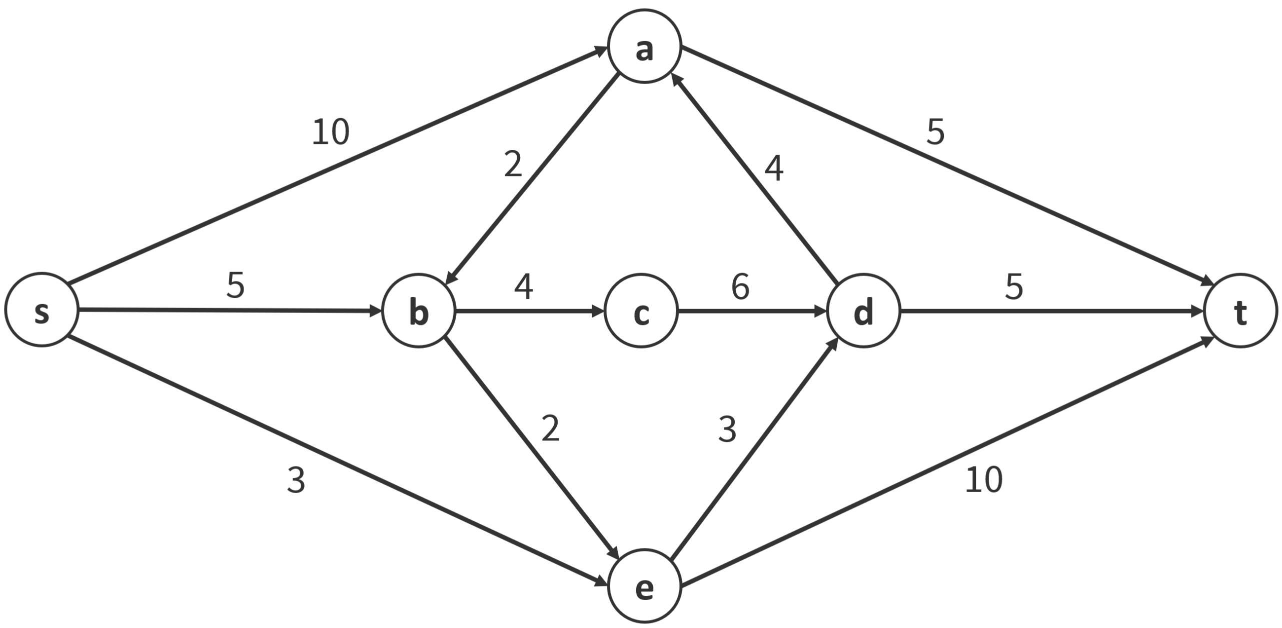
Section 7: Max Flow Min Cut and Applications

The goal for this week's section is to use max-flow and min-cut to model various problems. There are a few common tricks that allow us to cleverly solve problems that at first glance look like they are *not quite* the same as the "standard" problems. You might figure out the trick the first time, or you might not. But either way you should remember these tricks! If you've seen it once, it's much easier to use later, and these tricks are common in modeling problems with flows and cuts.

1. Go With the Flow

Using Ford-Fulkerson, find the maximum $s-t$ flow in the graph G below, the corresponding residual graph, and list out the corresponding minimum cut.



2. You're not a dummy...

You have three overfilled reservoirs and two underfilled reservoirs. You want to (as quickly as possible) move a total of 10,000 gallons of water from the overfilled reservoirs to the underfilled ones. You do not care how much comes from each of the three individual reservoirs (as long as the total is 10,000 gallons) nor how much arrives at each of the underfilled ones (again, as long as the total is 10,000 gallons). You have a map of (one-way) pipes connecting the reservoirs (in the form of a directed graph); each pipe has a maximum capacity in gallons per minute. You wish to find the way to route the water and the amount of time that will be required.

2.1. Read The Problem Carefully

Answer the usual quick-check questions:

- Are there any technical terms in the problem you don't know? Are there any words that look like normal words, but are actually technical terms?
- What is the input type?
- What is the output type?

2.2. Make a basic model

This sounds like a flow problem. From what you know so far, what would the flow model be? What parts of the problem have you represented successfully? What is still missing?

2.3. Brainstorm: How can you fix the missing piece?

Find a clever trick to represent the missing piece. The goal here is to do a reduction. By the end of this step, you should have a "standard" flow problem.

2.4. Correctness and running time

Explain why your algorithm is correct. For flow problems, the proof is usually just explaining how you've represented each part of the problem, and relying on the correctness of the flow algorithm.

3. Split Personality

You have been given a map of the water cleaning system for the city of Seattle. Water enters from a marked vertex, and flows through pipes (directed edges with specified capacities), through processing facilities, and back out to nature (marked as a specified sink vertex). The processing facilities are vertices in your graph. As the facilities process the water, they **also** have maximum capacities, which may be less than the total capacity entering or leaving the vertex. You wish to find the amount of water that can flow through this network while respecting both the facility and pipe capacities.

3.1. Read The Problem Carefully

Answer the usual quick-check questions:

- Are there any technical terms in the problem you don't know? Are there any words that look like normal words, but are actually technical terms?
- What is the input type?
- What is the output type?

3.2. Make a basic model

This sounds like a flow problem. From what you know so far, what would the most direct flow model be? What parts of the problem have you represented successfully? What is still missing?

3.3. Brainstorm: How can you fix the missing piece?

Find a clever trick to represent the missing piece. The goal here is to do a reduction. By the end of this step, you should have a "standard" flow problem.

3.4. Correctness and running time

Explain why your algorithm is correct. For flow problems, the proof is usually just explaining how you've represented each part of the problem, and relying on the correctness of the flow algorithm.

More problems!

4. Where in the Grid is Carmen Sandiego?

Carmen Sandiego is currently in an $n \times n$ grid. From any square she can move to the four adjacent squares (up, down, left, right). Her goal is to get to any square on any "edge" of the grid, from which she can escape. You wish to prevent her escape; you will place ACME agents on various locations on the grid. Each grid location u is labeled with a weight $w(u)$ which denotes the number of agents needed to block Carmen from that location. If that many agents are placed on that location, Carmen cannot pass through. You have k agents to place; determine whether you can catch Carmen (and if so how to place the agents).

(a) Describe a graph you can run a max-flow algorithm on. Be sure to mention edge directions and capacities.

(b) Describe how you'll tell whether an assignment is possible or not. If an assignment is possible, how do you read it from the result of the maximum flow algorithm?

(c) Briefly justify correctness. We aren't expecting a formal proof here, but you should have a sentence or two for each of the restrictions given in the problem.

(d) Describe the runtime of the algorithm. Briefly justify why the running time is the value that you state.