CSE 421
Introduction to Algorithms

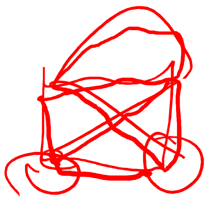
Lecture 26: Dealing with NP-completeness:
Approximation Algorithms

Some other NP-complete examples you should know

Hamiltonian-Cycle: **Given** a directed graph $G = (V, E)$. Is there a cycle in G that visits each vertex in V exactly once?

Hamiltonian-Path: **Given** a directed graph $G = (V, E)$. Is there a path p in G of length $n - 1$ that visits each vertex in V exactly once?

Same problems are also $\mathbf{NP}$-complete for undirected graphs



Note: If we asked about visiting each *edge* exactly once instead of each vertex, the corresponding problems are called **Euler Tour**, **Eulerian-Path** and are polynomial-time solvable.

Travelling-Salesperson Problem (TSP)

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Given: a set of n cities $v_1, \ldots, v_n$ and distance function d that gives distance $d(v_i, v_j)$ between each pair of cities

Find the shortest tour that visits all n cities.

$\text{HAMCYCLE} \le_P \text{DecisionTSP}$

v_i v_j $d(v_i, v_j) = 1$

$d(v_i, v_j) = 2$

ask $D = |V|$

DecisionTSP:

Given: a set of n cities $v_1, \ldots, v_n$ and distance function d that gives distance $d(v_i, v_j)$ between each pair of cities *and* a distance D

Is there a tour of total length at most D that visits all n cities?

NP-complete problems we've discussed

3SAT → Independent-Set → Clique

↓

Vertex-Cover → 01-Programming → Integer-Programming

↓

Set-Cover

→ 3Color

→ Subset-Sum

→ Hamiltonian-Cycle → DecisionTSP

→ Hamiltonian-Path

Some intermediate problems

Problems reducible to $\mathbf{NP}$ problems not known to be polytime:

Basis for the security of current cryptography:

- **Factoring:** Given an integer N in binary, find its prime factorization.
- **Discrete logarithm:** Given prime p in binary, and g and x modulo p.
  Find y such that $x \equiv g^y \pmod{p}$ if it exists.

  Best algorithms known are $2^{\widetilde{\Theta}(n^{1/3})}$ time.

Other famous ones:

- **Graph Isomorphism:** Given graphs G and H, can they be relabelled to be the same?
  Best algorithm now $n^{\Theta(\log^2 n)}$ (recently improved from $2^{\widetilde{\Theta}(n^{1/\cancel{3}2})}$) time.
- **Nash equilibrium:** Given a multiplayer game, find randomized strategies for each player so that no player could do better by deviating.

What to do if the problem you want to solve is NP-hard

1st thing to try:

- You might have phrased your problem too generally
  - e.g., In practice, the graphs that actually arise are far from arbitrary
    - Maybe they have some special characteristic that allows you to solve the problem in your special case
      - For example the **Independent-Set** problem is easy on “interval graphs”
        - Exactly the case for the Interval Scheduling problem!
  - Search the literature to see if special cases already solved

What to do if the problem you want to solve is NP-hard

2nd thing to try if your problem is a minimization or maximization problem

- Try to find a polynomial-time worst-case approximation algorithm
  - For a minimization problem
    - Find a solution with value $\leq \boldsymbol{K}$ times the optimum
  - For a maximization problem
    - Find a solution with value $\geq \mathbf{1}/\boldsymbol{K}$ times the optimum

Want $\boldsymbol{K}$ to be as close to $\mathbf{1}$ as possible.

Greedy Approximation for Vertex-Cover

On input $G = (V, E)$

$W \leftarrow \emptyset$

$E' \leftarrow E$

while $E' \neq \emptyset$

  select any $e = (u, v) \in E'$

  $W \leftarrow W \cup \{u, v\}$

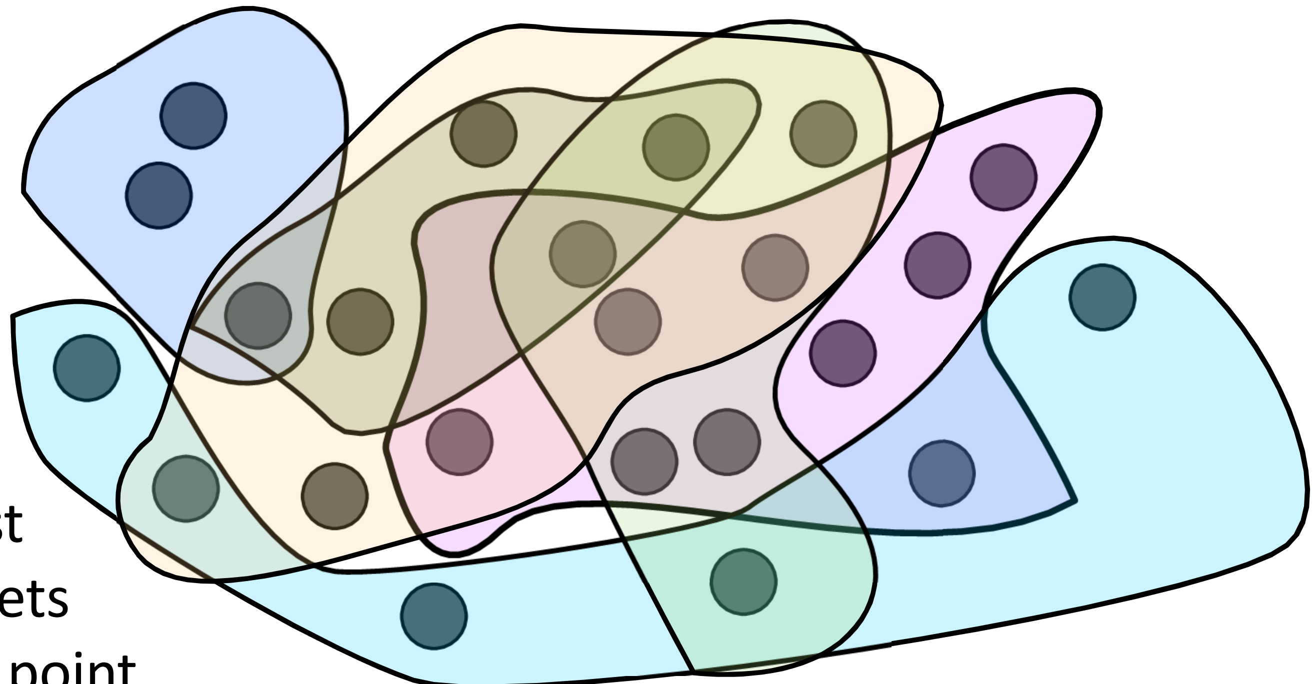
  $E' \leftarrow E' \setminus \{\text{edges } e \in E' \text{ that touch } u \text{ or } v\}$

This actually a better approximation factor than the greedy algorithm that repeatedly chooses the highest degree vertex remaining that you considered on Homework 3.

Claim: At most a factor 2 larger than the optimal vertex-cover size.

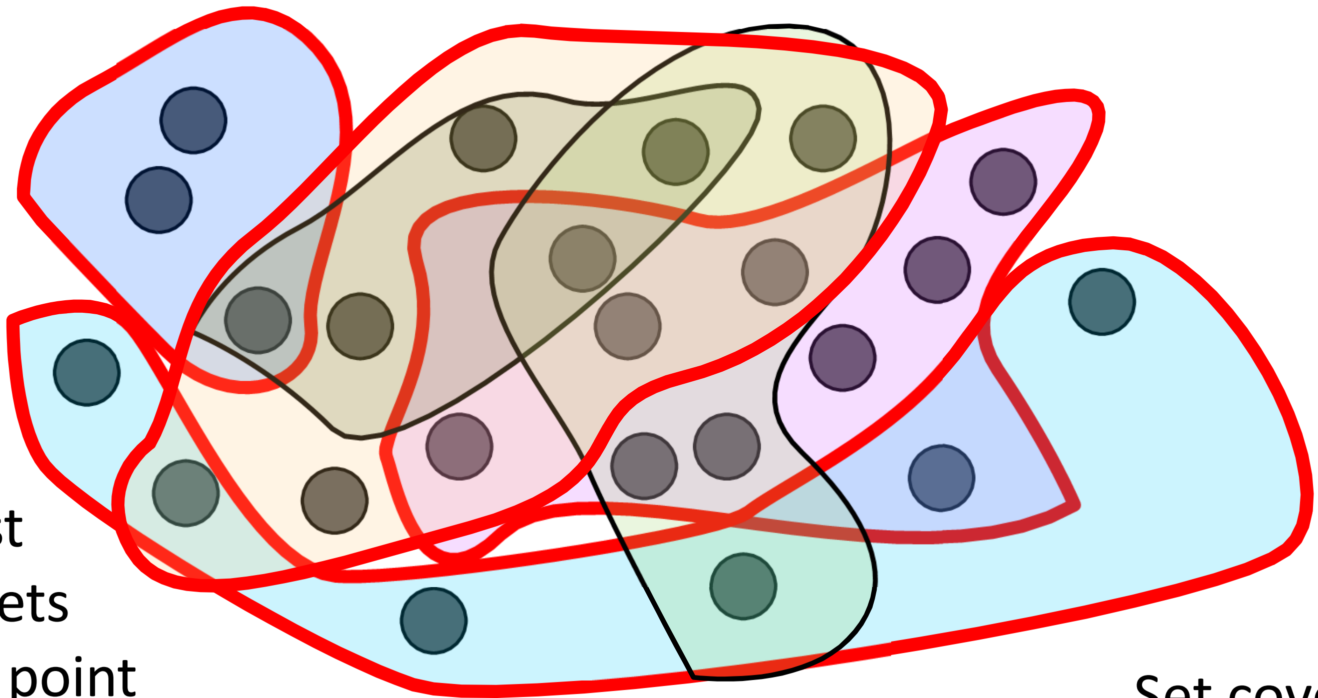
Proof: Edges selected don't share any vertices so any vertex-cover must choose at least one of u or v each time.

Set-Cover



Find smallest
collection of sets
containing every point

Set-Cover

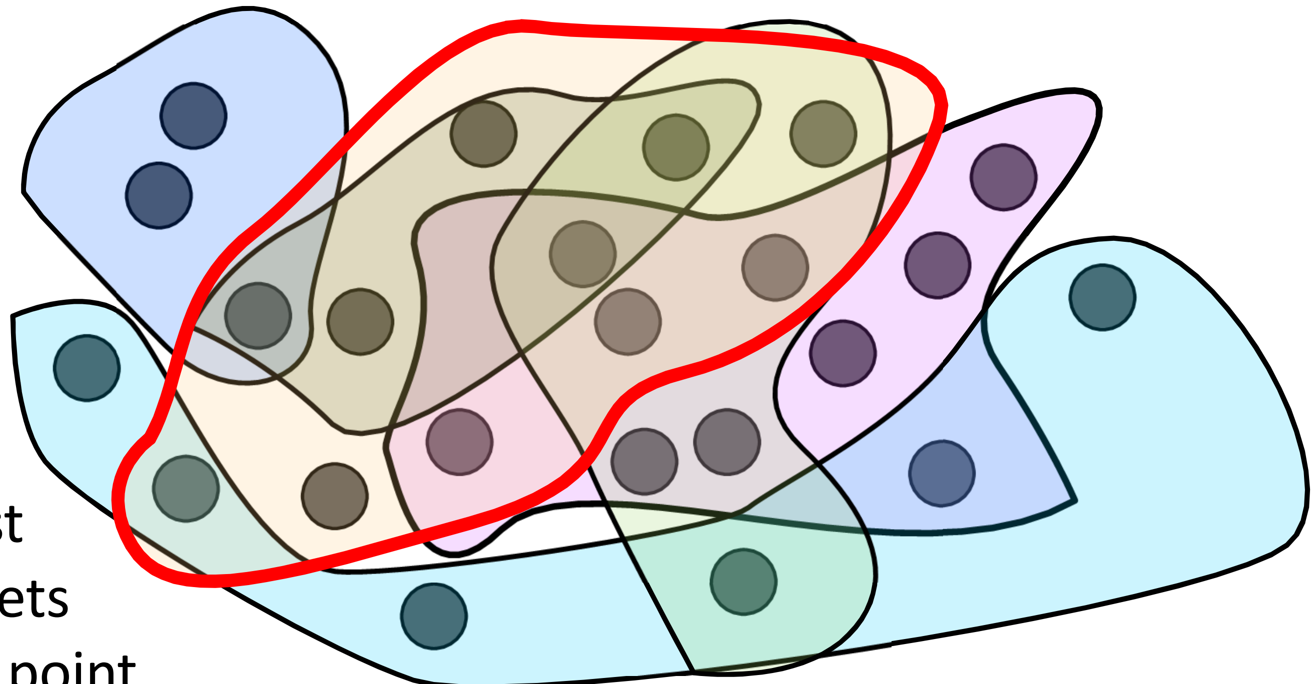


Find smallest collection of sets containing every point

Set cover size **4**

Set-Cover

Greedy Set Cover: Repeatedly choose the set that covers the most # of new elements



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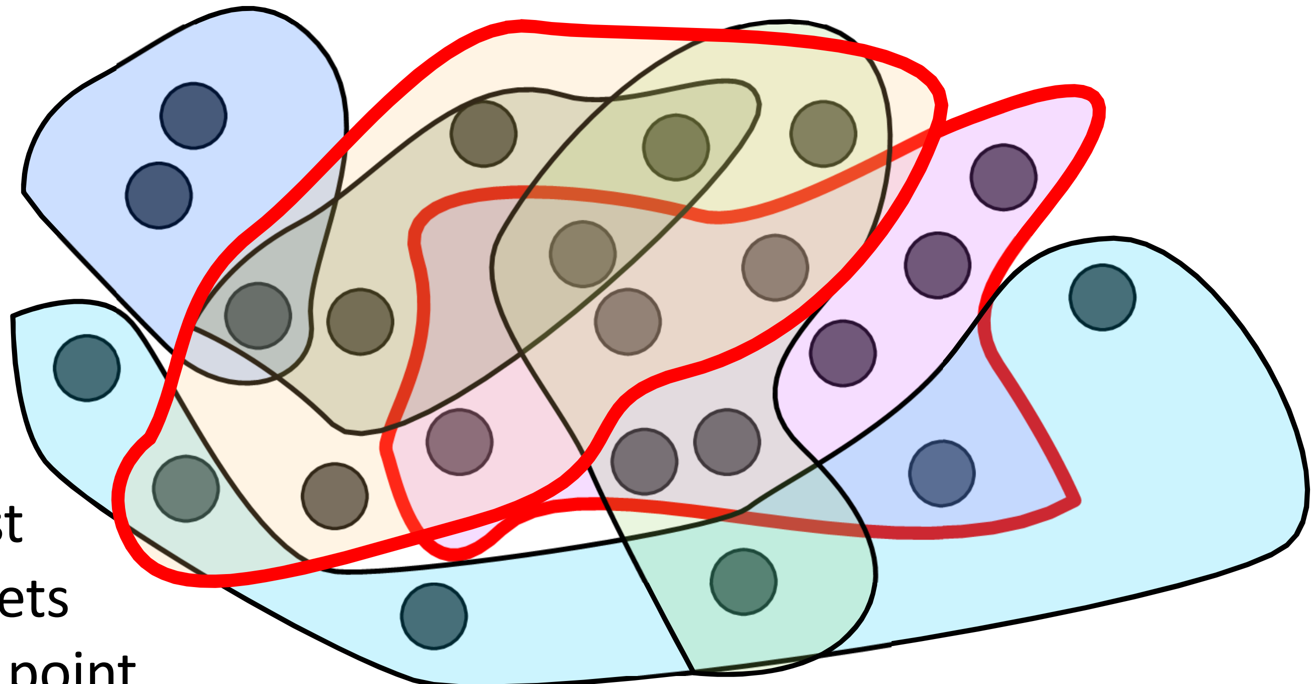

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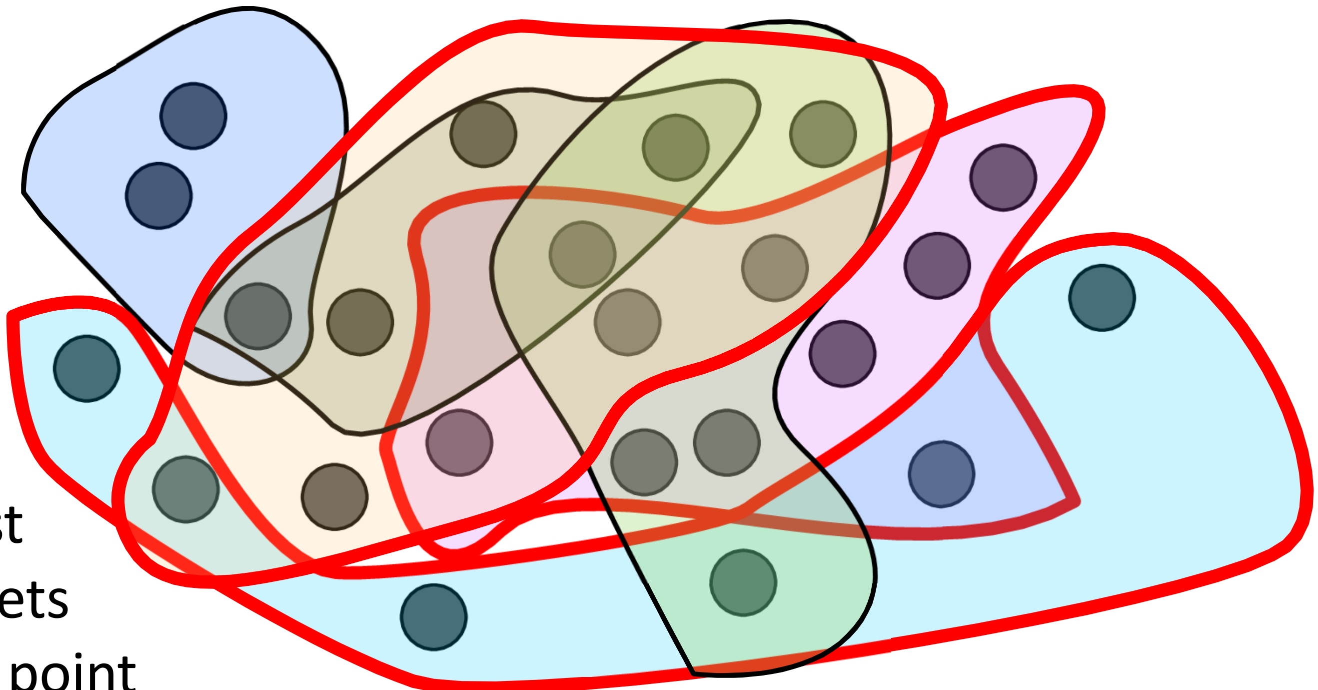

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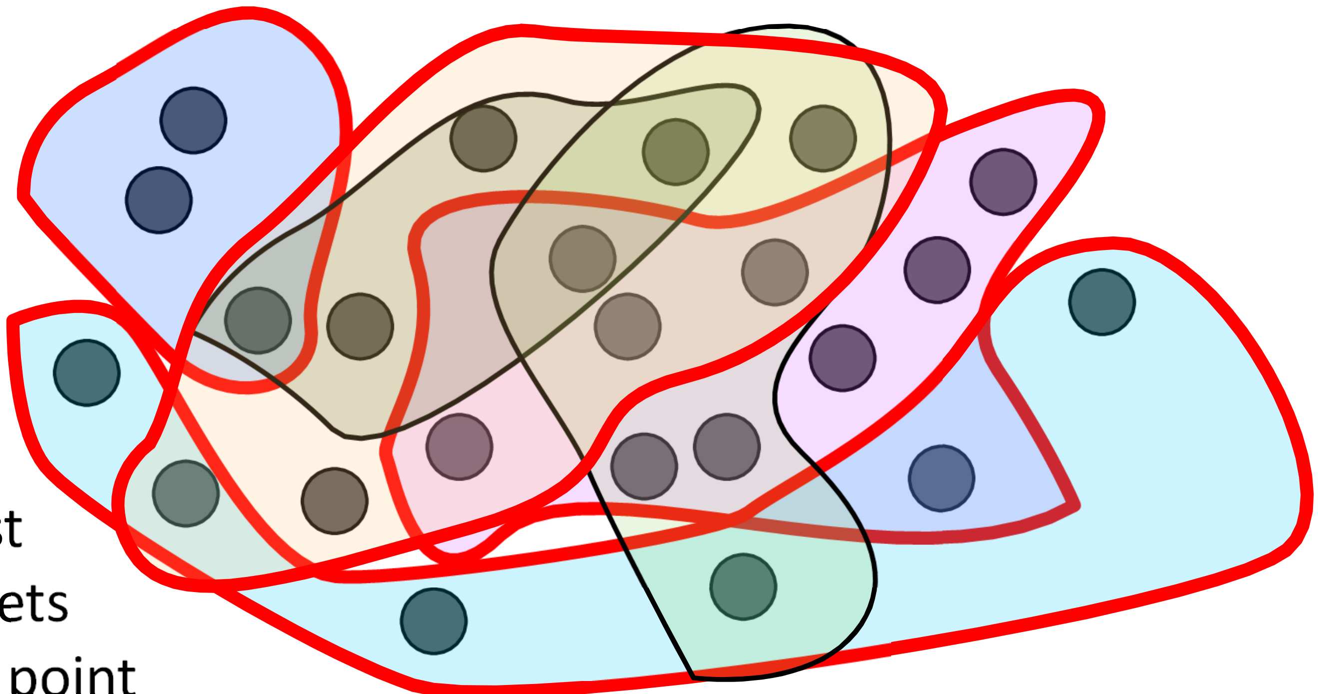
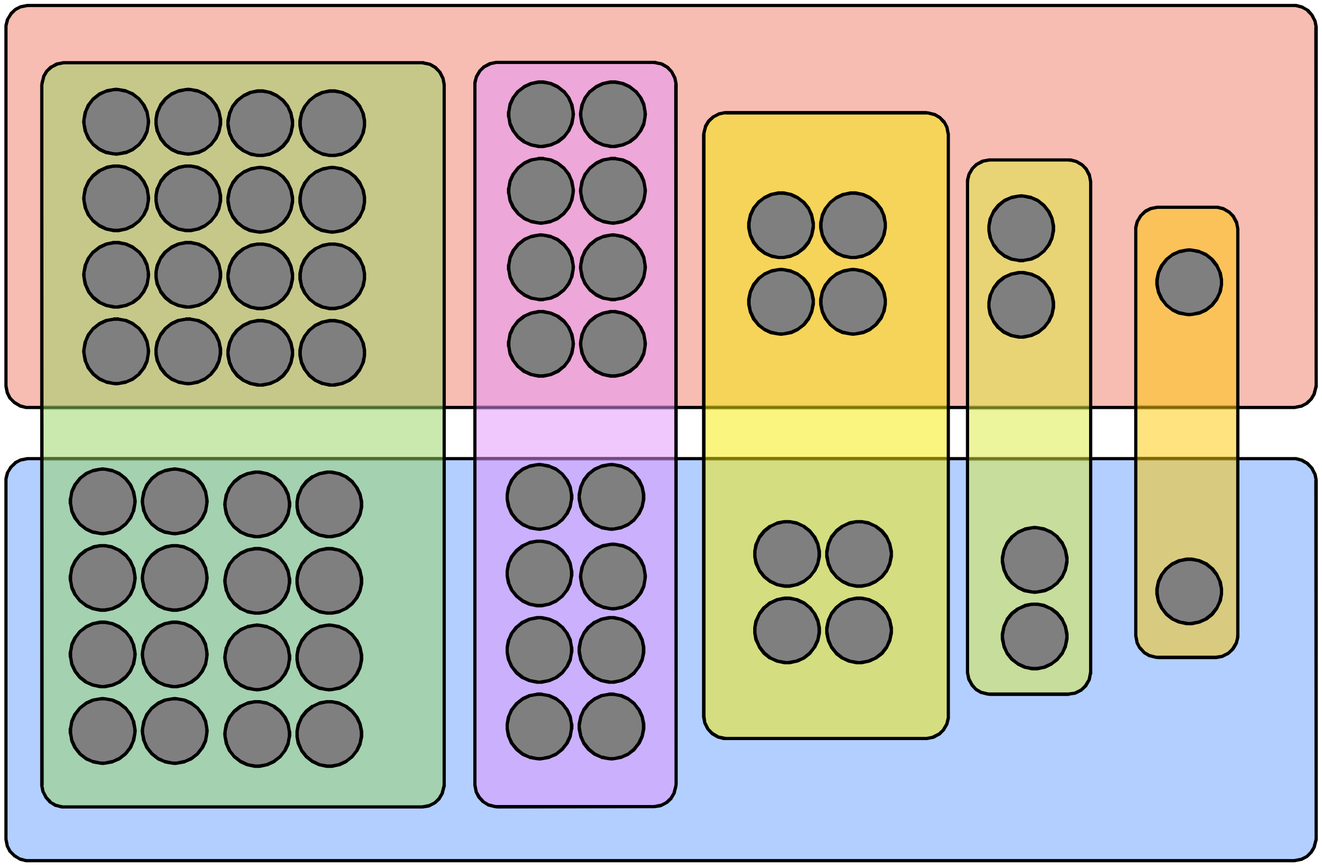


Find smallest collection of sets containing every point

Theorem: Greedy finds best cover up to a factor of $\ln \boldsymbol{n}$.

Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



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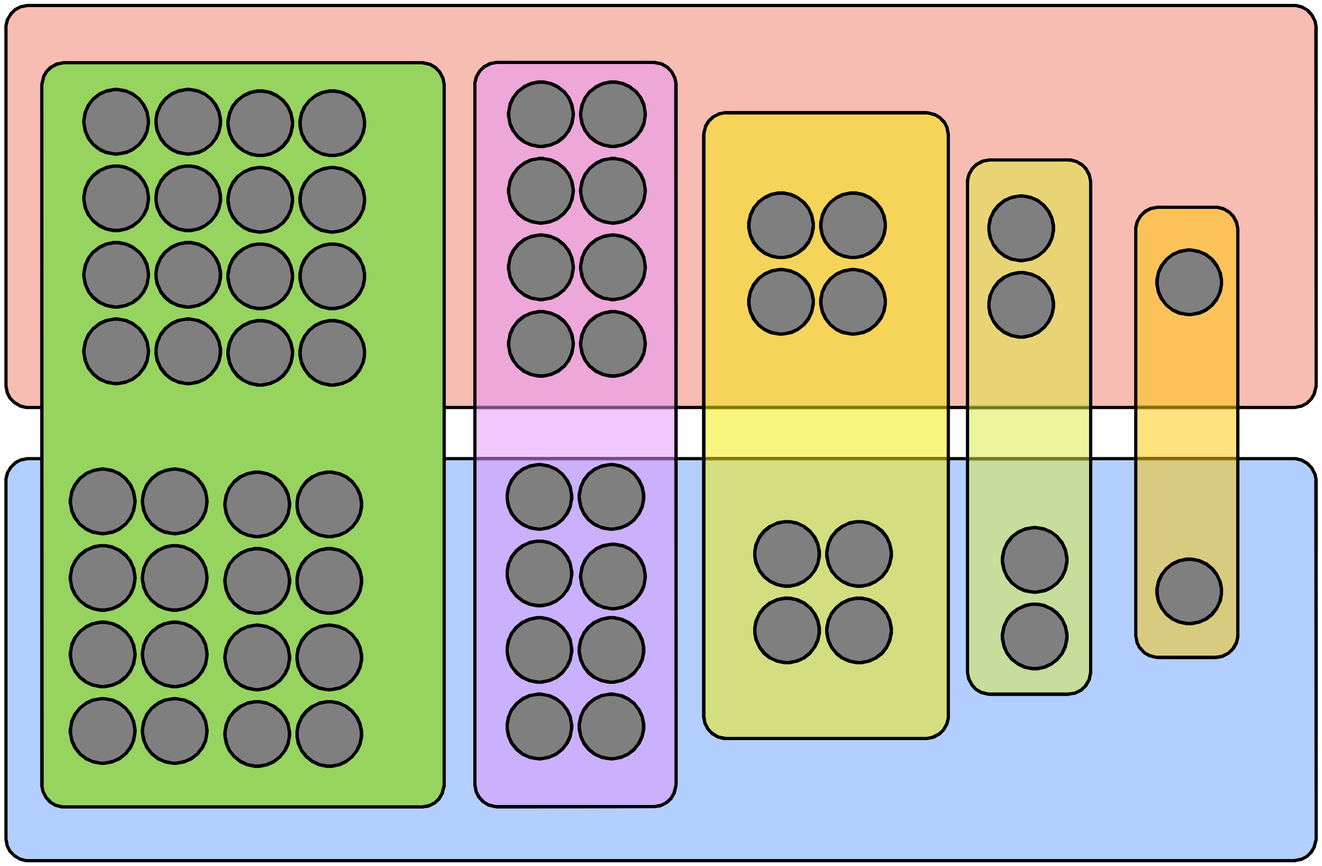

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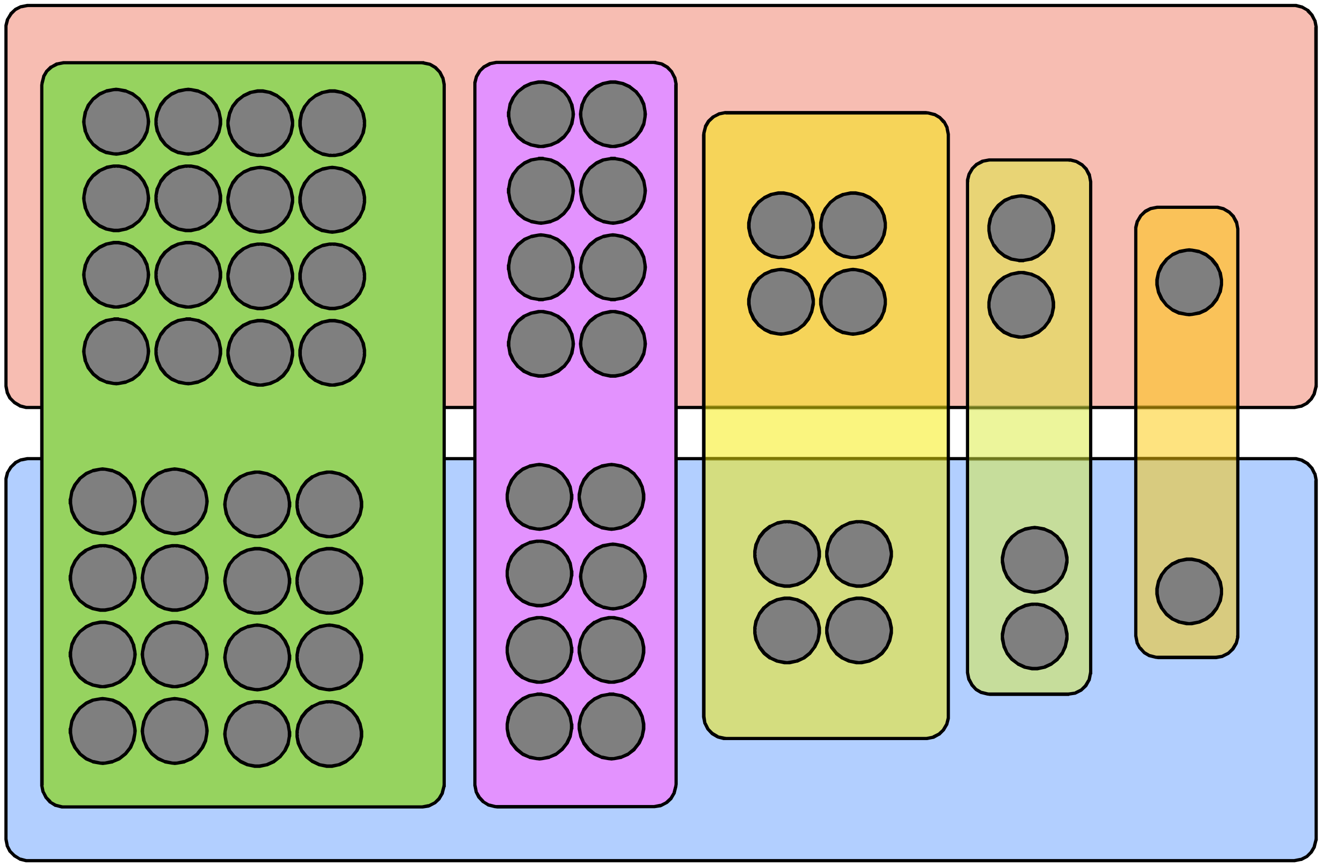

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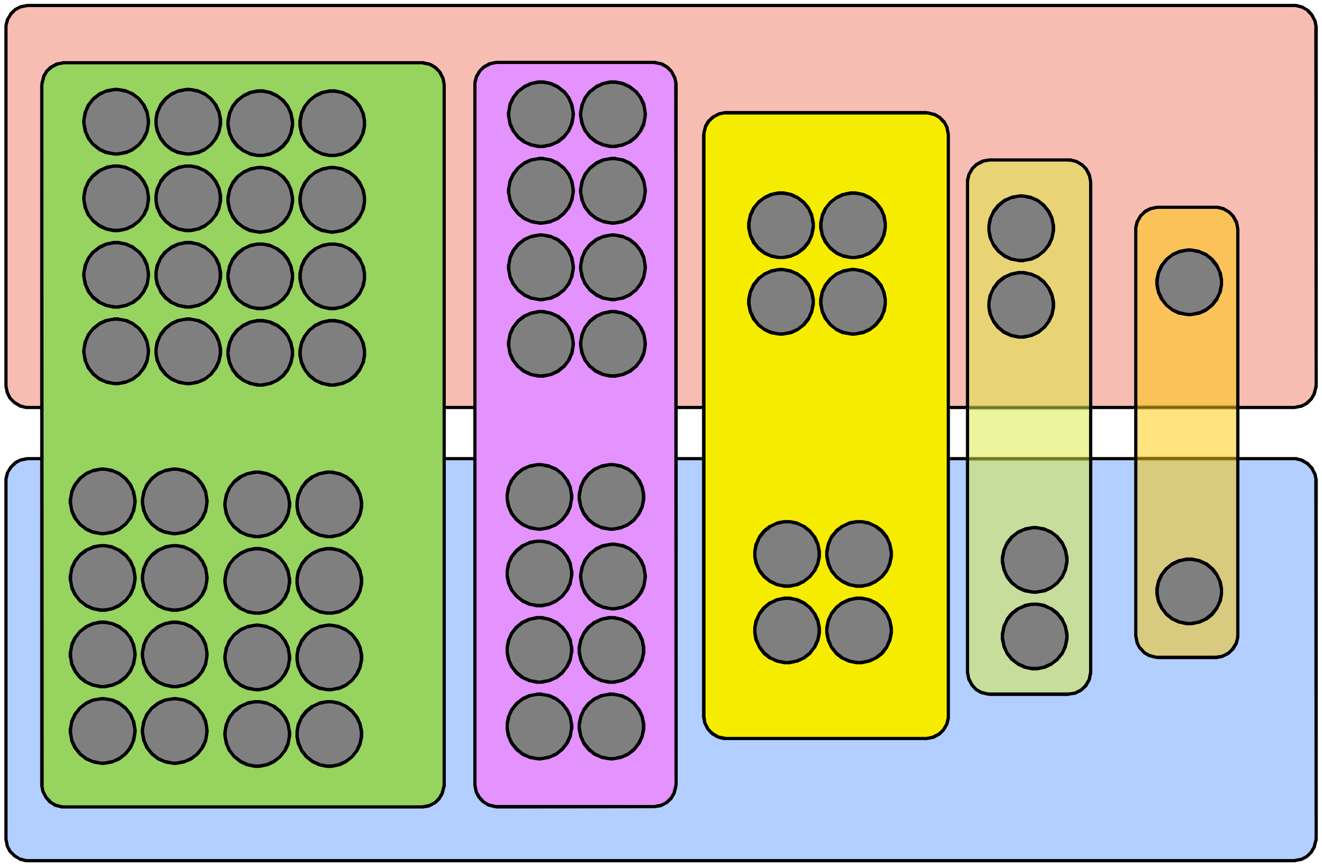

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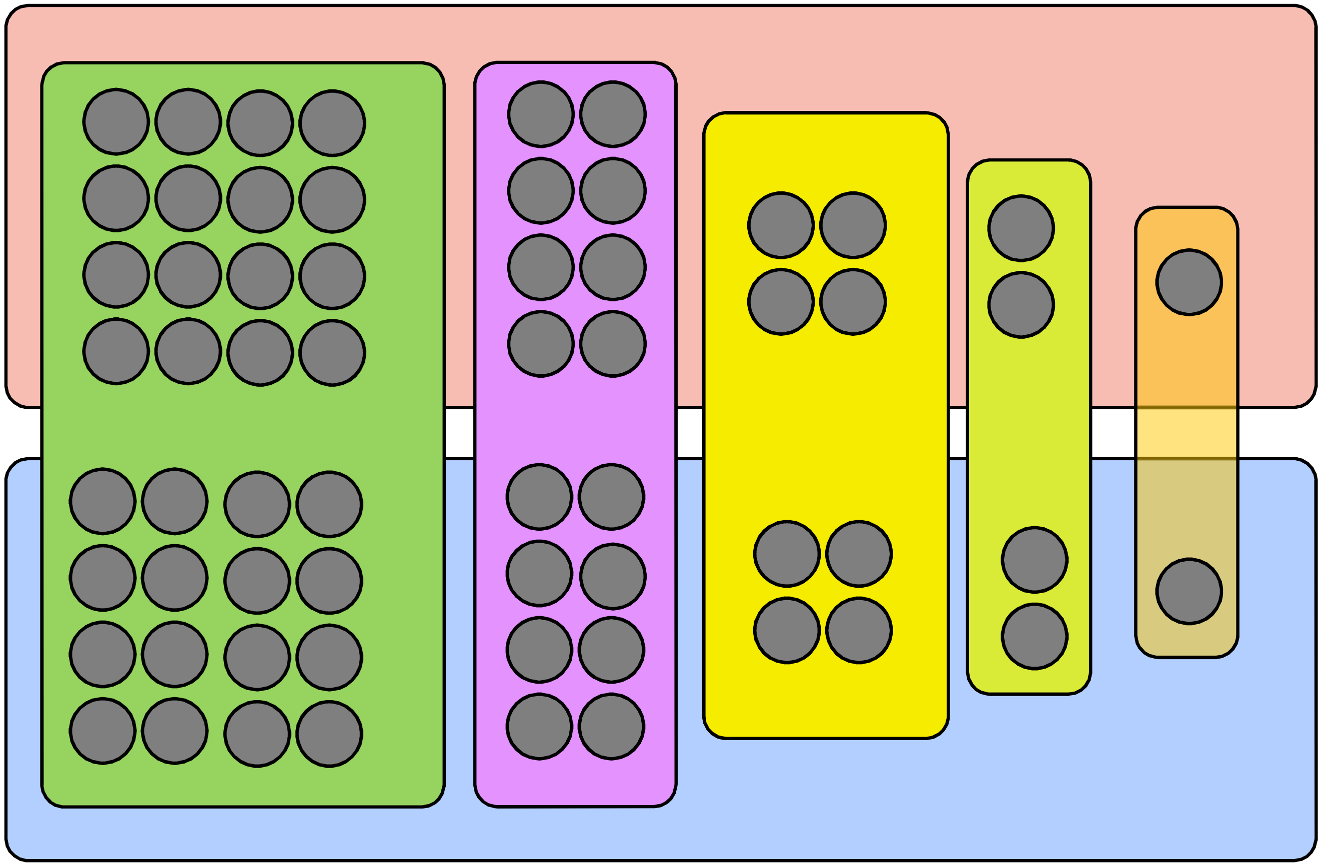

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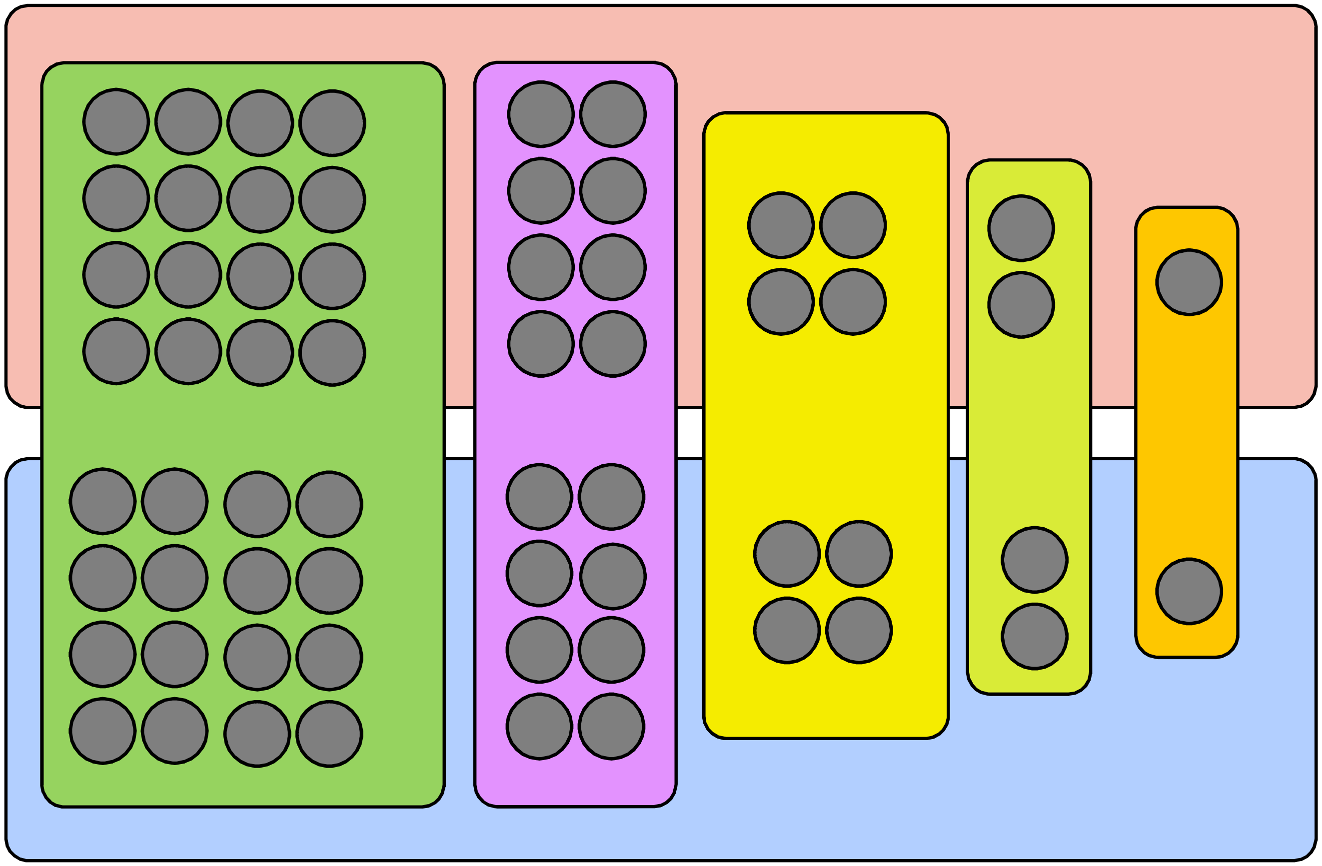
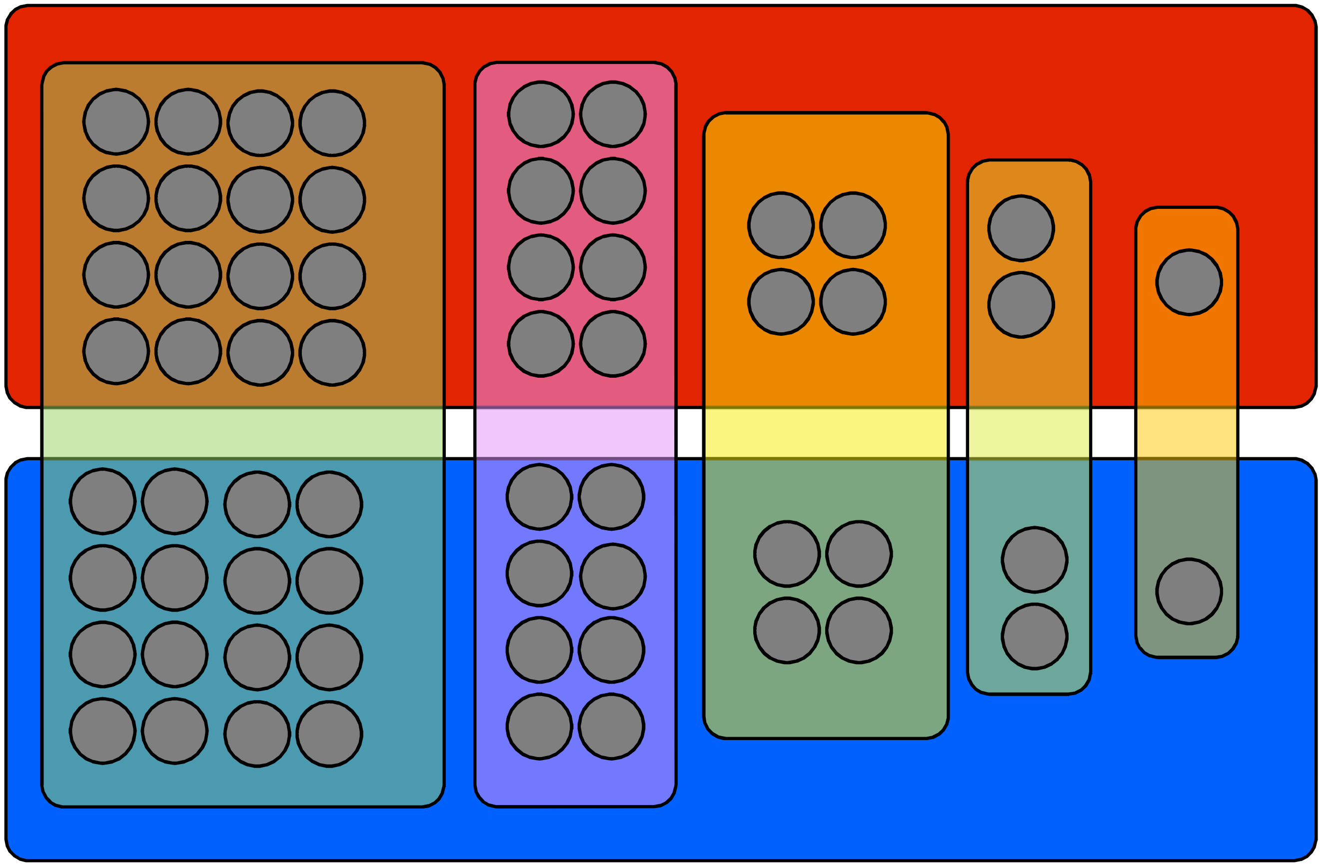


Greedy solution: **5** sets

Greedy solution: $\sim \log_2 n$ sets

Greedy Set Cover: Repeatedly choose the set that maximizes # new elements covered



Optimal:
2 sets

Greedy Approximation to Set-Cover

Theorem: If there is a set cover of size k then the greedy set cover has size $\leq k \ln n$.

Proof: Suppose that there is a set cover of size k.

At each step all elements remaining are covered by these k sets.

So always a set available covering $\geq 1/k$ fraction of remaining elts.

So # of uncovered elts after i sets $\leq \left(1 - \frac{1}{k}\right) \times$ # after $i - 1$ sets.

Total after t sets $\leq n\left(1 - \frac{1}{k}\right)^t < n \cdot e^{-t/k} = 1$ for $t = k \ln n$. ■

$$\boxed{1 - x < e^{-x} \text{ for } x > 0}$$

$e^{1/n}$ ↑

$e^x \geq x-1$

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Given: a set of $\boldsymbol{n}$ cities $\boldsymbol{v_1, \dots, v_n}$ and distance function $\boldsymbol{d}$ that gives distance $\boldsymbol{d(v_i, v_j)}$ between each pair of cities

Find the shortest tour that visits all $\boldsymbol{n}$ cities.

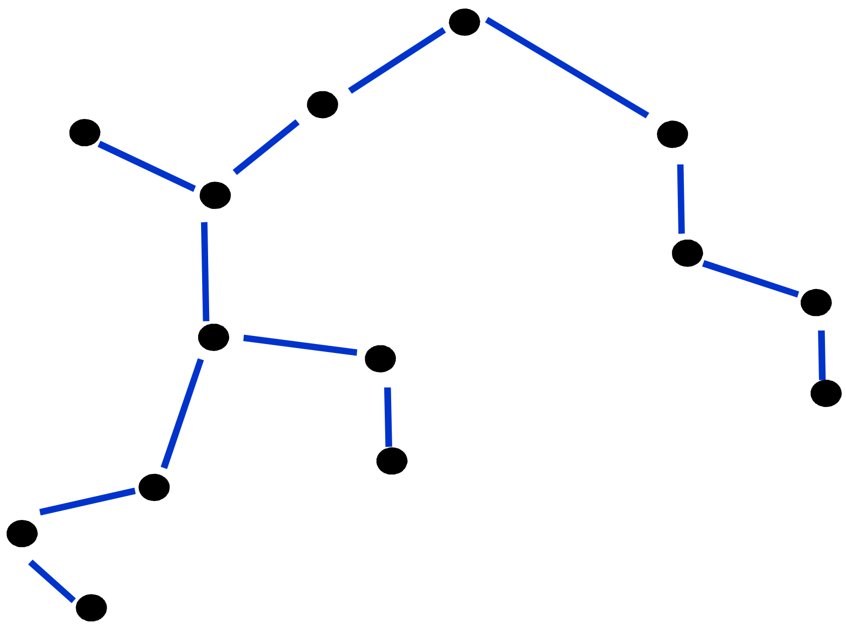
MetricTSP:

The distance function $\boldsymbol{d}$ satisfies the triangle inequality:

$$\boldsymbol{d(u,w) \leq d(u,v) + d(v,w)}$$

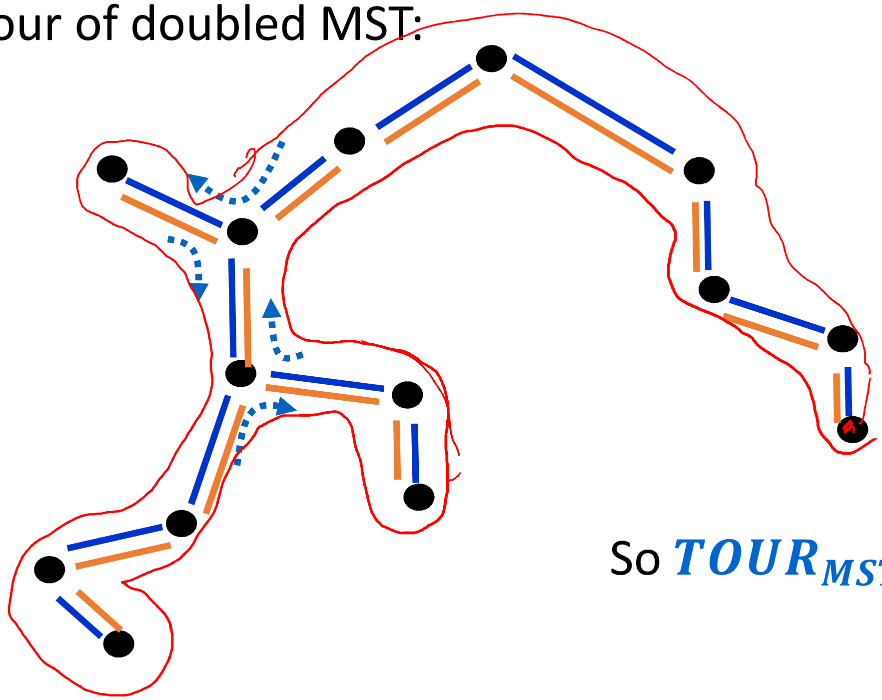
Proper tour: visit each city exactly once.

Minimum Spanning Tree Approximation: Factor of 2



TSP: Minimum Spanning Tree Factor 2 Approximation

Euler Tour of doubled MST:



Euler tour covers each edge twice
so $\boldsymbol{TOUR_{MST}(G) = 2\ MST(G)}$

Any tour contains a spanning tree
so $\boldsymbol{MST(G) \leq TOUR_{OPT}(G)}$

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This visits each node more than once, so not a proper tour.

Why did this work?

- We found an Euler tour on a graph that used the edges of the original graph (possibly repeated).
- The weight of the tour was the total weight of the new graph.
- Suppose now
  - All edges possible
  - Weights satisfy the triangle inequality (MetricTSP)

MetricTSP: Minimum Spanning Tree Factor 2 Approximation

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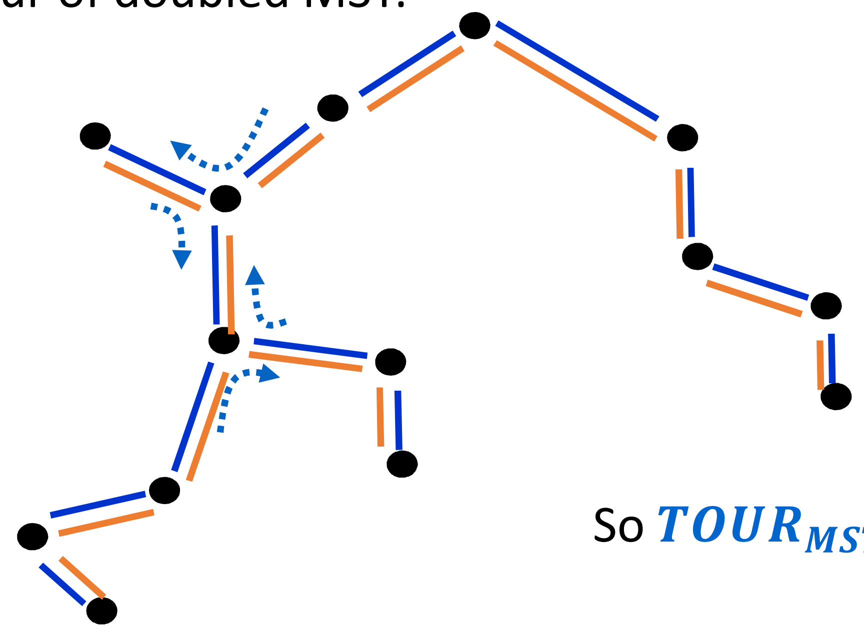

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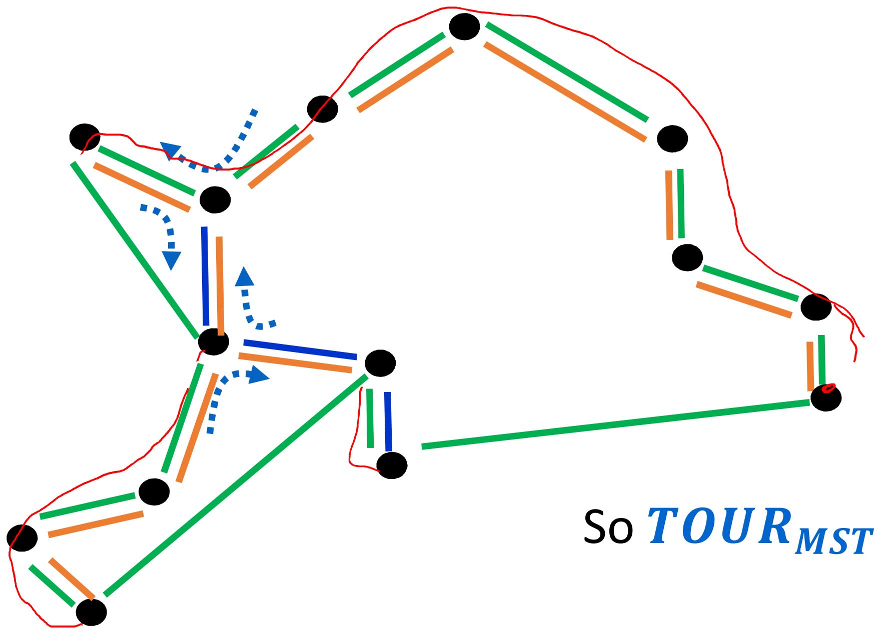
Instead: take shortcut to next unvisited vertex on the Euler tour
By triangle inequality this can only be shorter.

MetricTSP: Minimum Spanning Tree Factor 2 Approximation



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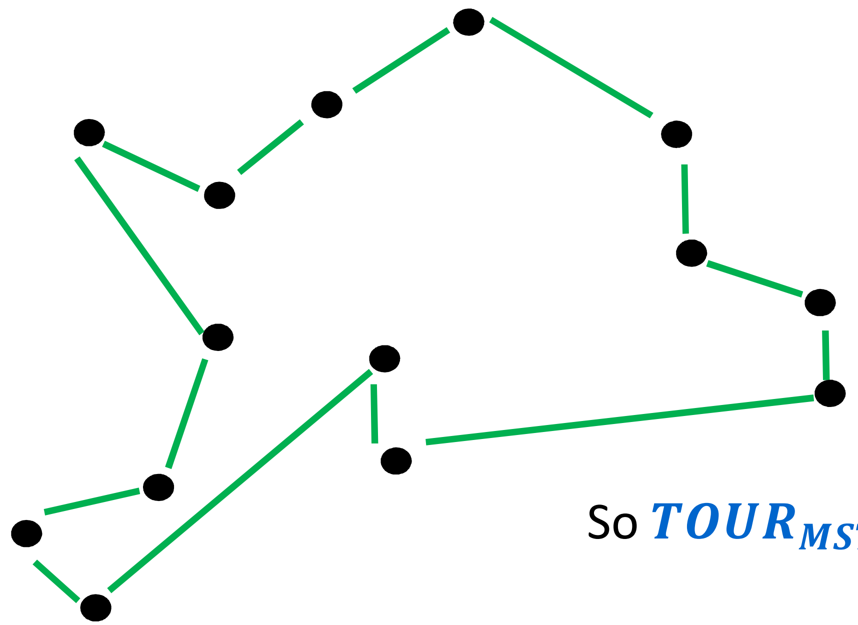
MetricTSP: Minimum Spanning Tree Factor 2 Approximation

Final:



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Christofides Algorithm: A factor 3/2 approximation

Any subgraph of the weighted complete graph that has an Euler Tour will work also!

Fact: To have an Euler Tour it suffices to have all degrees even.

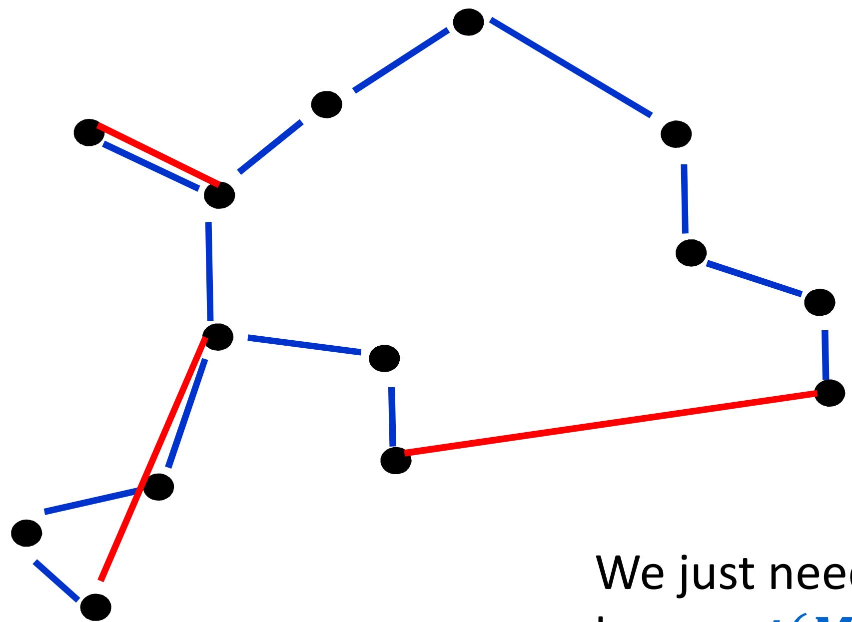
Christofides Algorithm:

- Compute an MST $\boldsymbol{T}$
- Find the set $\boldsymbol{O}$ of odd-degree vertices in $\boldsymbol{T}$
- Add a minimum-weight perfect matching* $\boldsymbol{M}$ on the vertices in $\boldsymbol{O}$ to $\boldsymbol{T}$ to make every vertex have even degree
  - There are an even number of odd-degree vertices!
- Use an Euler Tour $\boldsymbol{E}$ in $\boldsymbol{T} \cup \boldsymbol{M}$ and then shortcut as before

Theorem: $\boldsymbol{Cost(E) \leq 1.5\, TOUR_{OPT}}$

*Requires finding optimal matchings in general graphs, not just bipartite ones

Christofides Approximation

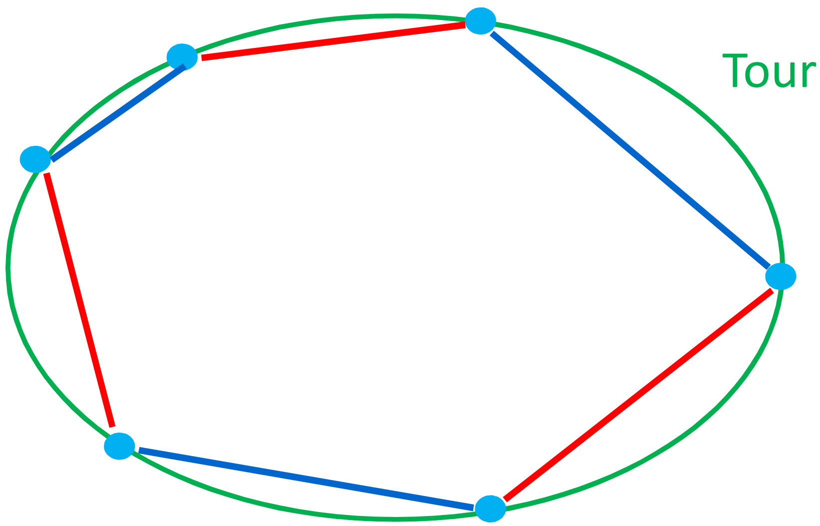


Any tour contains a spanning tree so $MST \leq TOUR_{OPT}$

We just need to show that the matching M has $cost(M) \leq TOUR_{OPT}/2$

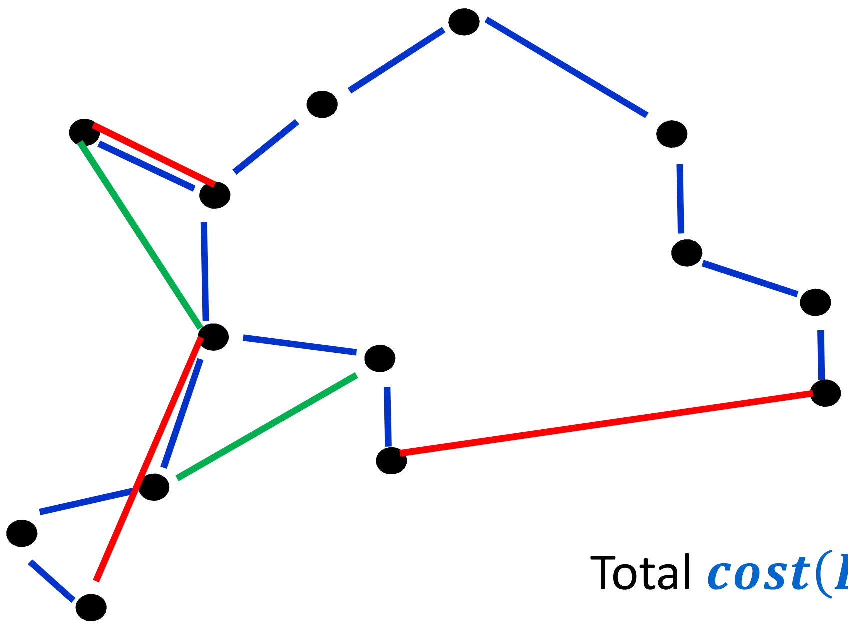
Christofides Approximation

Any tour costs at least the cost of two matchings M_1 and M_2 on O



$$2\,cost(M) \leq cost(M_1) + cost(M_2) \leq TOUR_{OPT}$$

Christofides Approximation Final Tour



Total $\boldsymbol{cost(E) \leq 3\ TOUR_{OPT}/2}$

Max-3SAT Approximation

Max-3SAT: Given a 3CNF formula $\boldsymbol{F}$ find a truth assignment that satisfies the maximum possible # of clauses of $\boldsymbol{F}$.

Observation: A single clause on 3 variables only rules out $\mathbf{1/8}$ of the possible truth assignments since each literal has to be false to be ruled out.

$\Rightarrow$ a random truth assignment will satisfy the clause with probability $\mathbf{7/8}$.

So in expectation, if $\boldsymbol{F}$ has $\boldsymbol{m}$ clauses, a random assignment satisfies $\mathbf{7}\boldsymbol{m}\mathbf{/8}$ of them.

A greedy algorithm can achieve this: Choose most frequent literal appearing in clauses that are not yet satisfied and set it to true.

If $\mathbf{P} \neq \mathbf{NP}$ no better approximation is possible

Knapsack Problem

Each item has a value v_i and a weight w_i.

Maximize $\sum_{i \in S} v_i$ with $\sum_{i \in S} w_i \leq W$.

Theorem: For any $\varepsilon > 0$ there is an algorithm that produces a solution within $(1 + \varepsilon)$ factor of optimal for the Knapsack problem with running time $O(n^2/\varepsilon^2)$

"Polynomial-Time Approximation Scheme" or PTAS

Algorithm: Maintain the high order bits in the dynamic programming solution.

Hardness of Approximation

Polynomial-time approximation algorithms for $\mathbf{NP}$-hard optimization problems can sometimes be ruled out unless $\mathbf{P} = \mathbf{NP}$.

Easy example:

Coloring: Given a graph $G = (V, E)$ find the smallest k such that G has a k-coloring.

Because $\mathbf{3}$-coloring is $\mathbf{NP}$-hard, no approximation ratio better than $\mathbf{4/3}$ is possible unless $\mathbf{P} = \mathbf{NP}$ because you would have to be able to figure out if a $\mathbf{3}$-colorable graph can be colored in $< \mathbf{4}$ colors. i.e. if it can be $\mathbf{3}$-colored.

- We now know a huge amount about the hardness of approximating $\mathbf{NP}$ optimization problems if $\mathbf{P} \neq \mathbf{NP}$.
- Approximation factors are very different even for closely related problems like **Vertex-Cover** and **Independent-Set**.

Approximation Algorithms/Hardness of Approximation

Research has classified many problems based on what kinds of polytime approximations are possible if $\mathbf{P} \neq \mathbf{NP}$

- **Best:** $(\mathbf{1} + \boldsymbol{\varepsilon})$ factor for any $\boldsymbol{\varepsilon} > \mathbf{0}$. (PTAS)
  - packing and some scheduling problems, TSP in plane
- Some fixed constant factor $> \mathbf{1}$. e.g. $\mathbf{2}, \mathbf{3}/\mathbf{2}, \mathbf{8}/\mathbf{7}, \mathbf{100}$
  - Vertex Cover, Max-3SAT, MetricTSP, other scheduling problems
  - Exact best factors or very close upper/lower bounds known for many problems.
- $\Theta(\log \boldsymbol{n})$ factor
  - Set Cover, Graph Partitioning problems
- **Worst:** $\Omega(\boldsymbol{n}^{\mathbf{1}-\boldsymbol{\varepsilon}})$ factor for every $\boldsymbol{\varepsilon} > \mathbf{0}$.
  - Clique, Independent-Set, Coloring