# CSE 421
# Introduction to Algorithms

## Lecture 17: Polynomial-Time MaxFlow/MinCut Algorithms

# Announcements

Midterm next **Wednesday, November 8, 6:00 – 7:30 pm in this room**

- See post on Important Midterm Information

- Links to sample midterm, practice problems, and reference sheet posted yesterday – (zoom

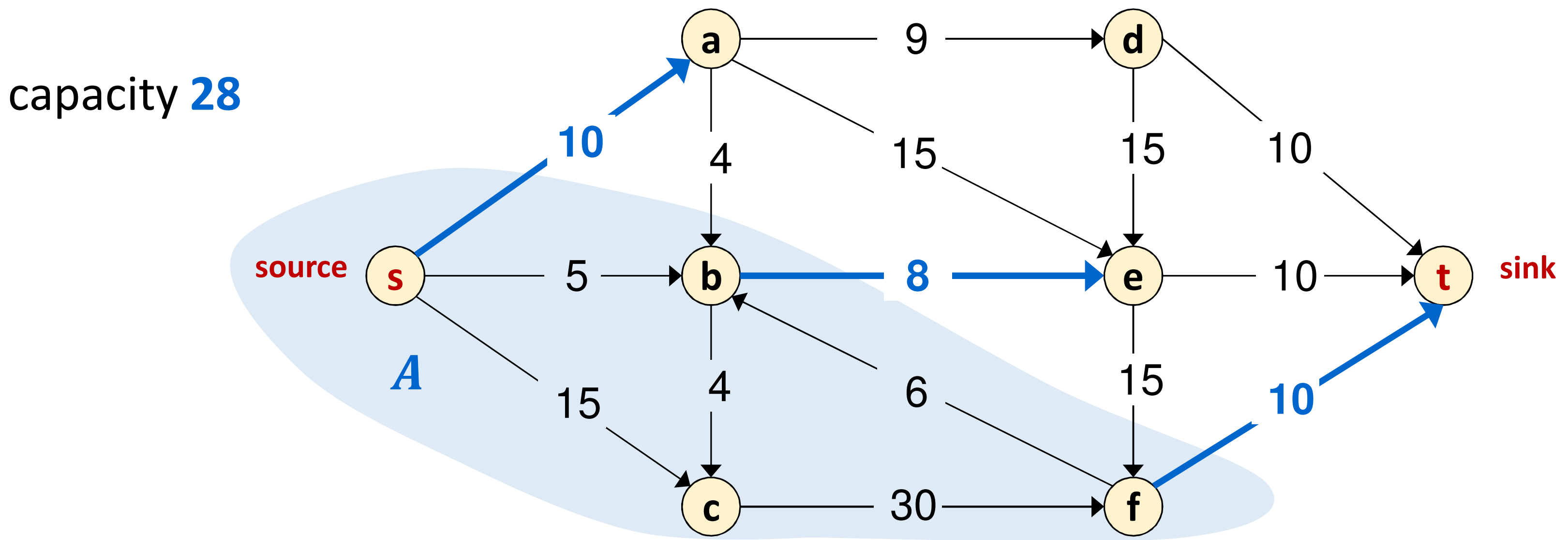
- Zoom review session for Q&A on Tuesday Nov 7 at 4:30 pm.

# Minimum Cut Problem

**Minimum s-t cut problem:**

**Given:** a flow network

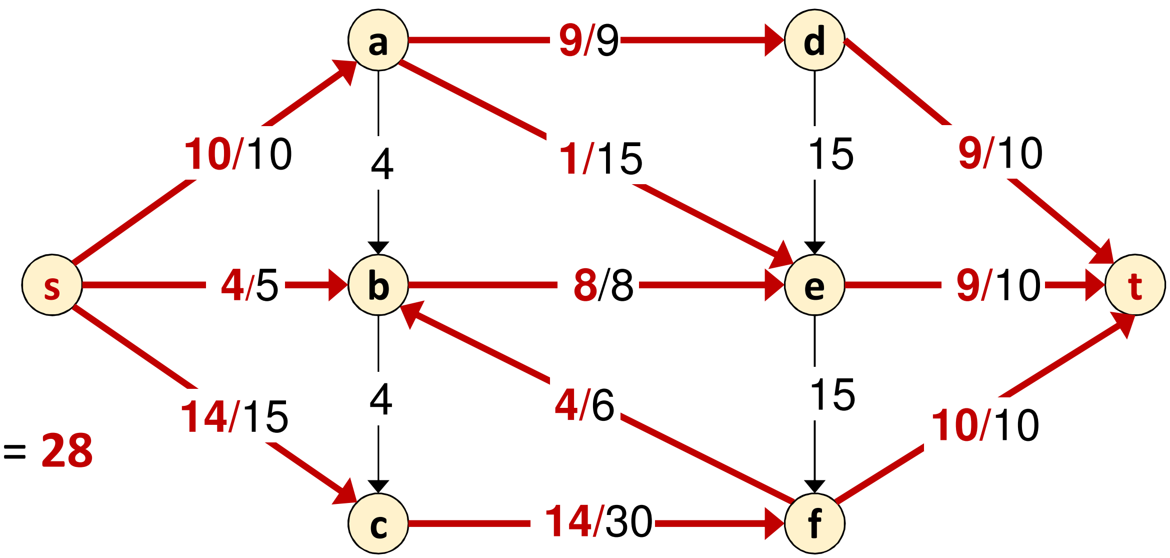
**Find:** an $s$-$t$ cut $(A, B)$ of minimum capacity $c(A,B) = \sum_{e \text{ out of } A} c(e)$



# Maximum Flow Problem

**Given:** a flow network

**Find:** an $s$-$t$ flow of maximum value




value = **28**

# Ford-Fulkerson Augmenting Path Algorithm

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Gf ← residual graph

   while (Gf has an s-t path P) {
      f ← Augment(f, c, P)
      update Gf
   }
   return f
}
```

```
Augment(f, c, P) {
   b ← bottleneck(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b
      else       f(eR)← f(eR) - b
   }
   return f
}
```

# MaxFlow/MinCut & Ford-Fulkerson Algorithm

**Augmenting Path Theorem:** Flow $f$ is a max flow $\Leftrightarrow$ there are no augmenting paths wrt $f$

**Max-Flow Min-Cut Theorem:** The value of the max flow equals the value of the min cut.
[Elias-Feinstein-Shannon 1956, Ford-Fulkerson 1956] **"MaxFlow = MinCut"**

**Flow Integrality Theorem:** If all capacities are integers then there is a maximum flow with all-integer flow values.

**Ford-Fulkerson Algorithm:** $O(m)$ per iteration. With integer capacities each at most $C$ need at most **MaxFlow** $< nC$ iterations for a total of $O(mnC)$ time.

# Ford-Fulkerson Efficiency

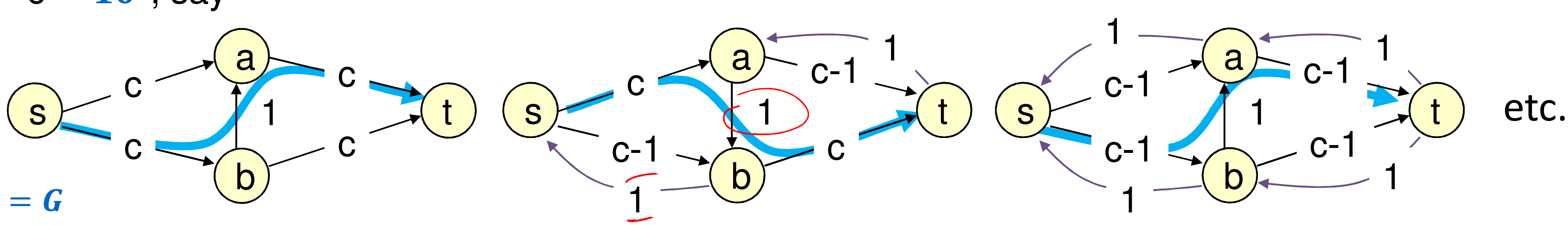
Worst case runtime $O(mnC)$ with integer capacities $\leq C$.

- $O(m)$ time per iteration.
- At most $nC$ iterations.
- This is "pseudo-polynomial" running time.

log C bits of input

- May take exponential time, even with integer capacities:

c $= 10^9$, say




$G_f = G$

# Choosing Good Augmenting Paths

# Polynomial-Time Variants of Ford-Fulkerson

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

**Goal:** Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

- Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]
  - Max bottleneck capacity.
  - Sufficiently large bottleneck capacity.
  - Fewest number of edges.

# Polynomial-Time MaxFlow: Capacity Scaling

General idea:

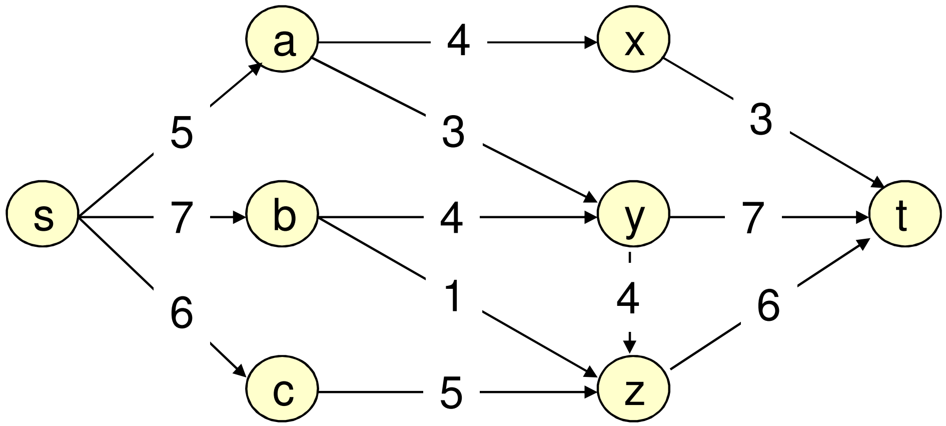
- Choose augmenting paths **P** with 'large' capacity.

- Can augment flows along a path **P** by any amount $\leq \textbf{bottleneck}(\boldsymbol{P})$
  - Ford-Fulkerson still works

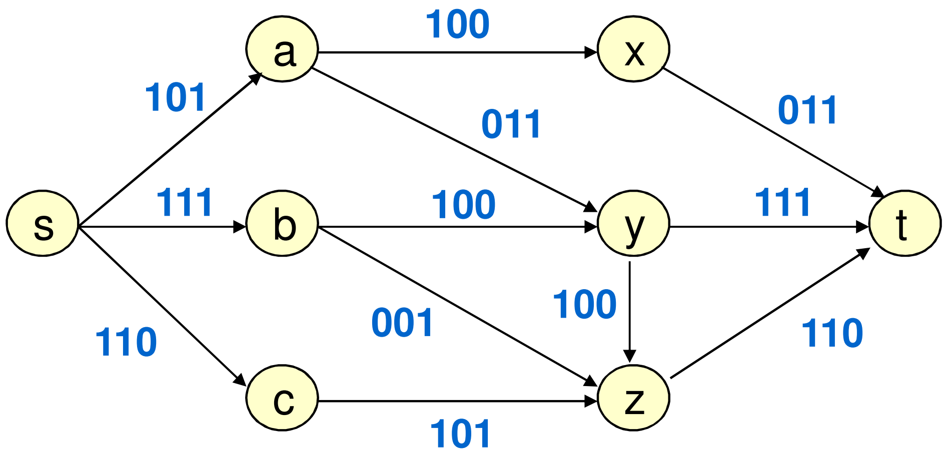
- Choose that amount to be "nice round number" (i.e. a big power of 2.)

- Get a flow that is maximum for the high-order bits first and then add more bits later

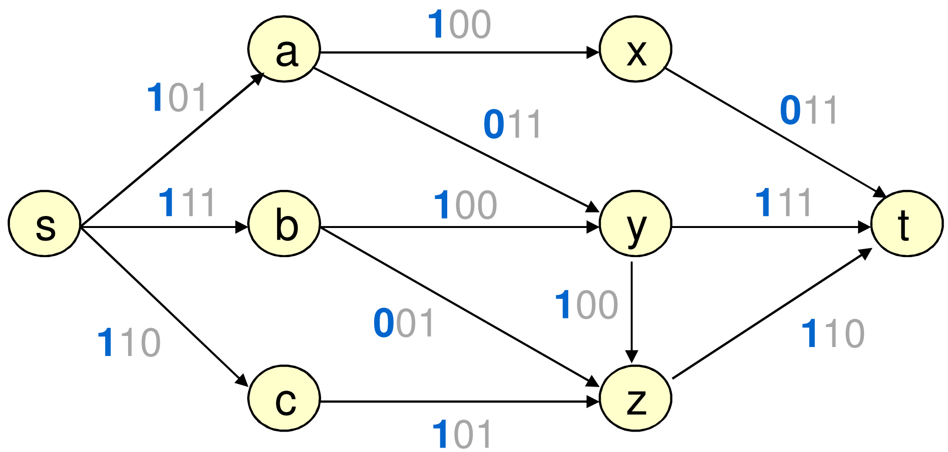
# Capacity Scaling



# Write Capacities in Binary



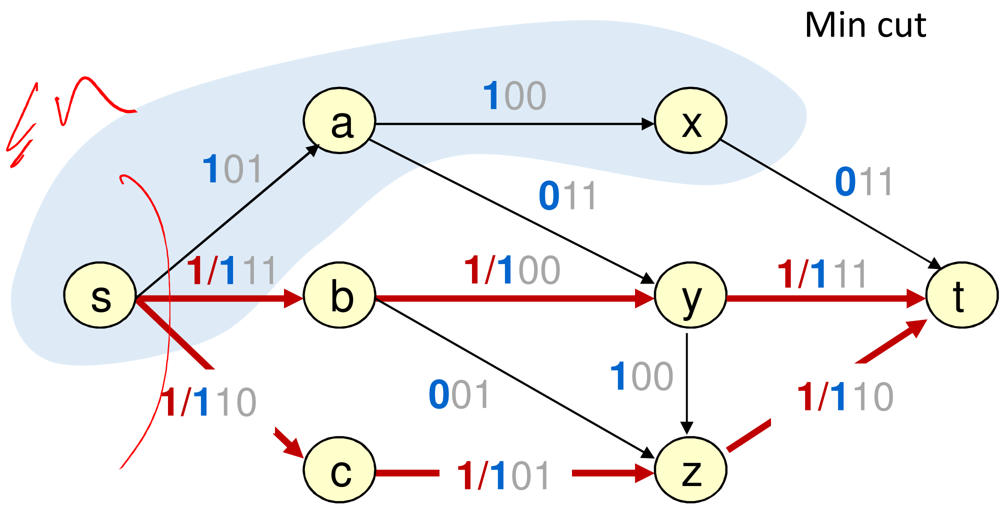
# Capacity Scaling Bit 1




Solve flow problem with capacities with just the high-order bit:

# Capacity Scaling Bit 1

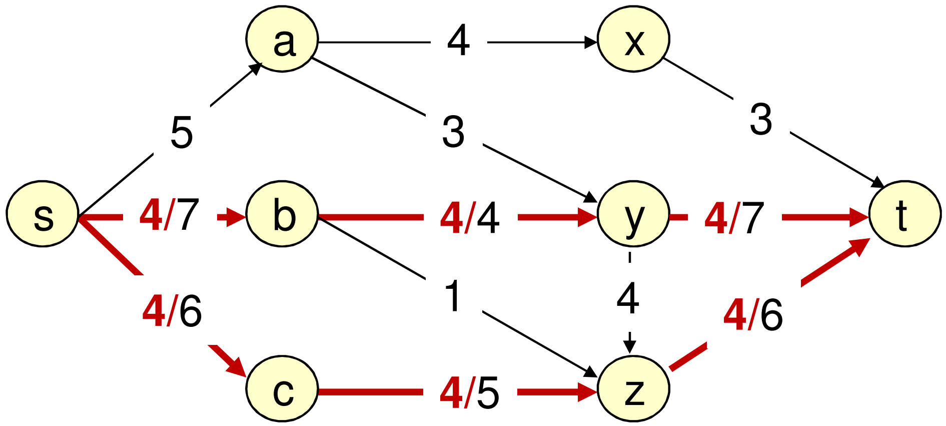
m edges
n vertices




Solve flow problem with capacities with just the high-order bit:

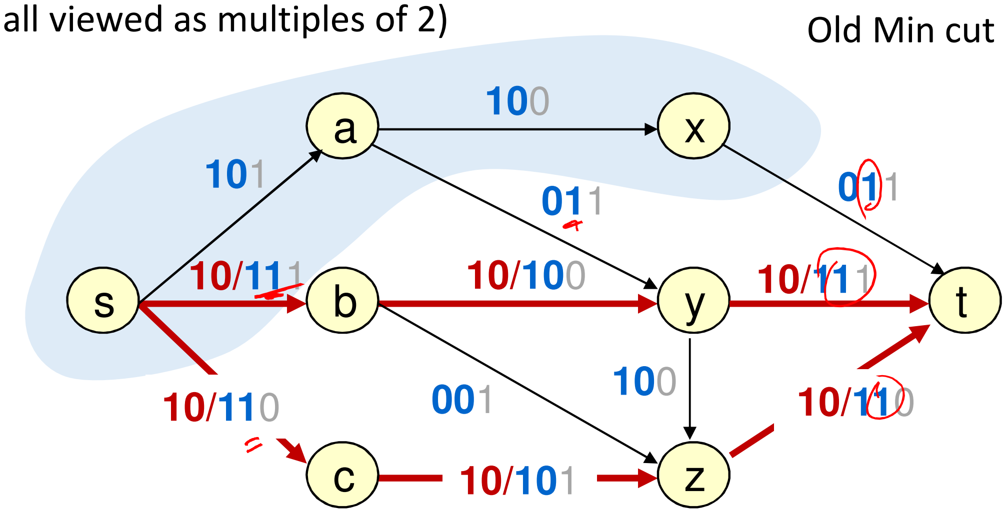
- Each edge has "capacity" $\leq \mathbf{1}$ (equivalent to $\mathbf{4}$ here)
- Time $O(\boldsymbol{mn})$

# Capacity Scaling Bit 1



# Capacity Scaling Bits 2 1 and 2

Add 0 bit to the end of the flows
Add bit 2 to capacities (all viewed as multiples of 2)




≤ m edges leaving that old cut
≤ +1 for edge

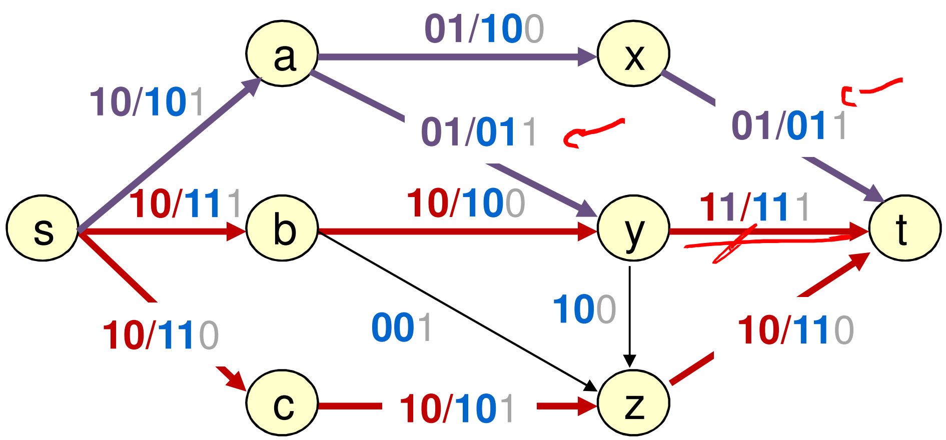
Solve flow problem with capacities with the **2** high-order bits:

- Capacity of old min cut goes up by $\leq 1$ per edge (equivalent to $2$ here) for a total residual capacity $\leq m$.

m capacities
O(m)

$O(m^2)$ steps

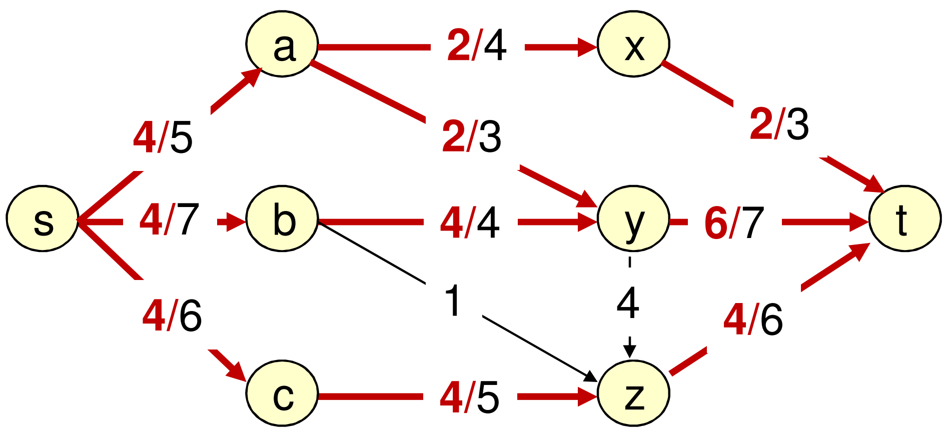
# Capacity Scaling Bit 2




Solve flow problem with capacities with the **2** high-order bits:

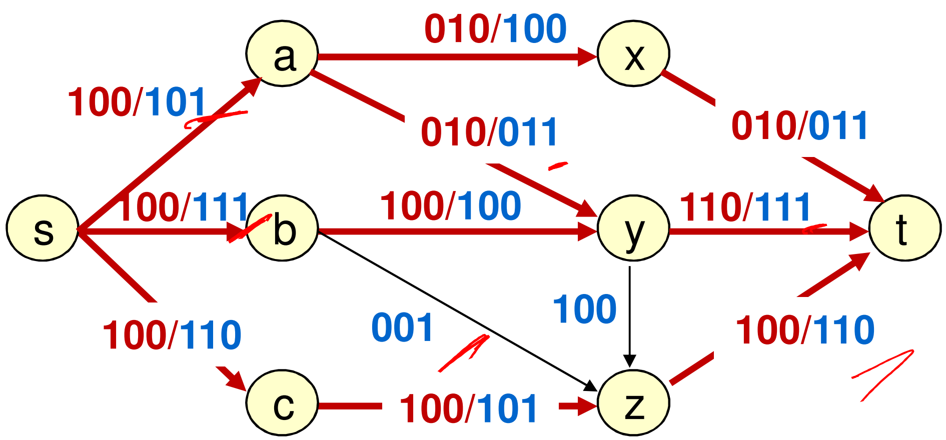
- Capacity of old min cut goes up by $\leq \mathbf{1}$ per edge (equivalent to $\mathbf{2}$ here) for a total residual capacity $\leq \boldsymbol{m}$.
- Time $O(\boldsymbol{m}^2)$ for $\leq \boldsymbol{m}$ iterations.

# Capacity Scaling Bits 1 and 2



# Capacity Scaling Bit 3

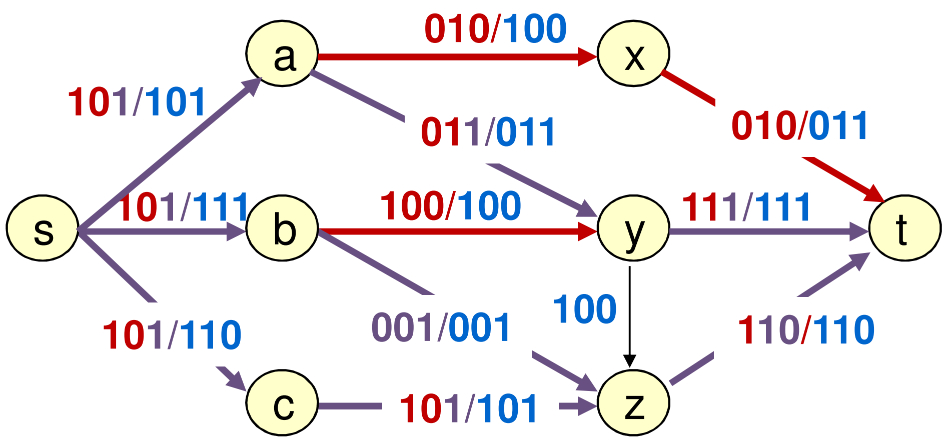
Add 0 bit to the end of the flows
Add bit 3 to capacities (all now multiples of 1)




Solve flow problem with capacities with all 3 bits:

- Capacity of old min cut goes up by $\leq 1$ per edge for a total residual capacity $\leq m$.

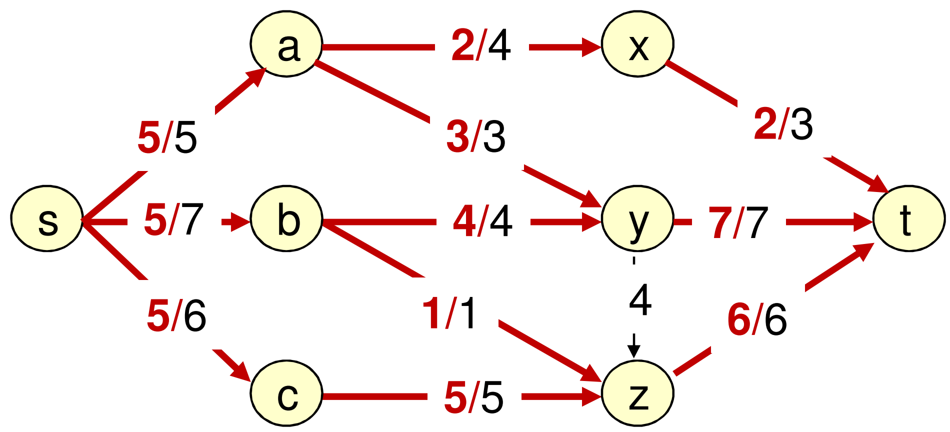
# Capacity Scaling Bit 3




Solve flow problem with capacities with all 3 bits:

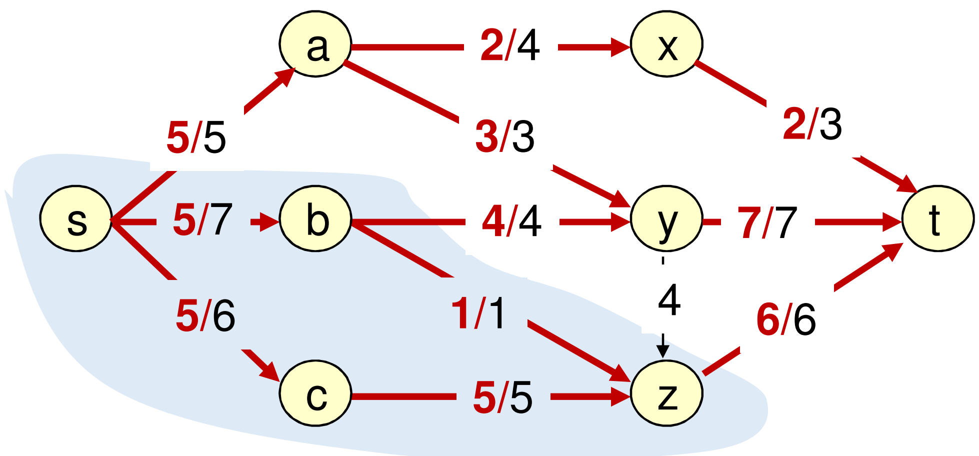
- Capacity of old min cut goes up by $\leq \mathbf{1}$ per edge for a total residual capacity $\leq \boldsymbol{m}$.
- Time $O(\boldsymbol{m}^2)$ for $\leq \boldsymbol{m}$ iterations.

# Capacity Scaling All Bits




Flow = **15**

# Capacity Scaling All Bits




Flow = **15**

Cut Value = **15**

Flow is a MaxFlow

# Total time for capacity scaling

- Number of rounds = $\lceil \log_2 \boldsymbol{C} \rceil$ where $\boldsymbol{C}$ is the largest capacity
- Time per round $O(\boldsymbol{m}^2)$
  - At most $\boldsymbol{m}$ augmentations per round
  - $O(\boldsymbol{m})$ time per augmentation

Total time $O(\boldsymbol{m}^2 \log \boldsymbol{C})$

Great! This is now polynomial time in the input size.

Can we get more?

- What about an algorithm with a number of arithmetic operations that doesn't depend on the size of the numbers?

# Polynomial-Time Variants of Ford-Fulkerson

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

**Goal:** Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

- Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]
  - Max bottleneck capacity.
  - Sufficiently large bottleneck capacity.
  - Fewest number of edges. (i.e. just run BFS to find an augmenting path.)

# Edmonds-Karp Algorithm (Ford-Fulkerson with BFS)

Use Breadth First Search as the search algorithm to find an $s$-$t$ path in $G_f$.

- Using any shortest augmenting path

**Theorem**: Ford-Fulkerson using BFS terminates in $O(m^2 n)$ time. [Edmonds-Karp, Dinitz]

"One of the most obvious ways to implement Ford-Fulkerson is always polynomial time"

Why might this be good intuitively?

- Longer augmenting paths involve more edges so may be more likely to hit a low residual capacity one which would limit the amount of flow improvement.

The proof uses a completely different idea…

# Edmonds-Karp Algorithm (Ford-Fulkerson with BFS)

$e$

$e^r$

**Analysis Focus:**

For any edge $e$ that could be in the residual graph $G_f$, (either an edge in $G$ or its reverse)

count # of iterations that $e$ is the first bottleneck edge on the augmenting path chosen by the algorithm.

**Claim:** This can't happen in more than $n/2$ iterations. for any fixed edge $e$:

**Proof:** Write $e = (u, v)$.

Show that each time it happens, the distance from $s$ to $u$ in the residual graph $G_f$ is at least $2$ more than it was the last time.

This would be enough since the distance is either $< n$ (or infinite and hence $u$ isn't reachable) so this can happen at most $n/2$ times.

# Distances in the Residual Graph

**Key Lemma:** Let $f$ be a flow, $G_f$ the residual graph, and $P$ be a shortest augmenting path. No vertex is closer to $s$ in the residual graph after augmenting along $P$.

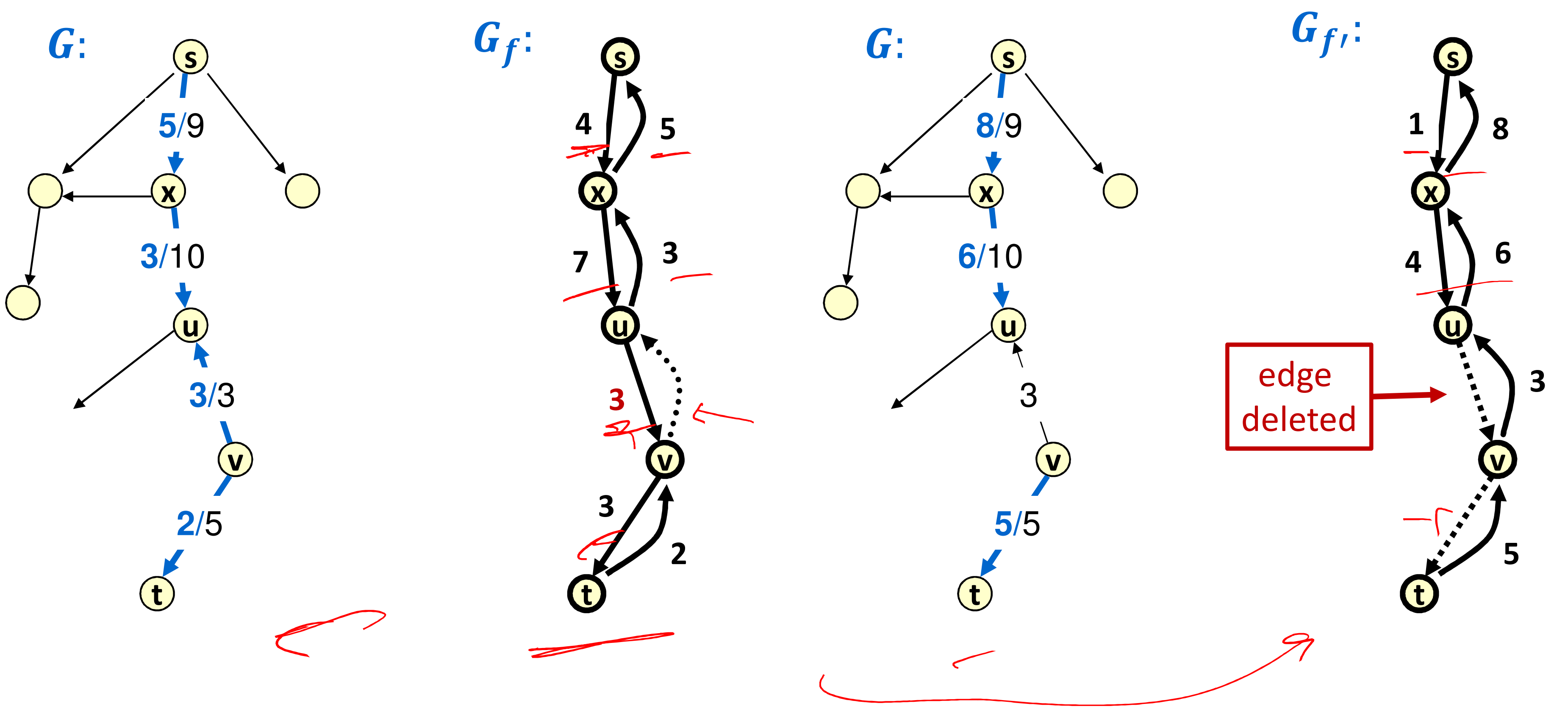
**Proof:** Augmenting along $P$ can only change the edges in $G_f$ by either:

1. Deleting a forward edge
   - Deleting any edge can never reduce distances
2. Add a backward edge $(v, u)$ that is the reverse of an edge $(u, v)$ of $P$
   - Since $P$ was a shortest path in $G_f$, the distance from $s$ to $v$ in $G_f$ is already more than the distance from $s$ to $u$. Using the new backward edge $(v, u)$ to get to $u$ would be an even longer path to $u$ so it is never on a shortest path to any node in the new residual graph. ■

u v P

$G_f$

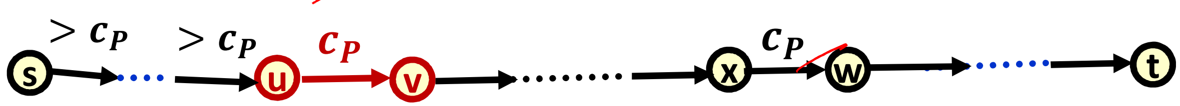
# Augmentation vs BFS



# First Bottleneck Edges in $G_f$

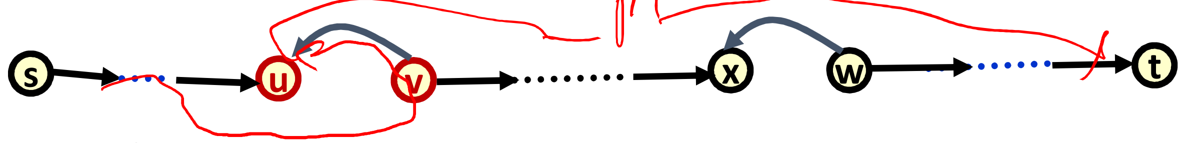
Shortest $s$-$t$ path $P$ in $G_f$

Write $c_P = \textbf{bottleneck}(P)$




$\boxed{d_f(s,v) = d_f(s,u) + 1 \text{ since } P \text{ is a shortest path.}}$

After augmenting along $P$, edge $(u,v)$ disappears; but will have edge $(v,u)$




distance is $\geq 2$ larger than before

For $(u,v)$ to be a first bottleneck edge later, it must get added back to the residual graph by augmenting along a shortest path $P'$ containing $(v,u)$ in $G_{f'}$ for some flow $f'$

Since $P'$ is shortest $d_{f'}(s,u) = d_{f'}(s,v) + 1 \geq d_f(s,v) + 1 = d_f(s,u) + 2$

The next time that $(u,v)$ is first bottleneck edge is even later so distance is at least as large! ■

# Edmonds-Karp Algorithm (Ford-Fulkerson with BFS)

**Analysis Focus:**

For any edge $e$ that could be in the residual graph $G_f$, (either an edge in $G$ or its reverse)

count # of iterations that $e$ is the first bottleneck edge on the augmenting path chosen by the algorithm.

**Claim:** This can't happen in more than $n/2$ iterations for any fixed edge $e$

**Claim ⇒ Theorem:**

Only $2m$ edges and $O(m)$ time per iteration so $O(m^2 n)$ time overall. ■

Which is better in practice $O(m^2 n)$ vs. $O(m^2 \log C)$?

# History & State of the Art for MaxFlow Algorithms

| # | year | discoverer(s) | bound |
|---|---|---|---|
| 1 | 1951 | Dantzig | $O(n^2 mU)$ |
| 2 | 1955 | Ford & Fulkerson | $O(nmU)$ |
| 3 | 1970 | Dinitz<br>Edmonds & Karp | $O(nm^2)$ |
| 4 | 1970 | Dinitz | $O(n^2 m)$ |
| 5 | 1972 | Edmonds & Karp<br>Dinitz | $O(m^2 \log U)$ |
| 6 | 1973 | Dinitz<br>Gabow | $O(nm \log U)$ |
| 7 | 1974 | Karzanov | $O(n^3)$ |
| 8 | 1977 | Cherkassky | $O(n^2 \sqrt{m})$ |
| 9 | 1980 | Galil & Naamad | $O(nm \log^2 n)$ |
| 10 | 1983 | Sleator & Tarjan | $O(nm \log n)$ |
| 11 | 1986 | Goldberg & Tarjan | $O(nm \log(n^2/m))$ |
| 12 | 1987 | Ahuja & Orlin | $O(nm + n^2 \log U)$ |
| 13 | 1987 | Ahuja et al. | $O(nm \log(n\sqrt{\log U}/(m+2)))$ |
| 14 | 1989 | Cheriyan & Hagerup | $E(nm + n^2 \log^2 n)$ |
| 15 | 1990 | Cheriyan et al. | $O(n^3/\log n)$ |
| 16 | 1990 | Alon | $O(nm + n^{8/3} \log n)$ |
| 17 | 1992 | King et al. | $O(nm + n^{2+\epsilon})$ |
| 18 | 1993 | Phillips & Westbrook | $O(nm(\log_{m/n} n + \log^{2+\epsilon} n))$ |
| 19 | 1994 | King et al. | $O(nm \log_{m/(n \log n)} n)$ |
| 20 | 1997 | Goldberg & Rao | $O(m^{3/2} \log(n^2/m) \log U)$<br>$O(n^{2/3} m \log(n^2/m) \log U)$ |

Source: Goldberg & Rao, FOCS '97

| | | | |
|---|---|---|---|
| 21 | 2013 | Orlin | $O(mn)$ |
| 22 | 2014 | Lee & Sidford | $m\sqrt{n} \log^{O(1)} n \log U$ |
| 23 | 2016 | Madry | $m^{10/7} U^{1/7} \log^{O(1)} n$ |
| 24 | 2021 | Gao, Liu, & Peng | $m^{3/2-1/328} \log^{O(1)} n \log U$ |
| 25 | 2022 | van den Brand et al. | $m^{3/2-1/58} \log^{O(1)} n \log U$ |
| 26 | 2022 | Chen et al. | $m^{1+o(1)} \log U$ |

Tables use ***U*** instead of ***C*** for the upper bound on capacities

Methods: Augmenting Paths – increase flow to capacity
Preflow-Push – decrease flow to get flow conservation
Linear Programming – randomized high probability

$\tilde{O}(f(n))$ $\big(O(f(n)\log^c(n))$ for some $c\big)$ $\log^c n$