 1` if the  $i^{th}$  city and the  $j^{th}$  city are directly connected, and `isConnected[i][j] = 0` otherwise. Return the total number of provinces.

- (g) You are given an integer array `nums`. You are initially positioned at the array's first index, and each element in the array represents your maximum jump length at that position. Return `true` if you can reach the last index or `false` otherwise.

- (h) Write an efficient algorithm that searches for a value `target` in an  $m \times n$  integer matrix. This matrix has the following properties

- (i) Integers in each row are sorted in ascending order from left to right.
  - (ii) Integers in each column are sorted in ascending order from top to bottom.
- (j) You are given an integer array `height` of length  $n$ . There are  $n$  vertical lines drawn such that the two endpoints of the  $i^{\text{th}}$  line are  $(i, 0)$  and  $(i, \text{height}[i])$ . Find two lines that together with the x-axis form a container, such that the container contains the most water. Return the maximum amount of water a container can store.
- (k) There is a group of  $n$  people labeled from 0 to  $n - 1$  where each person has a different amount of money and a different level of quietness. You are given an array `richer` where `richer[i] = [ai, bi]` indicates that  $a_i$  has more money than  $b_i$  and an integer array `quiet` where `quiet[i]` is the quietness of the  $i^{\text{th}}$  person. All the given data in `richer` are logically correct (i.e., the data will not lead you to a situation where  $x$  is richer than  $y$  and  $y$  is richer than  $x$  at the same time). Return an integer array `answer` where `answer[x] = y` if  $y$  is the least quiet person (that is, the person  $y$  with the smallest value of `quiet[y]`) among all people who definitely have equal to or more money than the person  $x$ .
- (l) Given an integer array `nums`, return an integer array `counts` where `counts[i]` is the number of smaller elements to the right of `nums[i]`.
- (m) You are given several boxes with different colors represented by different positive numbers. You may experience several rounds to remove boxes until there is not box left. Each time you can choose some continuous boxes with the same color (i.e., composed of  $k$  boxes,  $k \geq 1$ ), remove them and get  $k * k$  points. Return the maximum points you can get.
- (n) There are  $n$  rooms labeled from 0 to  $n - 1$  and all the rooms are locked except for room 0. Your goal is to visit all the rooms. However, you cannot enter a locked room without having its key. When you visit a room, you may find a set of distinct keys in it. Each key has a number on it, denoting which room it unlocks, and you can take all of them with you to unlock the other rooms. Given an array `rooms` where `rooms[i]` is the set of keys that you can obtain if you visit room  $i$ , return `true` if you can visit all the rooms, or `false` otherwise.

- (n) Given an integer  $n$ , return the least number of perfect square numbers that sum to  $n$ . A perfect square is an integer that is the square of an integer, in other words, it is the product of some integer with itself. For example, 1, 4, and 9 are perfect squares while 3 and 11 are not.
- (o) You are given a network of  $n$  locations labeled from 1 to  $n$ . You are also given times, a list of travel times such that  $\text{times}[i] = (u_i, v_i, w_i)$ , where  $u_i$  is the source,  $v_i$  is the destination and  $w_i$  is the time it takes for a signal to travel from the source to the destination. We will send a signal from a given location  $k$ . Return the minimum time it takes for all  $n$  locations to receive the signal. If it is impossible for all  $n$  locations to receive the signal, return -1.
- (p) You are given two integers  $n$  and  $k$  and two integer arrays `speed` and `efficiency` both of length  $n$ . There are  $n$  engineers numbered from 1 to  $n$ . `speed[i]` and `efficiency[i]` represent the speed and efficiency of the  $i$ th engineer respectively. Choose at most  $k$  different engineers out of the  $n$  engineers to form a team with the maximum performance. The performance of a team is the sum of their engineers' speeds multiplied by the minimum efficiency among their engineers. Return the maximum performance of this team.
- (q) There are  $n$  piles of stones arranged in a row. The  $i$ th pile has `stones[i]` stones. A move consists of merging exactly  $k$  consecutive piles into one pile, and the cost of this move is equal to the total number of stones in these  $k$  piles. Return the minimum cost to merge all piles of stones into one pile.
- (r) A series of highways connect  $n$  cities numbered from 0 to  $n - 1$ . You are given a 2D integer array `highways` where `highways[i] = [city1i, city2i, tolli]` indicates that there is a highway that connects `city1i` and `city2i`, allowing a car to go from `city1i` to `city2i` and vice versa for a cost of `tolli`. You are also given an integer `discounts` which represents the number of discounts you have. You can use a discount to travel across the  $i$ th highway for a cost of `tolli/2` (integer division). Each discount may only be used once, and you can only use at most one discount per highway. Return the minimum total cost to go from city 0 to city  $n - 1$ , or -1 if it is not possible to go from city 0 to city  $n - 1$ .
- (s) A subsequence of a string is a new string that is formed from the original

string by deleting some (can be none) of the characters without disturbing the relative positions of the remaining characters. (i.e., "ace" is a subsequence of "abcde" while "aec" is not). Given two strings source and target, return the minimum number of subsequences of source such that their concatenation equals target. If the task is impossible, return -1.

## 5. DP on Trees

You are given a tree  $T = (V, E)$  with nonnegative edge weights. You want to determine the diameter of the tree, which is the longest distance between any two nodes. The goal is to design a DP which runs in  $\mathcal{O}(n)$  where  $n = |V|$ .

### 5.1. Write the Dynamic Program

- (a) Formulate the problem recursively – what are you looking for (in English!!), and what parameters will you need as you're doing the calculation? It will be useful to fix a root node in the tree first, then every node in the tree has a list of children nodes you can access.
- (b) Write a recurrence for solving the problem you defined in the last part (the recurrence is for the answer, **not** the running time).
- (c) What is your final answer (e.g. what parameters for the recurrence do you need? Is it a single value or the max/min of a set of values?)?
- (d) Give a brief justification for why your recurrence is correct. You do not need a formal inductive proof, but your intuition will likely resemble one.

### 5.2. Analyze the Dynamic Program

- (a) Describe a memoization structure for your algorithm.
- (b) Describe a filling order for your memoization structure.
- (c) State and justify the running time of an iterative solution.