CSE 421
Introduction to Algorithms

Lecture 15: Network Flow

Announcements

Midterm Reminder:

- Date:
  - Next **Wednesday, November 8, 6:00 – 7:30 pm in this room**
  - Exam designed for a regular class time-slot but this includes extra time to finish.
- Coverage:
  - Up to the end of last Thursday's section on Dynamic Programming

- Sample midterm for practice problems and length coming soon.
  - Will include "summary sheet" available to you on the midterm.
- This week's section will focus on review problems.

- Zoom review session for Q&A on Tuesday Nov 7 at 4:30 pm.

Maximum Flow and Minimum Cut

Max flow and min cut:

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

Nontrivial applications / reductions:

- Data mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Strip mining.
- Network reliability.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- many many more …

Origins of Max Flow and Min Cut Problems

Max Flow problem formulation:

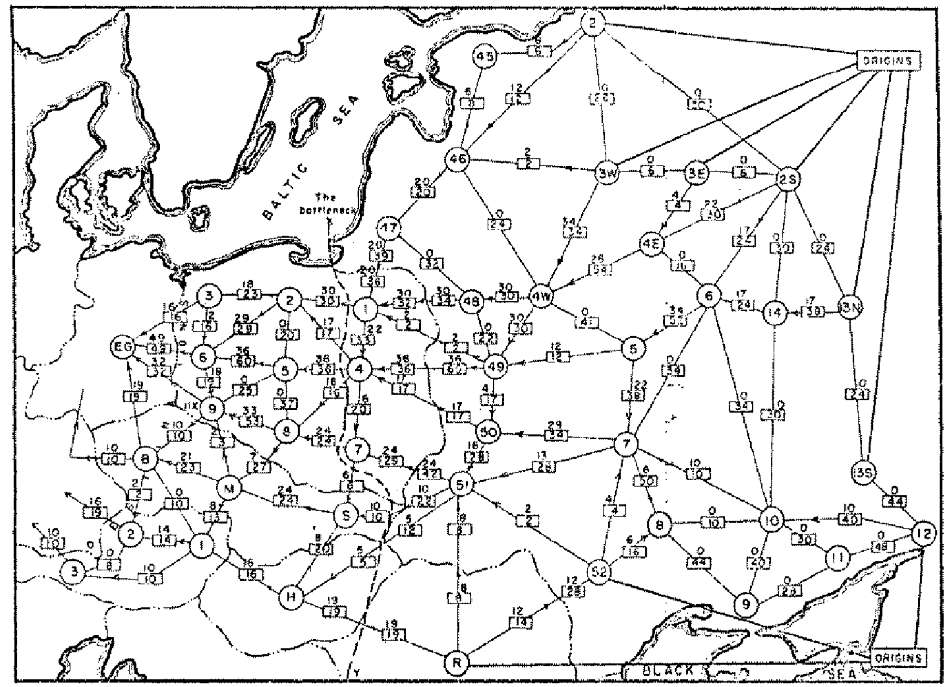
- [Tolstoy 1930] Rail transportation planning for the Soviet Union

Min Cut problem formulation:

- Cold War: US military planners want to find a way to cripple Soviet supply routes
- [Harris 1954] Secret RAND corp report for US Air Force

[Ford-Fulkerson 1955] Problems are equivalent

Soviet Rail Network 1955

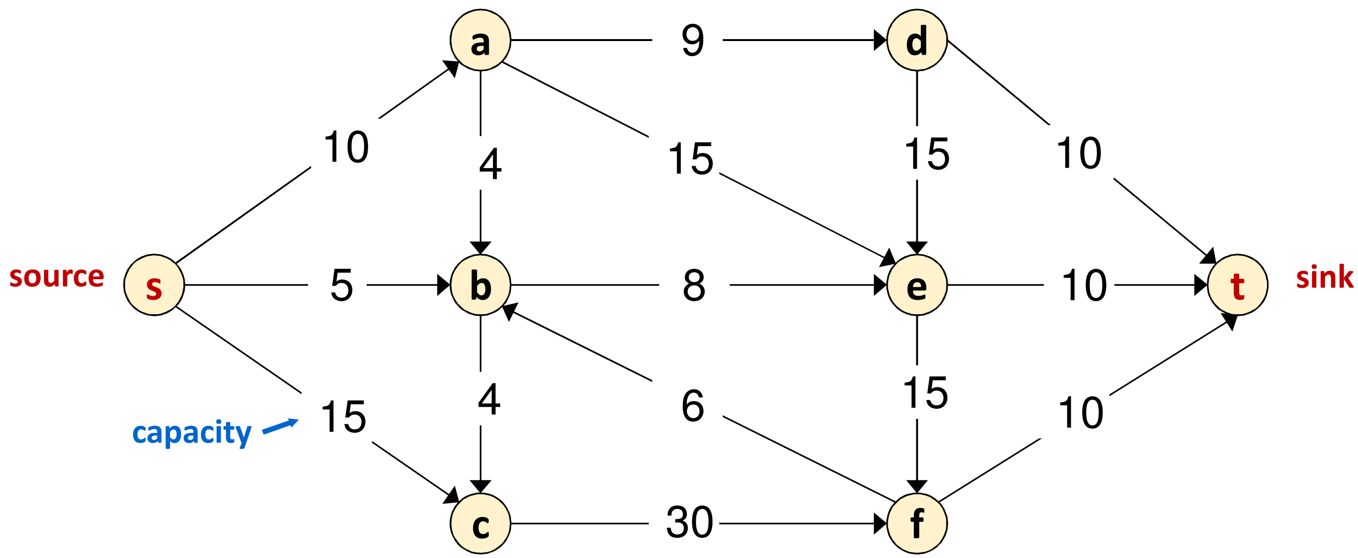


Reference: *On the history of the transportation and maximum flow problems*. Alexander Schrijver in Math Programming, 91: 3, 2002.

Flow Network

Flow network:

- Abstraction for material *flowing* through the edges.
- $G = (V, E)$ directed graph, no parallel edges.
- Two distinguished nodes: s = source, t = sink.
- $c(e)$ = capacity of edge $e \geq 0$.



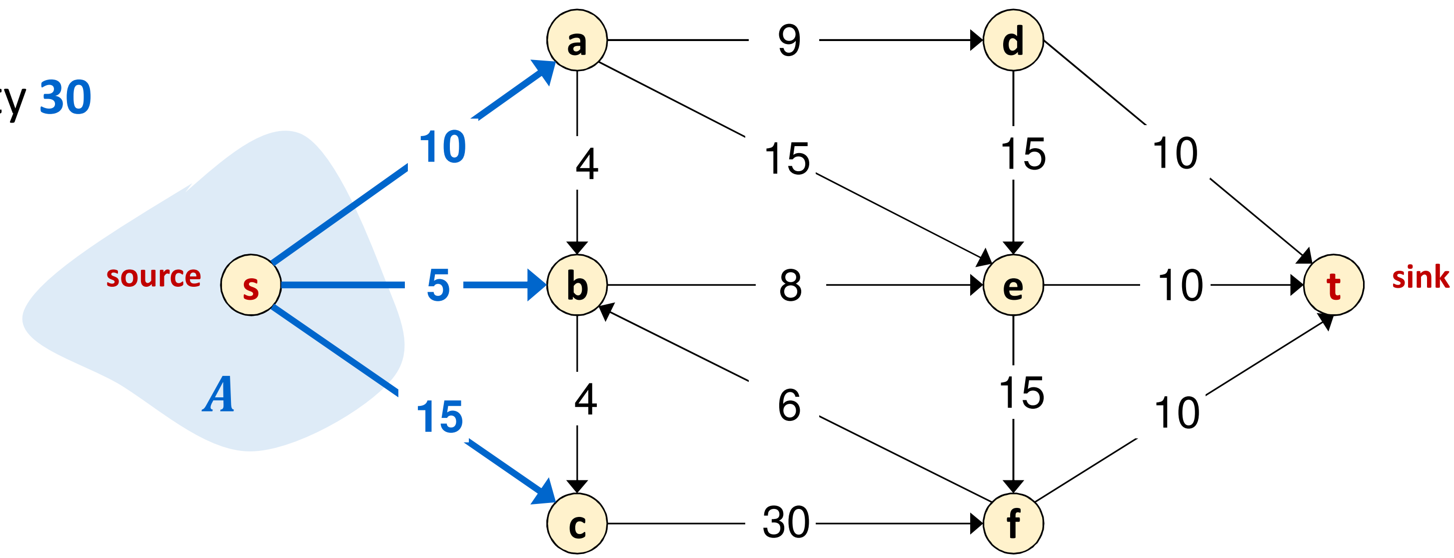
Cuts

Defn: An $\boldsymbol{s}$-$\boldsymbol{t}$ cut is a partition $(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{V}$ with $\boldsymbol{s} \in \boldsymbol{A}$ and $\boldsymbol{t} \in \boldsymbol{B}$.

The capacity of cut $(\boldsymbol{A}, \boldsymbol{B})$ is

$$\boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B}) = \sum_{\boldsymbol{e} \text{ out of } \boldsymbol{A}} \boldsymbol{c}(\boldsymbol{e})$$

capacity **30**



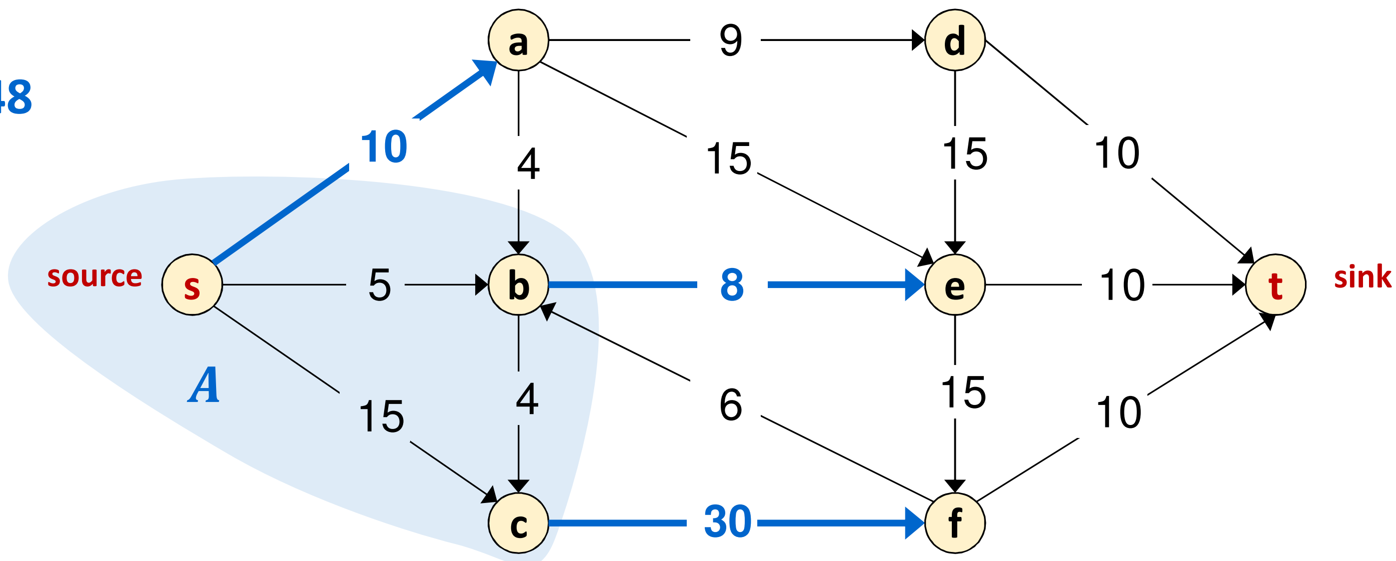
Cuts

Defn: An s-t cut is a partition (A, B) of V with $s \in A$ and $t \in B$.

The capacity of cut (A, B) is

$$c(A,B) = \sum_{e \text{ out of } A} c(e)$$

capacity **48**



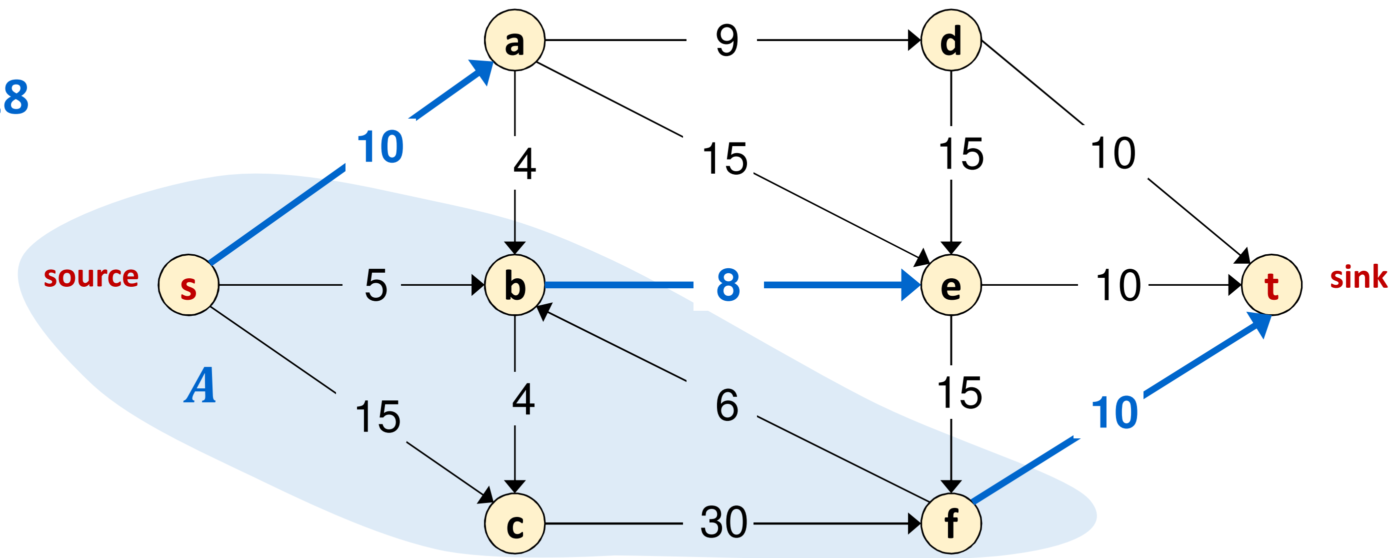
Minimum Cut Problem

Minimum s-t cut problem:

Given: a flow network

Find: an s-t cut of minimum capacity

capacity **28**



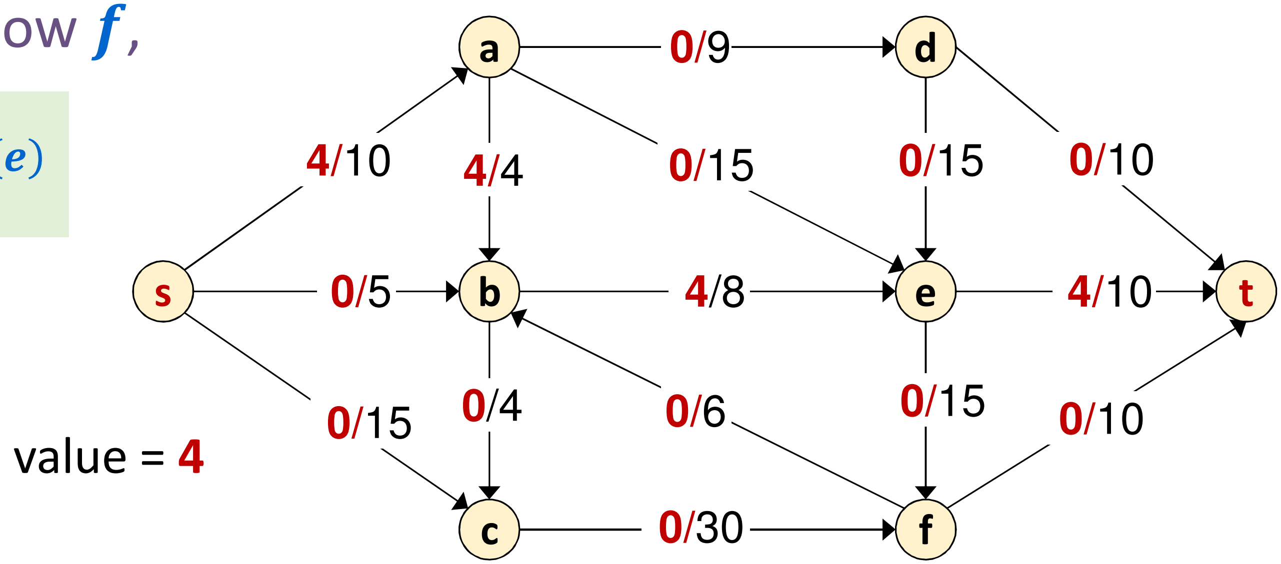
Flows

Defn: An ***s*-*t* flow** in a flow network is a function $f\colon E \to \mathbb{R}$ that satisfies:

- For each $e \in E$: $0 \le f(e) \le c(e)$ [capacity constraints]
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [flow conservation]

Defn: The value of flow f,

$$v(f) = \sum_{e \text{ out of } s} f(e)$$



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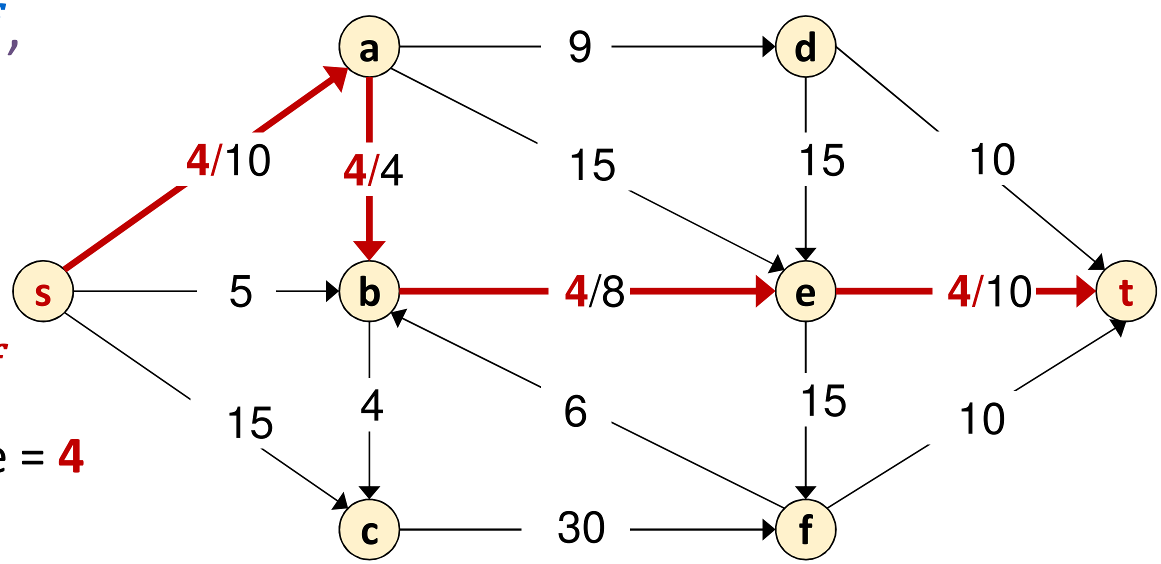

Only show non-zero values of f

value = 4

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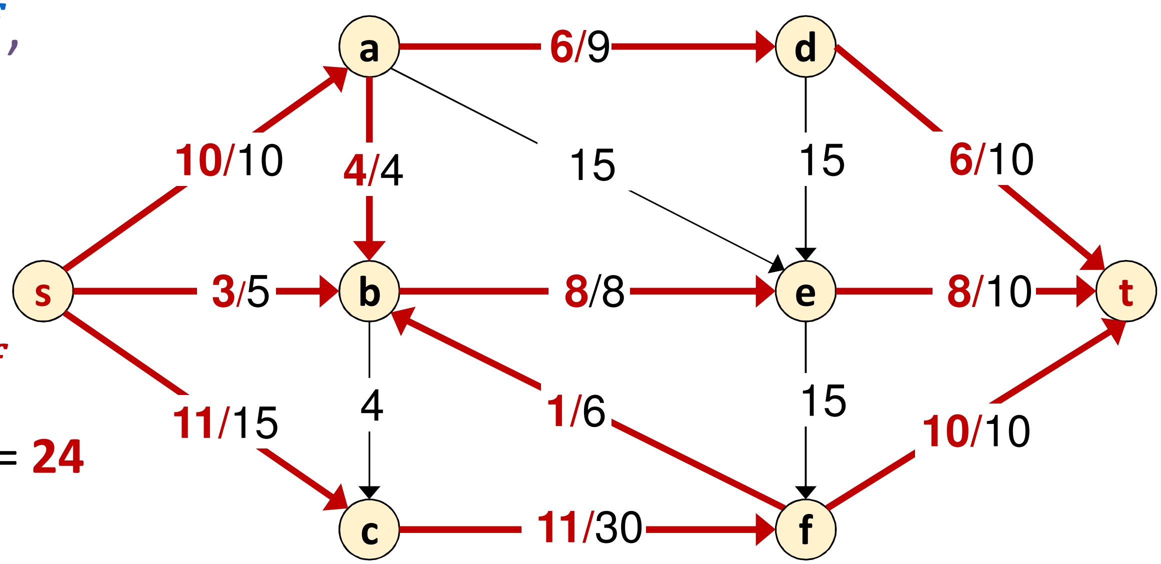
Only show non-zero values of f

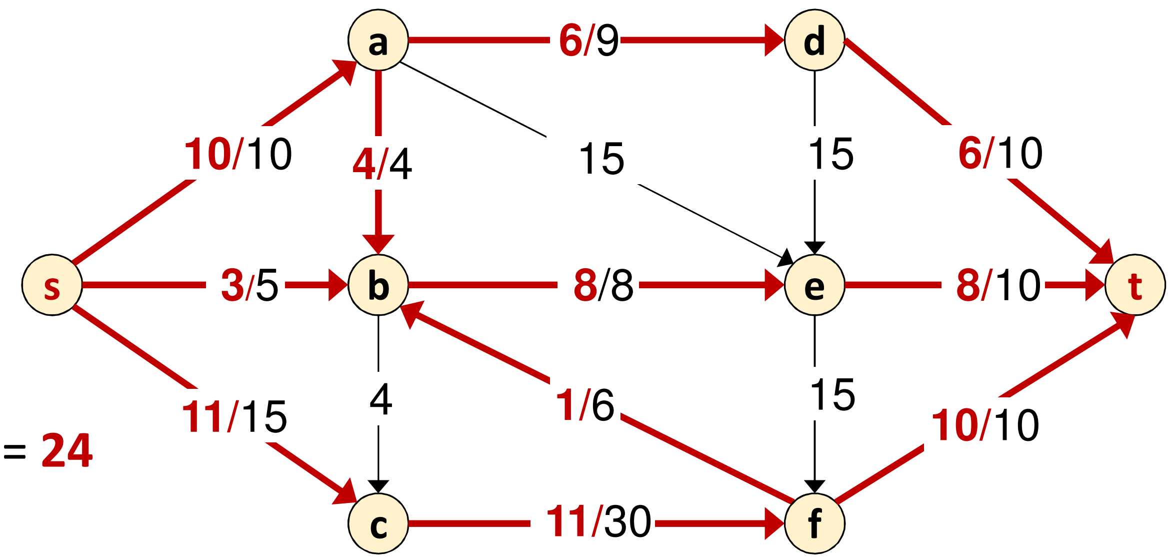
value = **24**



Maximum Flow Problem

Given: a flow network

Find: an s-t flow of maximum value

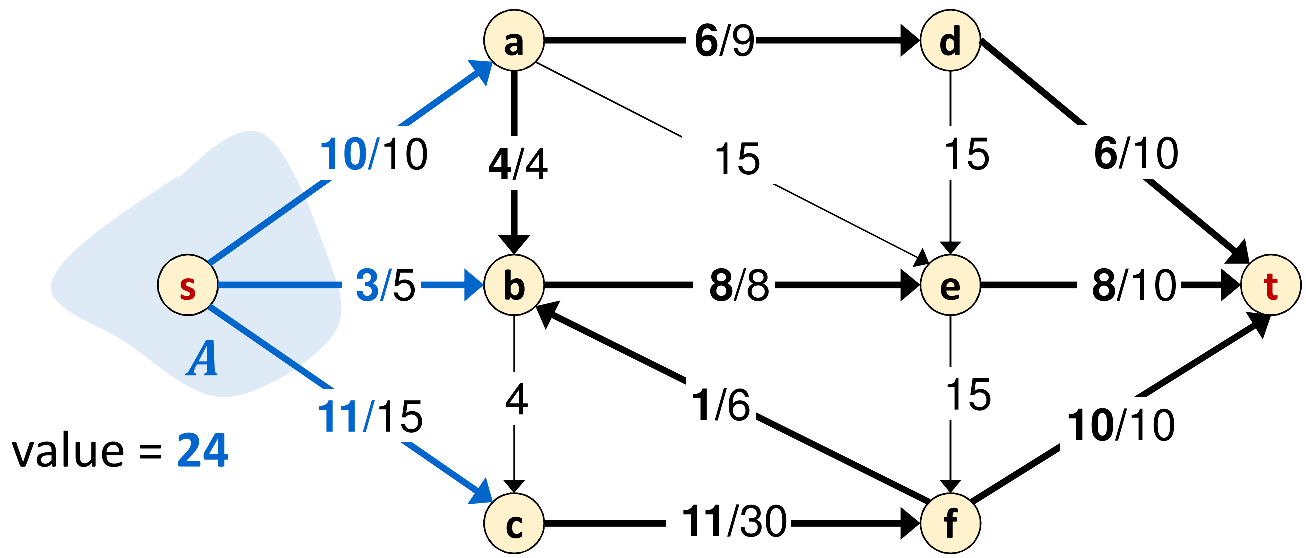


value = **24**

Maximum Flow Problem

Flow Value Lemma: Let f be any s-t flow and (A, B) be any s-t cut.
The net value of the flow sent across the cut equals $v(f)$:

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = v(f)$$



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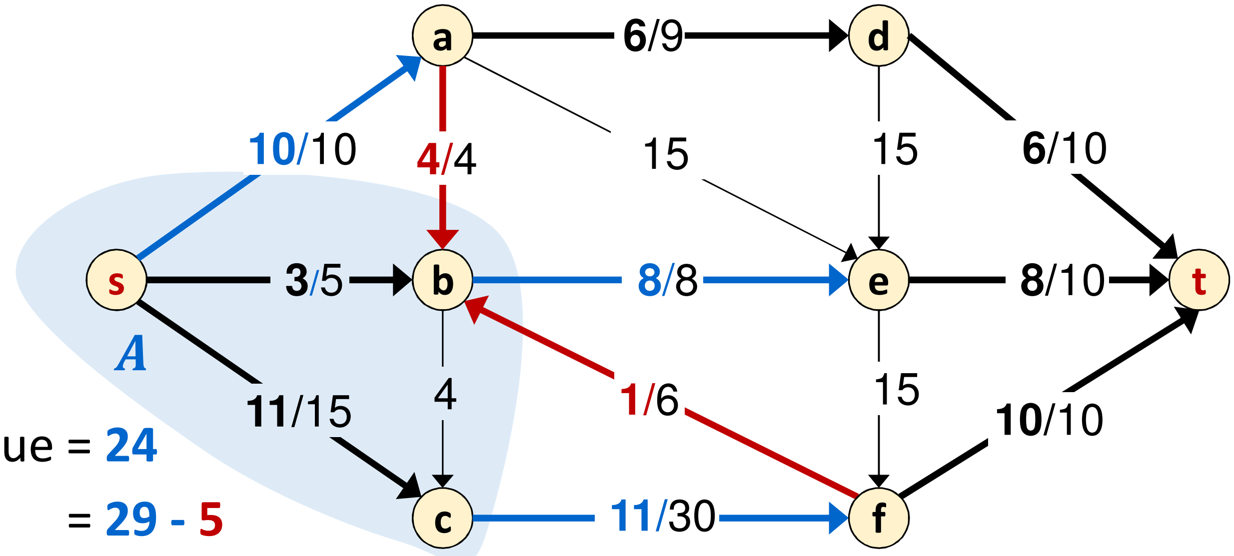


value = **24**

= **29 - 5**

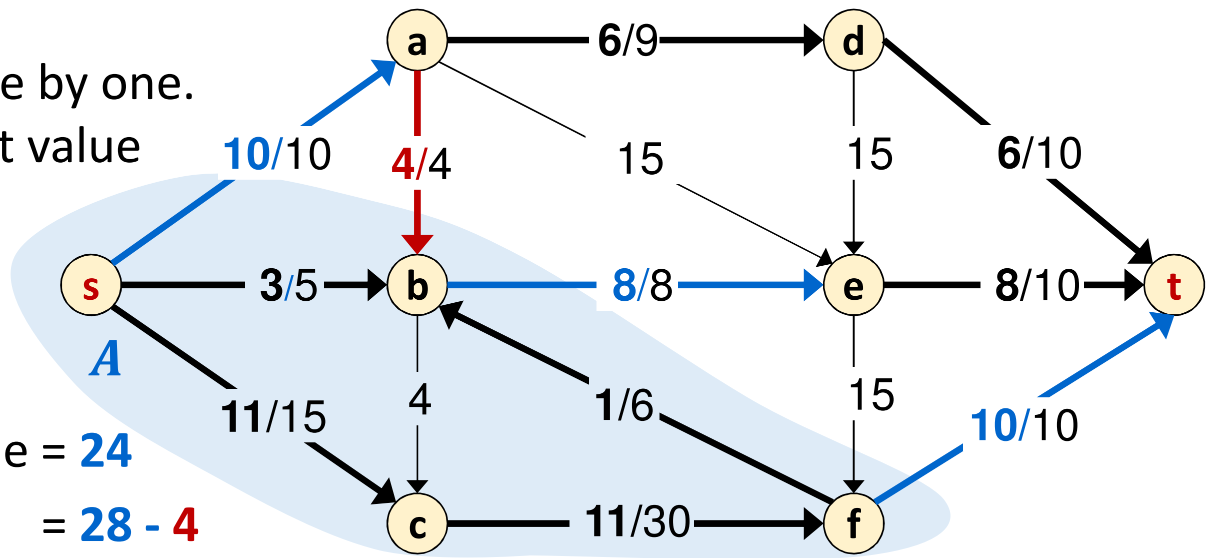
Maximum Flow Problem

Flow Value Lemma: Let f be any s-t flow and (A, B) be any s-t cut.
The net value of the flow sent across the cut equals $v(f)$:

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = v(f)$$

Why is it true?

- Add vertices to s side one by one.
- By flow conservation, net value doesn't change



value = **24**

= **28 - 4**

Maximum Flow Problem

Flow Value Lemma: Let f be any s-t flow and (A, B) be any s-t cut. The net value of the flow sent across the cut equals $v(f)$:

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = v(f)$$

Proof:

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

$= 0$. No edges into s since it is a source

$$= \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ into } s} f(e) + \sum_{v \in A - \{s\}} \left[\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right]$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$= \mathbf{0}$ by flow conservation. ■

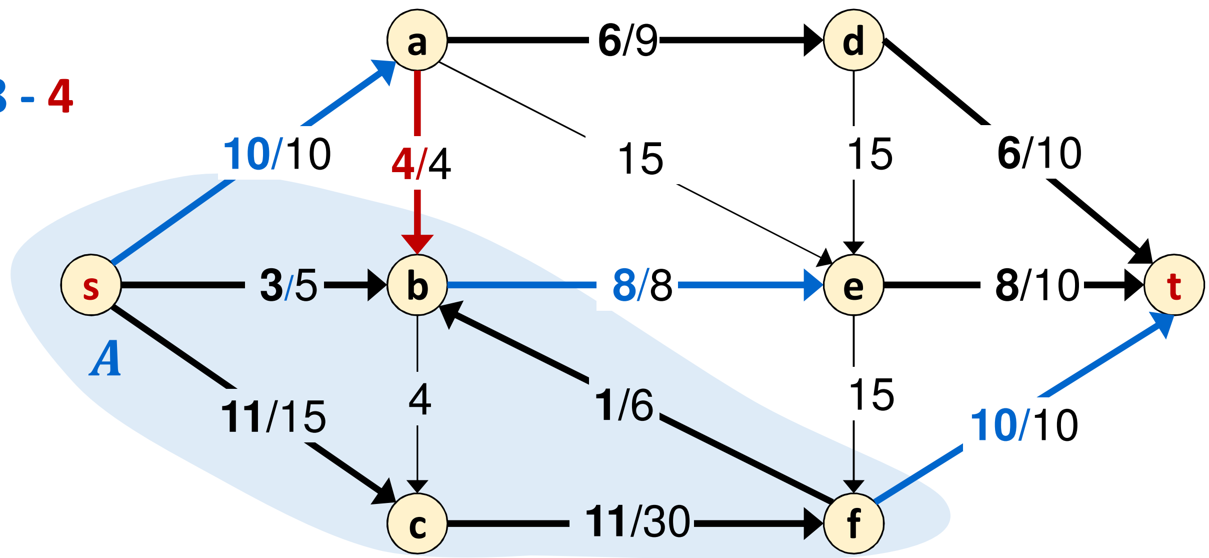
Contributions from internal edges of A cancel.

Flows and Cuts

Weak Duality: Let f be any s-t flow and (A, B) be any s-t cut. The value of the flow is at most the capacity of the cut; i.e., $v(f) \le c(A, B)$:

Value of flow = **24** = **28 - 4**

Capacity of cut = **28**



Flows and Cuts

Weak Duality: Let f be any s-t flow and (A, B) be any s-t cut. The value of the flow is at most the capacity of the cut; i.e., $v(f) \leq c(A, B)$.

Proof:

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \\ &\leq \sum_{e \text{ out of } A} f(e) && \text{since } f(e) \geq 0 \\ &\leq \sum_{e \text{ out of } A} c(e) && \text{since } f(e) \leq c(e) \\ &= c(A, B) \end{aligned}$$

$\blacksquare$

Certificate of Optimality

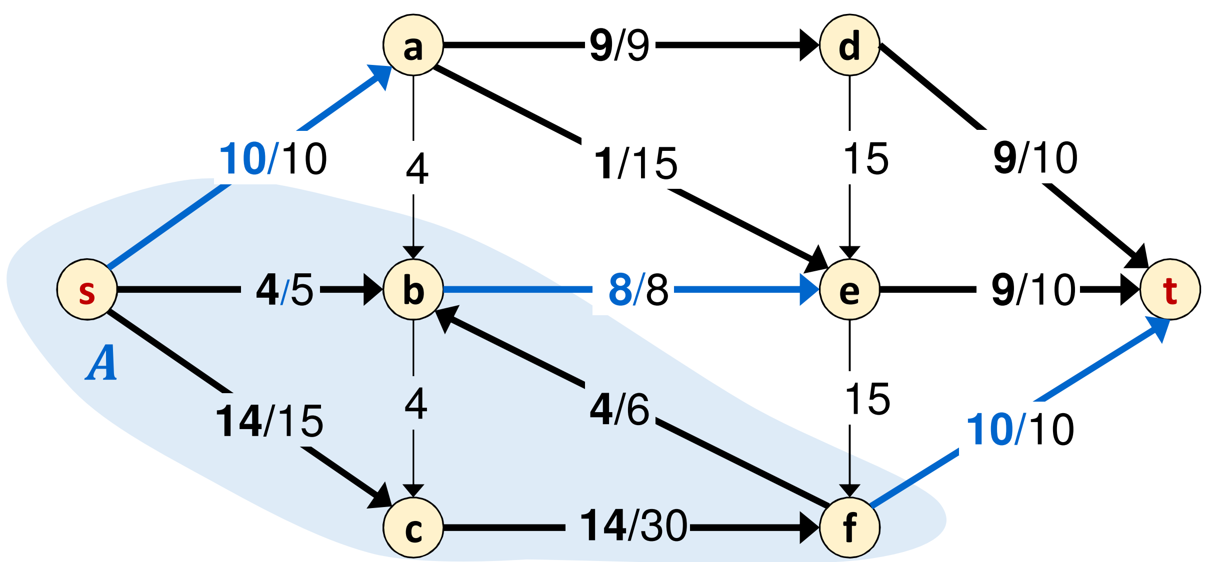
Corollary: Let f be any s-t flow and (A, B) be any s-t cut.

If $v(f) = c(A, B)$ then f is a max flow and (A, B) is a min cut.

Value of flow = **28**

Capacity of cut = **28**

Both are optimal!



Towards a Max Flow Algorithm

What about the following greedy algorithm?

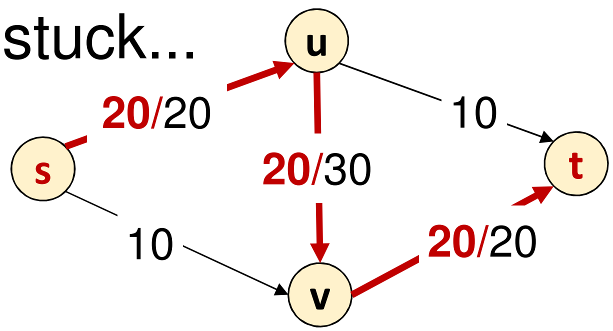
- Start with $\boldsymbol{f(e) = 0}$ for all edges $\boldsymbol{e \in E}$.
- While there is an $\boldsymbol{s}$-$\boldsymbol{t}$ path $\boldsymbol{P}$ where each edge has $\boldsymbol{f(e) < c(e)}$.
  - "Augment" flow along $\boldsymbol{P}$; that is:
    - Let $\boldsymbol{\alpha} = \min_{\boldsymbol{e \in P}}(\boldsymbol{c(e) - f(e)})$
    - Add $\boldsymbol{\alpha}$ to flow on every edge $\boldsymbol{e}$ along path $\boldsymbol{P}$. (Adds $\boldsymbol{\alpha}$ to $\boldsymbol{v(f)}$.)

Towards a Max Flow Algorithm

What about the following greedy algorithm?

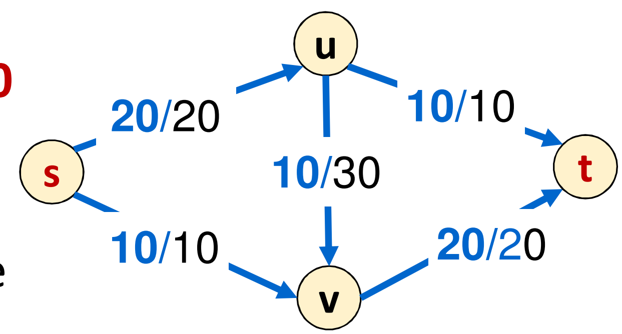
- Start with $f(e) = 0$ for all edges $e \in E$.
- While there is an s-t path P where each edge has $f(e) < c(e)$.
  - "Augment" flow along P; that is:
    - Let $\alpha = \min_{e \in P}(c(e) - f(e))$
    - Add α to flow on every edge e along path P. (Adds α to $v(f)$.)

Can get stuck...



Has flow value **20** and no path P

but **30** is possible

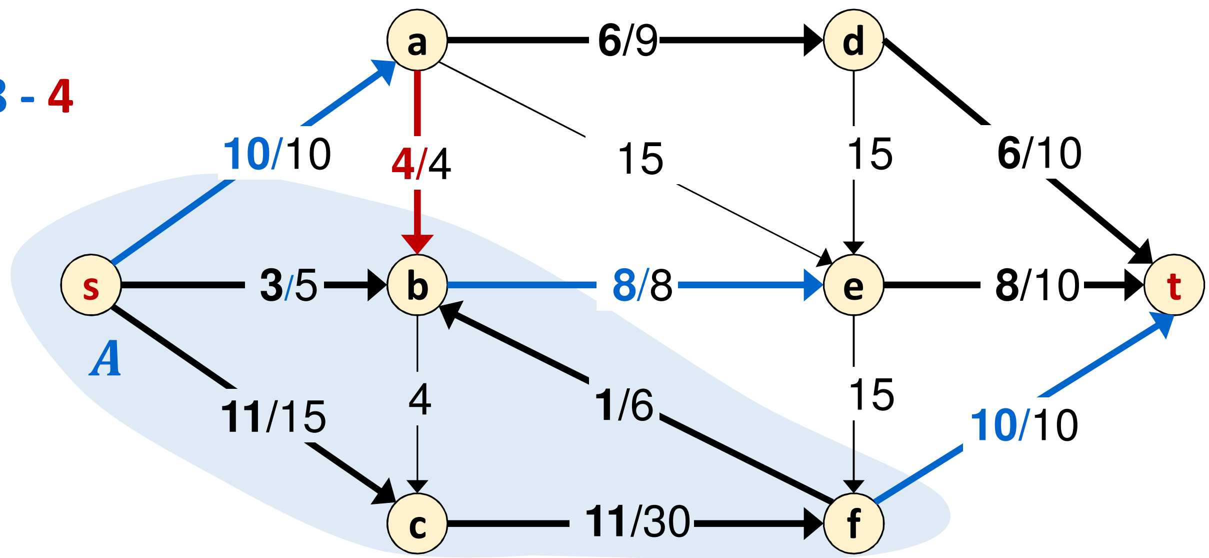


Another Stuck Example

On every s-t path there is some edge with $f(e) = c(e)$:

Value of flow = **24** = **28** - **4**

Capacity of cut = **28**



Flows and cuts so far

Let $\boldsymbol{f}$ be any $\boldsymbol{s}$-$\boldsymbol{t}$ flow and $(\boldsymbol{A}, \boldsymbol{B})$ be any $\boldsymbol{s}$-$\boldsymbol{t}$ cut:

Flow Value Lemma: The net value of the flow sent across $(\boldsymbol{A}, \boldsymbol{B})$ equals $\boldsymbol{v}(\boldsymbol{f})$.

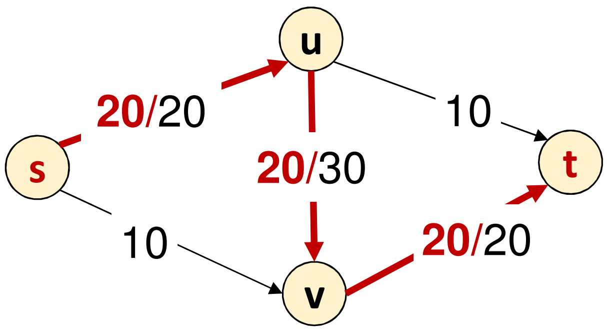
Weak Duality: The value of the flow is at most the capacity of the cut;
i.e., $\boldsymbol{v}(\boldsymbol{f}) \leq \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B})$.

Corollary: If $\boldsymbol{v}(\boldsymbol{f}) = \boldsymbol{c}(\boldsymbol{A}, \boldsymbol{B})$ then $\boldsymbol{f}$ is a maximum flow and $(\boldsymbol{A}, \boldsymbol{B})$ is a minimum cut.

Augmenting along paths using a greedy algorithm can get stuck.

Next idea: Ford-Fulkerson Algorithm, which applies greedy ideas to a “residual graph” that lets us reverse prior flow decisions from the basic greedy approach to get optimal results!

Greed Revisited: Residual Graph & Augmenting Paths

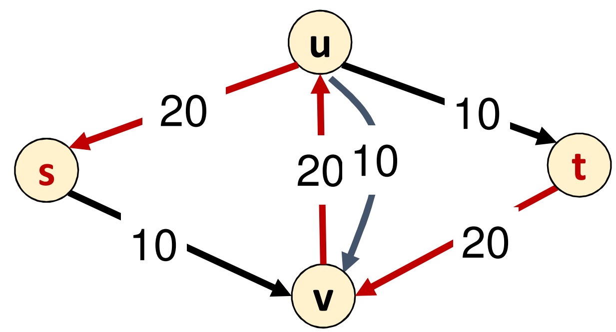


The only way we could route more flow from **s** to **t** would be to reduce the flow from **u** to **v** to make room for that amount of extra flow from **s** to **v**.
But to conserve flow we also would need to increase the flow from **u** to **t** by that same amount.

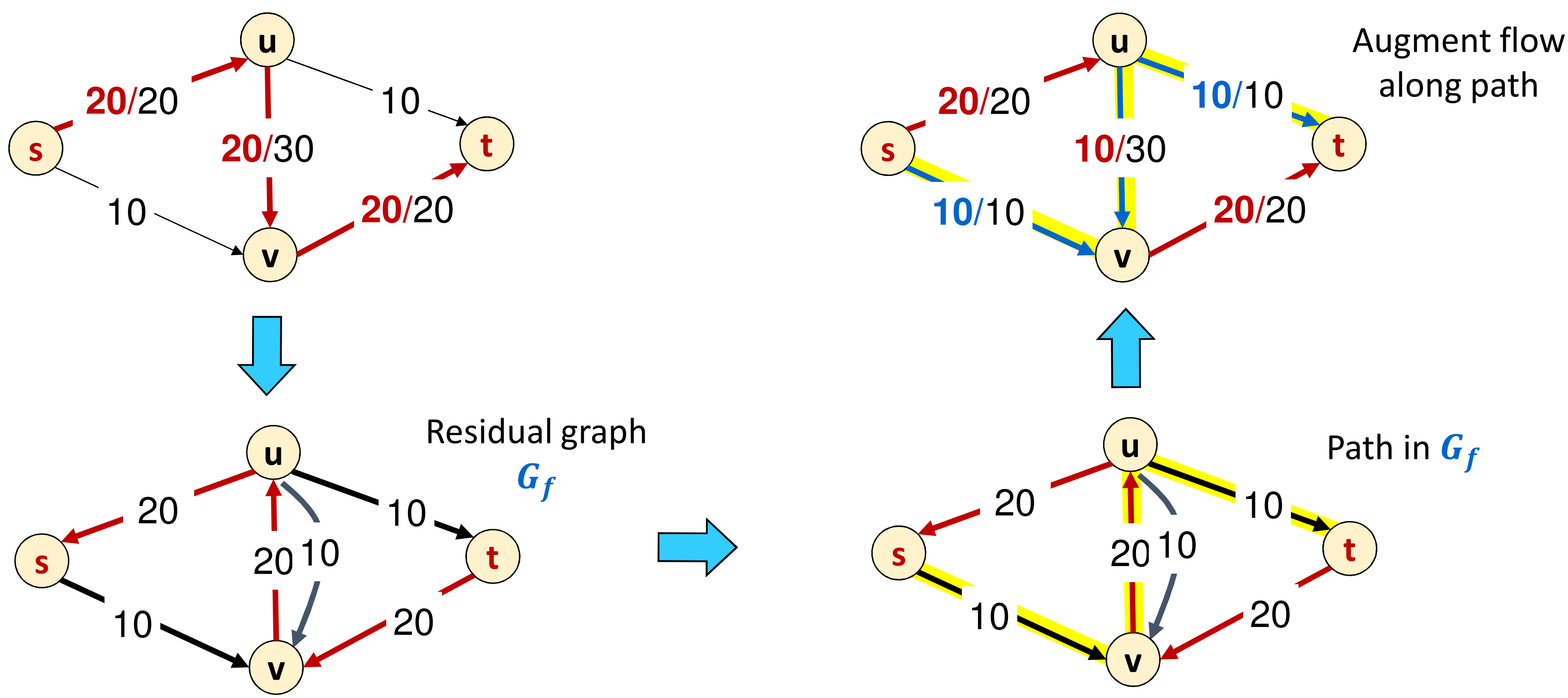
Suppose that we took this flow f as a baseline, what changes could each edge handle?

- We could add up to 10 units along **sv** or **ut** or **uv**
- We could reduce by up to 20 units from **su** or **uv** or **vt**

This gives us a residual graph G_f of possible changes where we draw reducing as "sending back".



Greed Revisited: Residual Graph & Augmenting Paths



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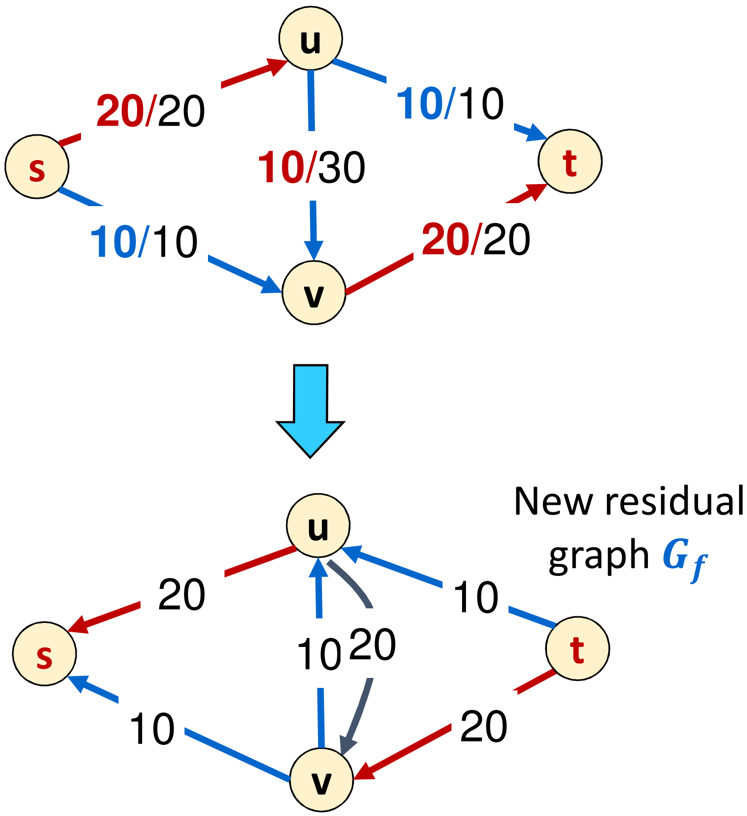

No path can even leave s!