CSE 421
Introduction to Algorithms

Lecture 11: Dynamic Programming

Algorithmic Paradigms

Greedy: Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer: Break up a problem into sub-problems (each typically a constant factor smaller), solve each sub-problem *independently*, and combine solution to sub-problems to form solution to original problem.

Dynamic programming: Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

Algorithm Design Techniques

Dynamic Programming:

- Technique for making building solution to a problem based on solutions to smaller subproblems (recursive ideas).
- The subproblems just have to be smaller, but don’t need to be a constant-factor smaller like divide and conquer.
- Useful when *the same subproblems show up over and over again*
- The final solution is simple iterative code when the following holds:
  - *The parameters to all the subproblems are predictable in advance*

Dynamic Programming History

Bellman. [1950s] Pioneered the systematic study of dynamic programming.

Etymology

- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.

> "it's impossible to use dynamic in a pejorative sense"
> "something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography.*

Dynamic Programming Applications

Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, compilers, systems, ….

Some famous dynamic programming algorithms.

- Unix **`diff`** for comparing two files.
- Viterbi for hidden Markov models.
- Smith-Waterman for genetic sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

Three Steps to Dynamic Programming

1. Formulate the answer as a recurrence relation or recursive algorithm

2. Figure out the possible values of parameters in the recursive calls.
    - This should be “small”, i.e., bounded by a low-degree polynomial
    - Can use memoization to store a cache of previously computing values

3. Specify an order of evaluation for the recurrence so that you already have the partial results stored in memory when you need them.

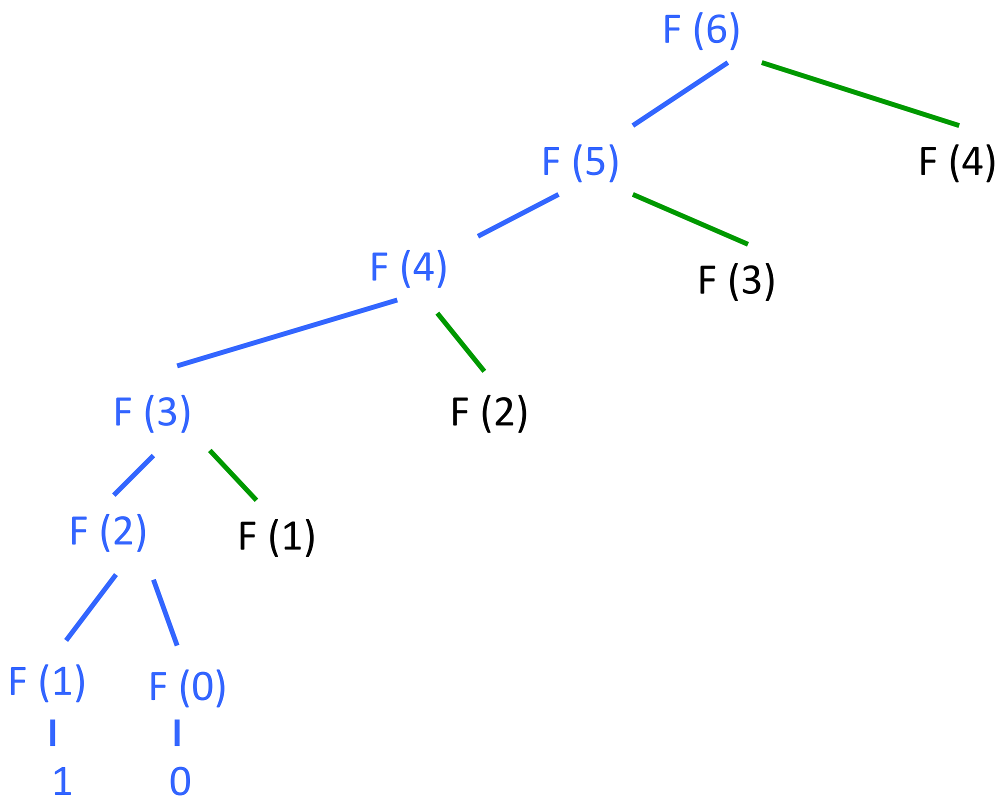
A Simple Case: Computing Fibonacci Numbers

Recall $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0, F_1 = 1$

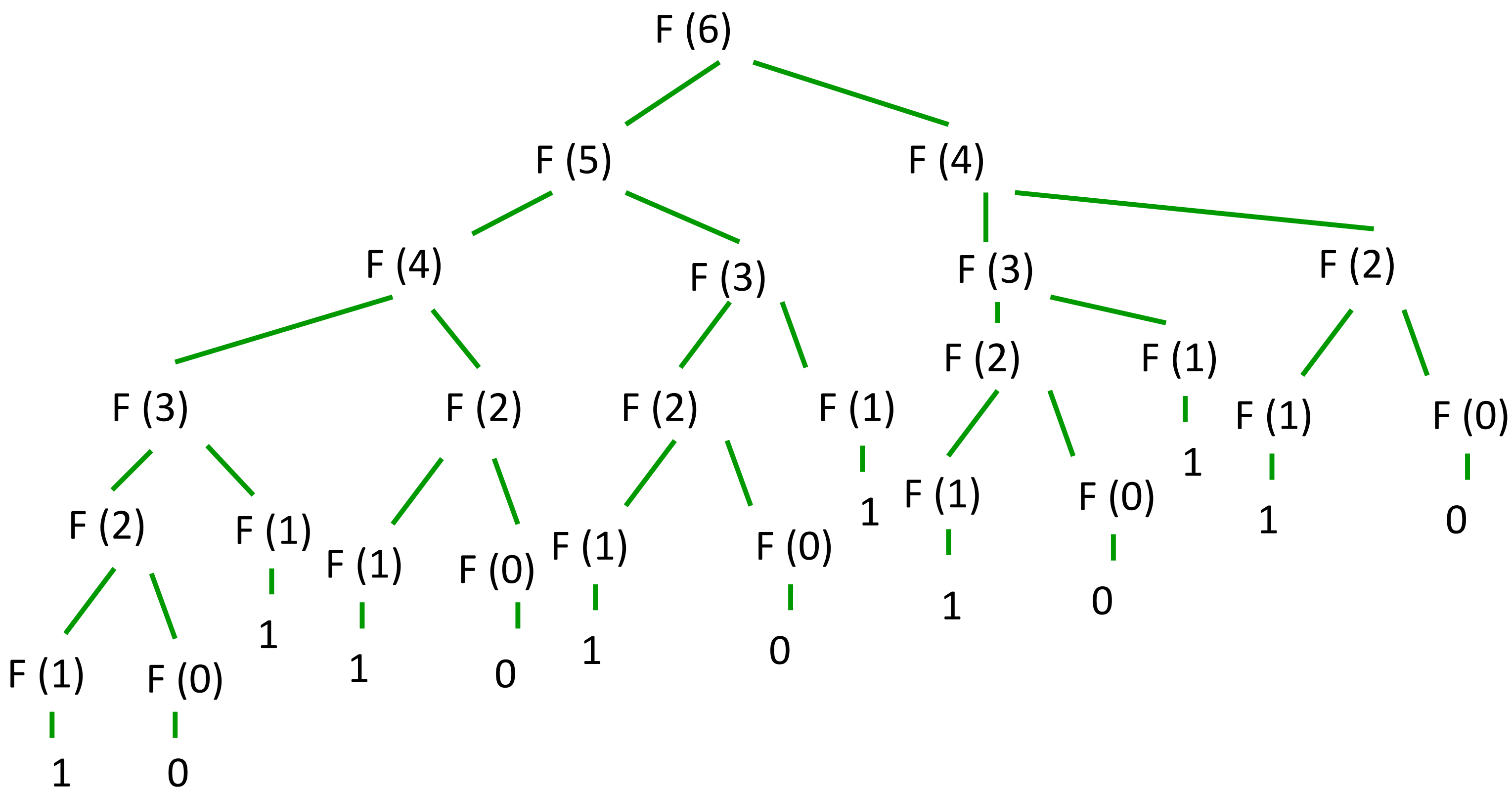
The obvious recursive algorithm direct from this recurrence is

```
F(n){
      if n=0 return(0)
      else if n=1 return(1)
      else return(F(n-1)+F(n-2))
}
```

Let's start tracking the call tree...



The full call tree has $> F_n$ leaves (exponential in n)



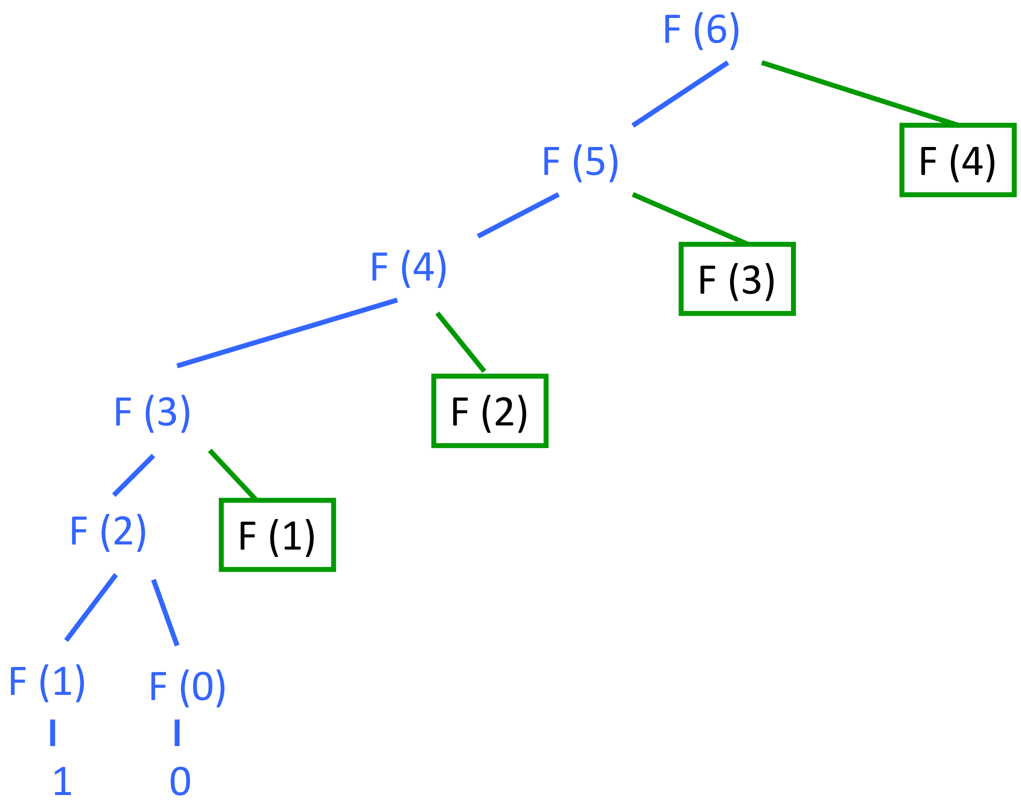
Memoization (Caching)

Remember all values from previous recursive calls in a cache

- the parameters and
- The values returned on those parameters

Before each recursive call, test to see if value has already been computed for those parameters

Memoization



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   - Produces iterative code

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Fibonacci: Dynamic Programming Version

```
FiboDP(n):
      F[0]←0
      F[1]←1
      for i←2 to n {
            F[i]←F[i-1]+F[i-2]
      }
      return(F[n])
```

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    - Produces iterative code

Once you have an iterative DP solution: see if you can save space...

Fibonacci: Space-Saving Dynamic Programming

```
FiboDP(n):
      prev←0
      curr←1
      for i←2 to n {
            temp←curr
            curr←curr+prev
            prev←temp
      }
      return(curr)
```

Dynamic Programming

When is dynamic programming useful?

- For optimization problems this typically requires that the "Principle of optimality" hold for the problem

  "Optimal solutions to the sub-problems suffice for optimal solution to the whole problem"

Weighted Interval Scheduling

Input: Like interval scheduling each request i has start and finish times s_i and f_i.
Each request i also has an associated value or weight v_i

v_i might be

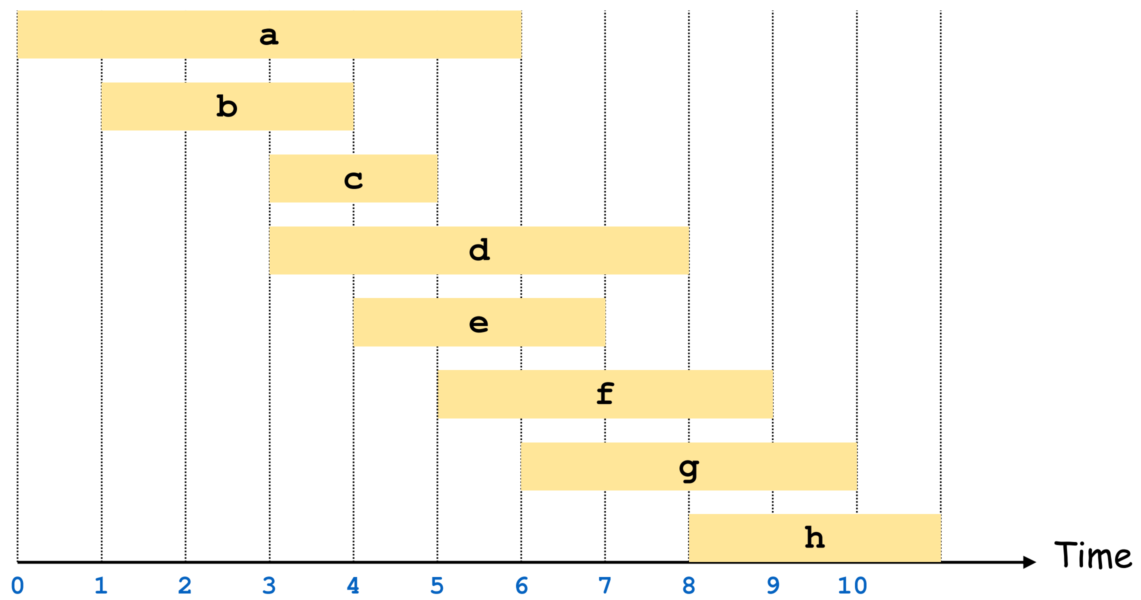
- the amount of money we get from renting out the resource
- the amount of time the resource is being used ($v_i = f_i - s_i$)

Find: A maximum-weight compatible subset of requests.

Weighted Interval Scheduling

Input: Set of jobs with start times, finish times, and weights

Goal: Find maximum weight subset of mutually compatible jobs.



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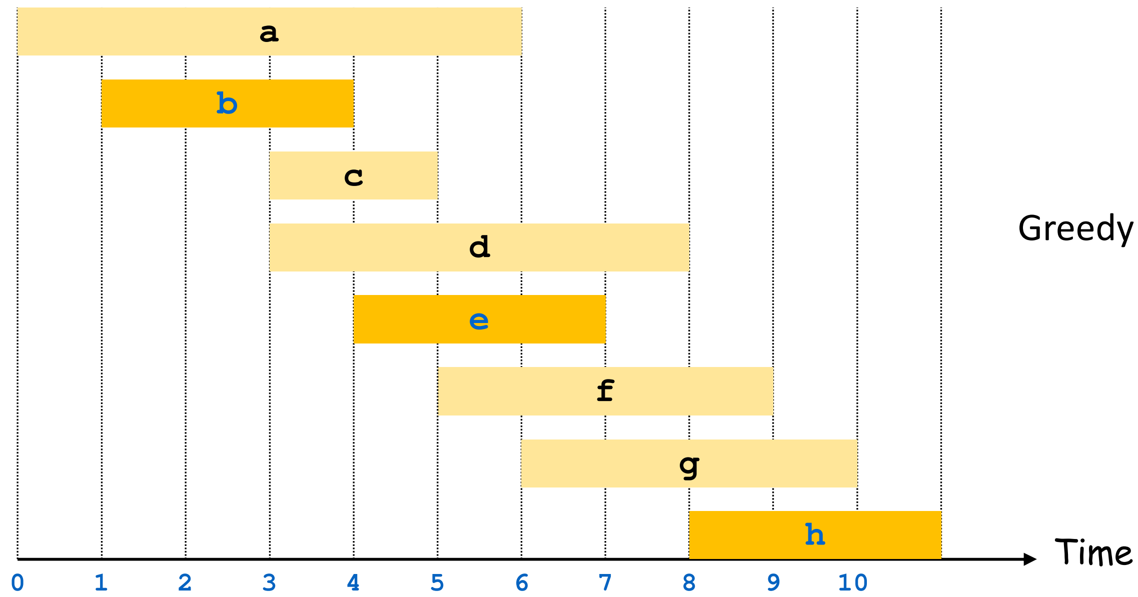

Greedy by finish times would give 9

Weighted Interval Scheduling

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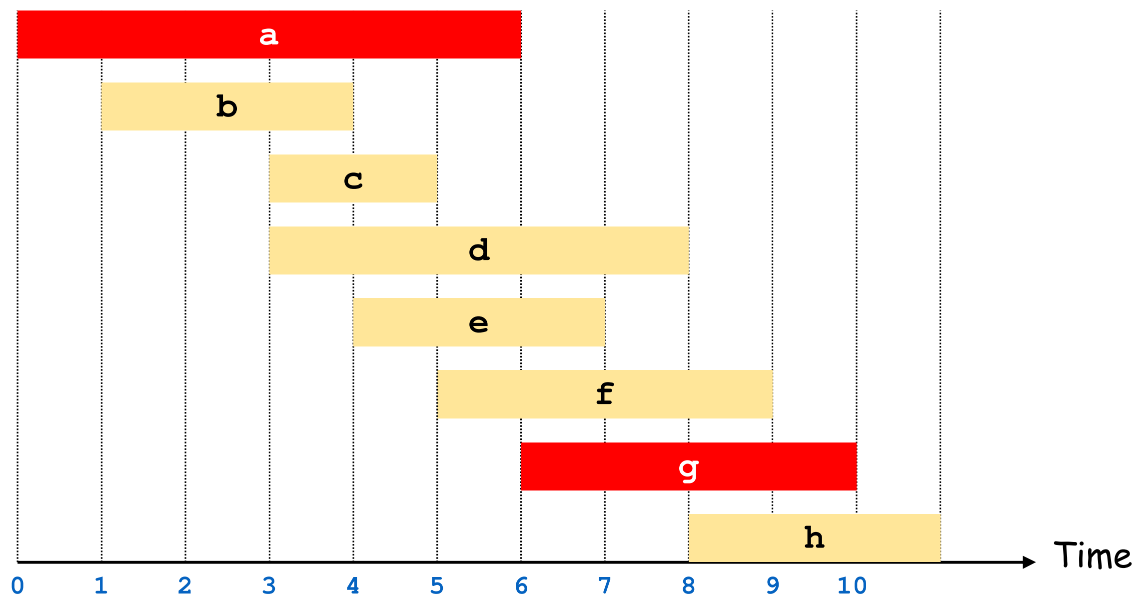


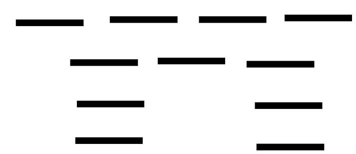
Optimal yields 10

Greedy Algorithms for Weighted Interval Scheduling?

- What criterion should we try?
  - Earliest start time s_i
    - Doesn't work

  - Shortest request time $f_i - s_i$
    - Doesn't work

  - Fewest conflicts
    - Doesn't work

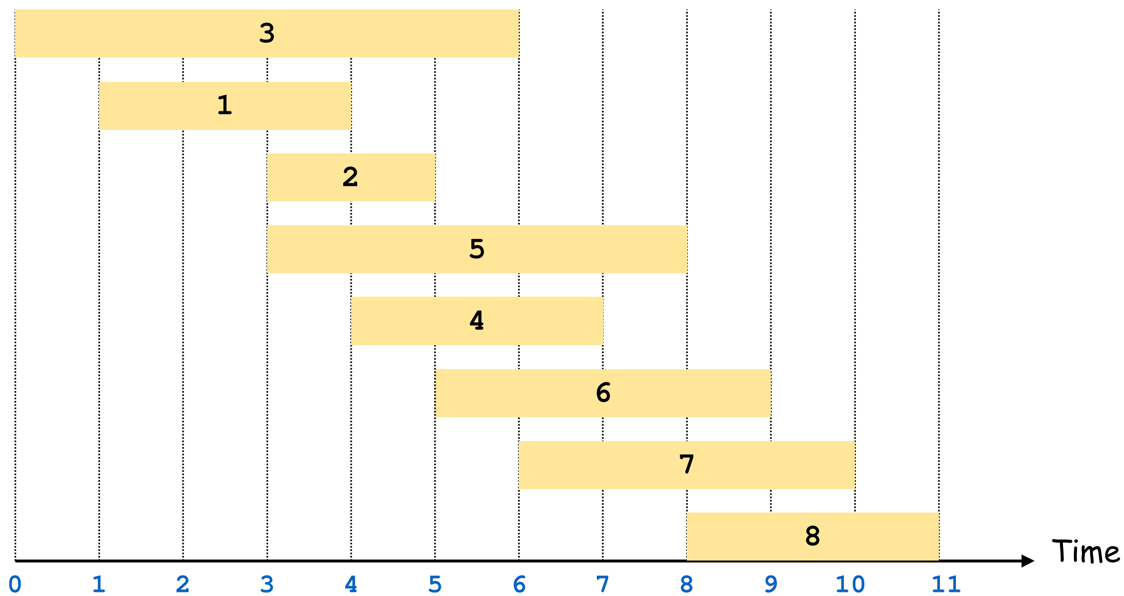


  - Earliest finish time f_i
    - Doesn't work

  - Largest value/weight v_i
    - Doesn't work

Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.



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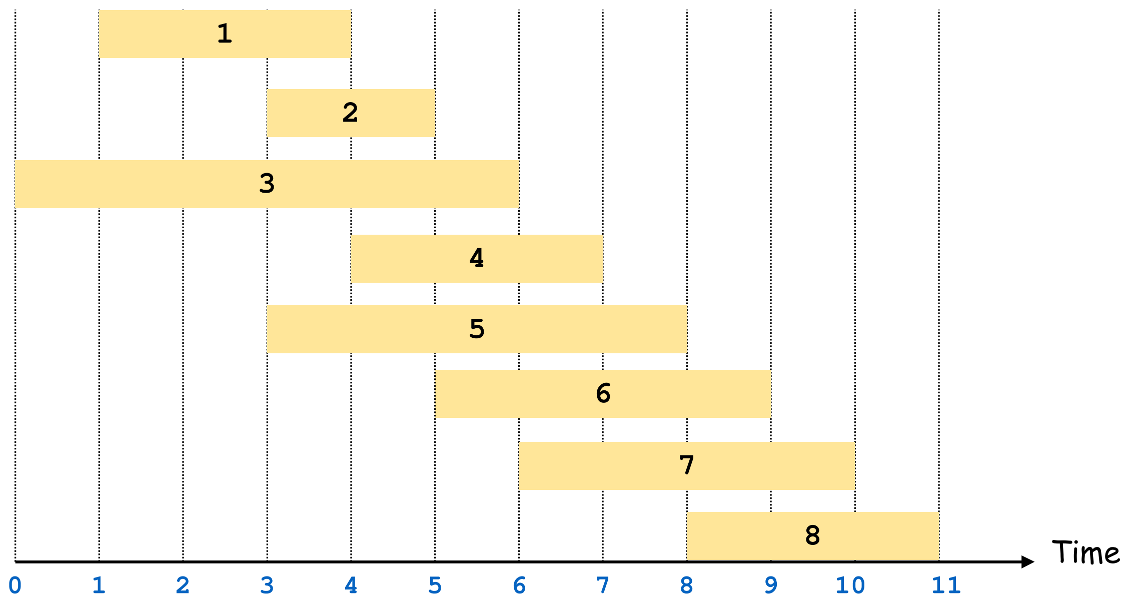


Towards Dynamic Programming: Step 1 – Recursive Algorithm

Suppose that we first sort the requests by finish time $\boldsymbol{f_i}$ so $\boldsymbol{f_1} \leq \boldsymbol{f_2} \leq \ldots \leq \boldsymbol{f_n}$.

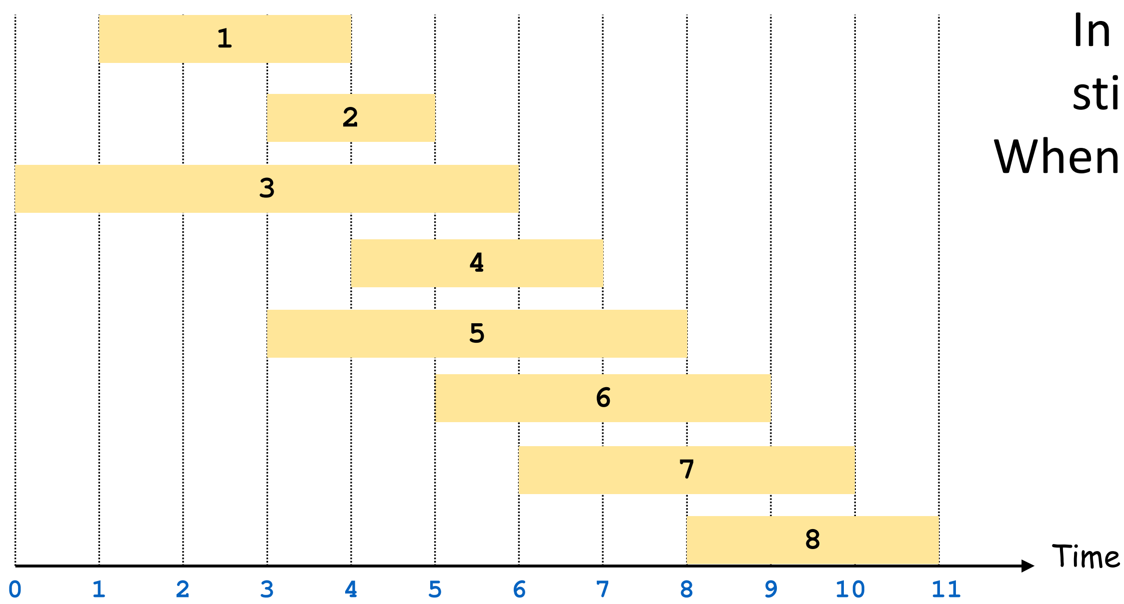
We now want

- a recursive solution that makes calls to smaller problems and
- the indices for those smaller problems to be convenient,

so we first focus on the options for the *last* request, request $\boldsymbol{n}$.

Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.



There are two cases we need to compare:

- When we don't include request n.
  - In this case all the other requests are still fair game
- When we do include request n.

Weighted Interval Scheduling

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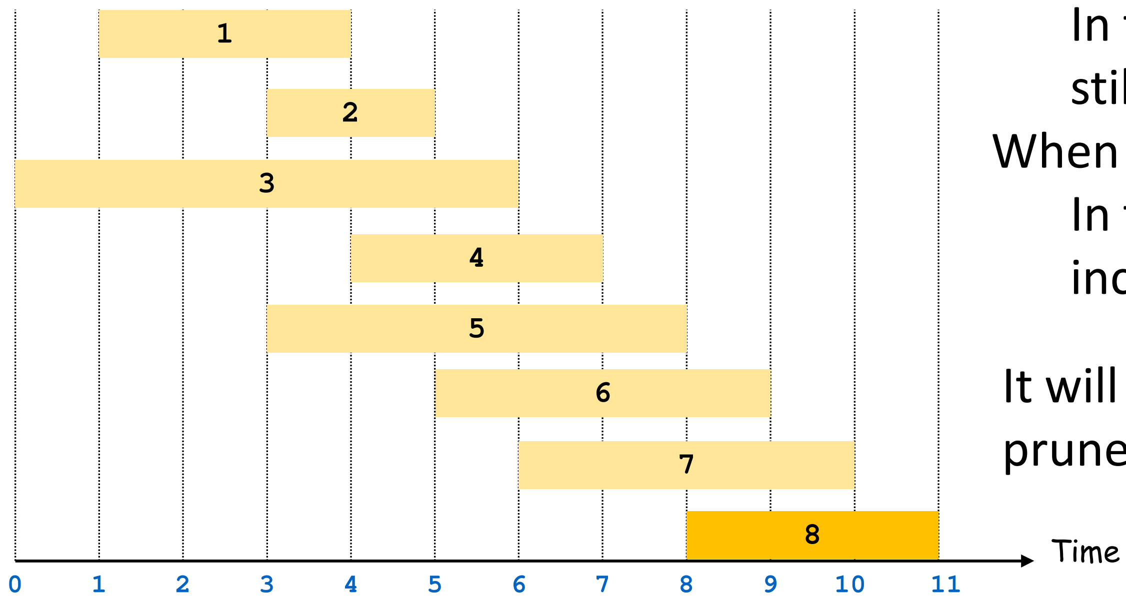


There are two cases we need to compare:

- When we don't include request n.
  - In this case all the other requests are still fair game
- When we do include request n.
  - In this case we need to rule out some incompatible requests.

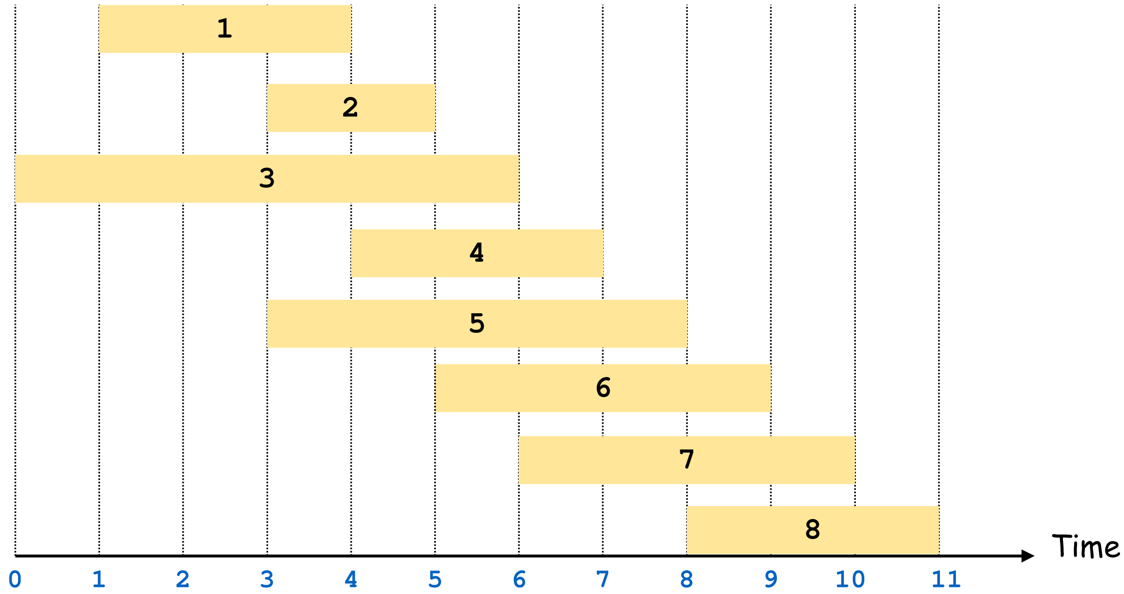
It will be convenient to be able to prune incompatible requests quickly…

Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

Example: $p(8) = 5, p(7) = 3, p(2) = 0$



j	$p(j)$
1	0
2	0
3	0
4	1
5	0
6	2
7	3
8	5

Structure of the subproblems

Notation: OPT(j) = value of optimal solution to the problem consisting of job requests $1, 2, \dots, j$.

Case 1: OPT selects job j

- It can't use incompatible jobs $p(j) + 1, \dots, j - 1$
- It must include an optimal solution to problem consisting of remaining compatible jobs $1, \dots, p(j)$.

Optimal substructure

Case 2: OPT doesn't select job

- It must include an optimal solution to problem consisting of remaining compatible jobs $1, \dots, j - 1$

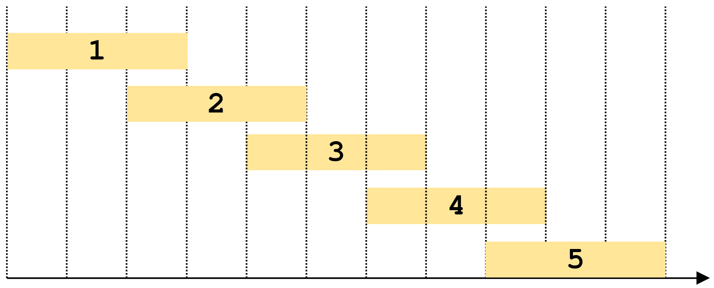
$$\text{OPT}(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\{v_j + \text{OPT}(p(j)), \text{OPT}(j-1)\} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Recursive Solution

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

Compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max(v_j + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

Weighted Interval Scheduling: Recursive Solution

This recursive algorithm can be very bad…



Suppose that $p(j) = j - 2$ for every $j \geq 2$.

- Then **Compute-Opt**(j) calls **Compute-Opt**($j - 1$) and **Compute-Opt**($j - 2$)
- This is the same exponential run-time as the recursive Fibonacci code!

Weighted Interval Scheduling: Step 2 Memoization

Memoization: Store results of each sub-problem in a cache; lookup as needed.

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.
Compute p(1), p(2), …, p(n)

for j = 1 to n
   M[j] = empty        global array
M[0] = 0

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max(v_j + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

Weighted Interval Scheduling: Step 3

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   - Produces iterative code

Recursion for $\mathbf{OPT}[j]$ only needs values of $\mathbf{OPT}[i]$ for $0 \leq i < j$.

- So we can evaluate them in order $j = 0, 1, 2, \ldots, n$

Weighted Interval Scheduling: Iterative Solution

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   OPT[0] = 0
   for j = 1 to n
      OPT[j] = max(v_j + OPT[p(j)], OPT[j-1])
}
```

$O(n \log n)$

$O(n)$

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

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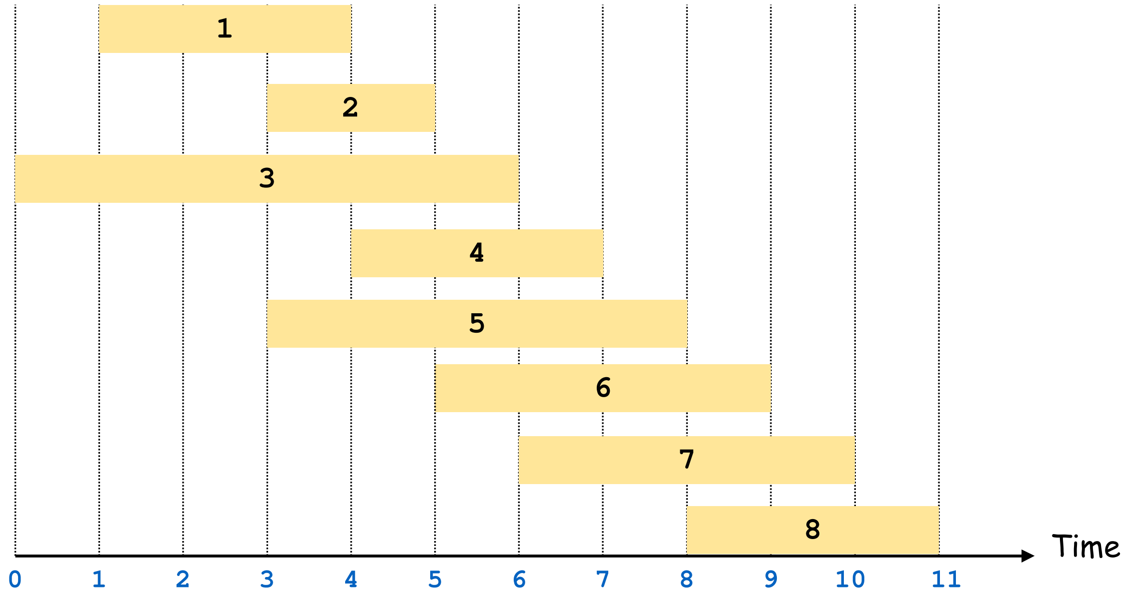

j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	
2	2	0	
3	6	0	
4	3	1	
5	5	0	
6	4	2	
7	4	3	
8	3	5	

Weighted Interval Scheduling: Iterative Solution

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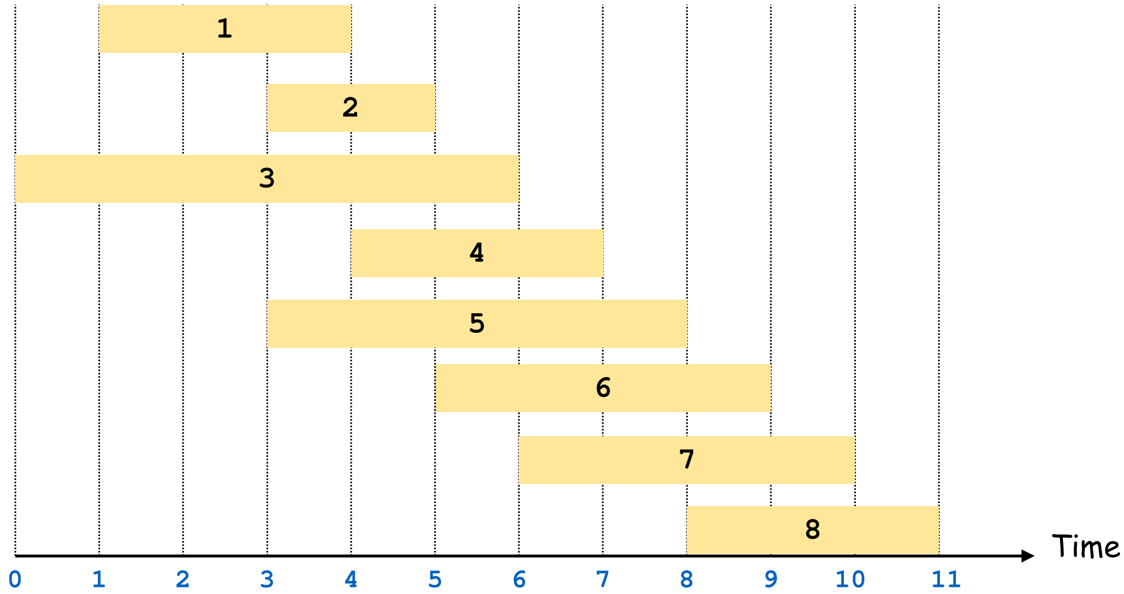

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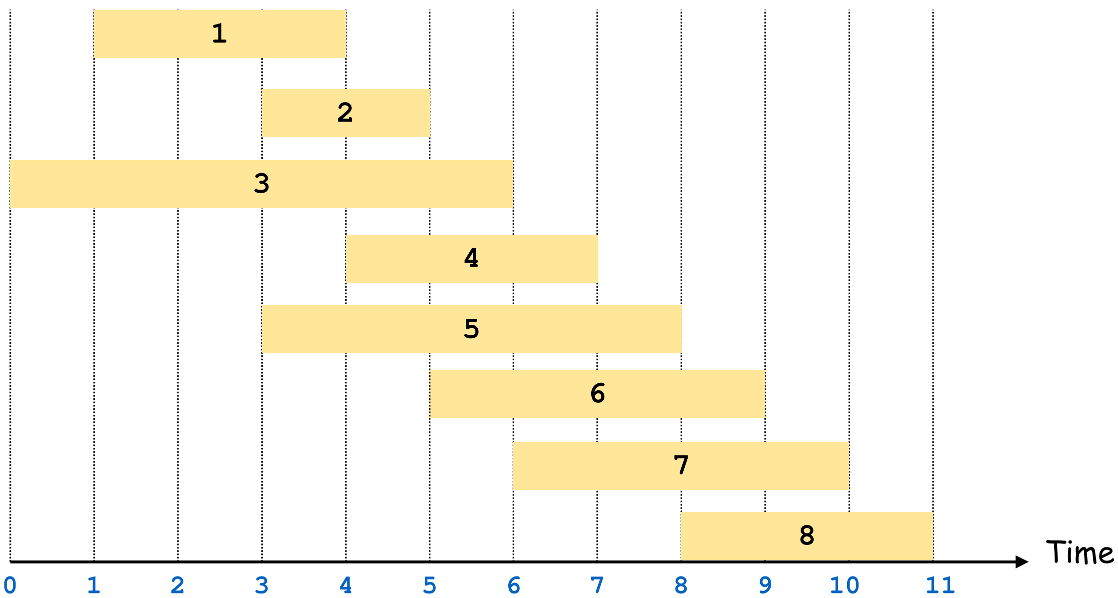

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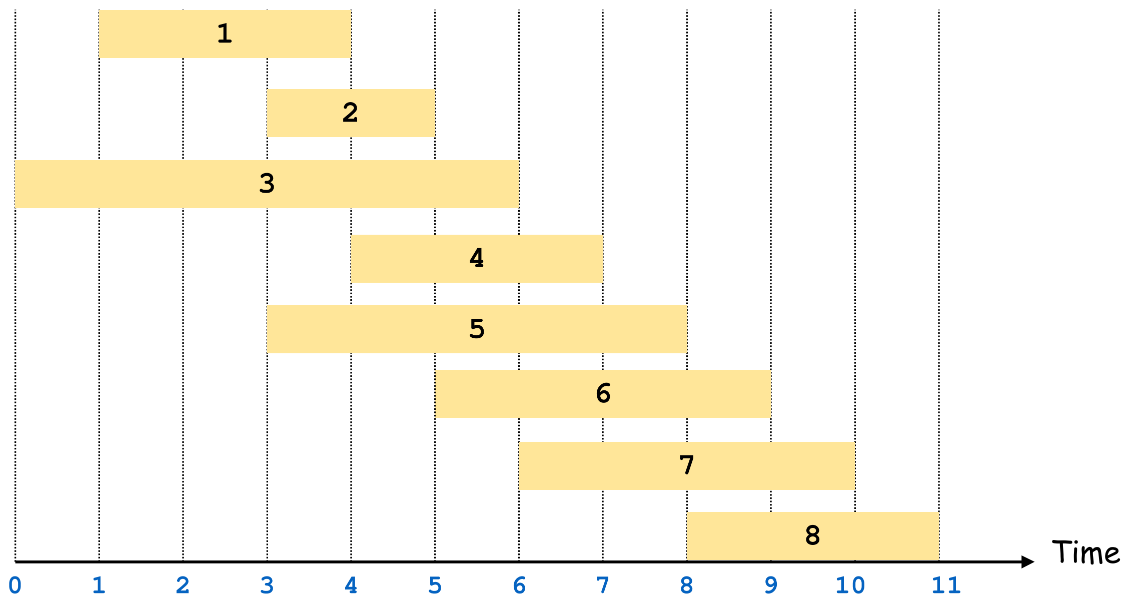

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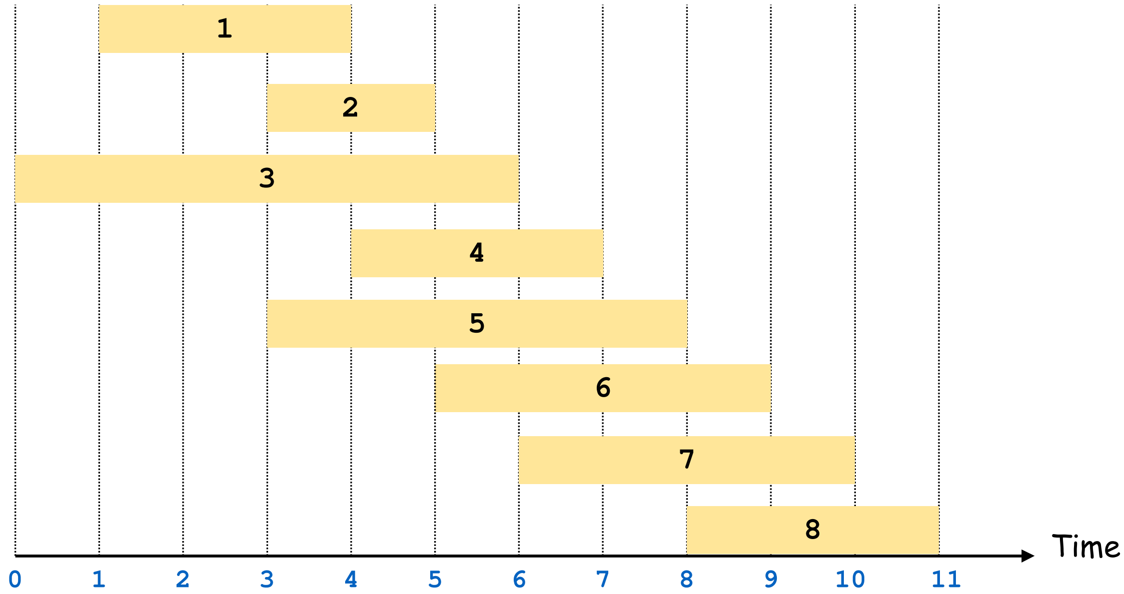

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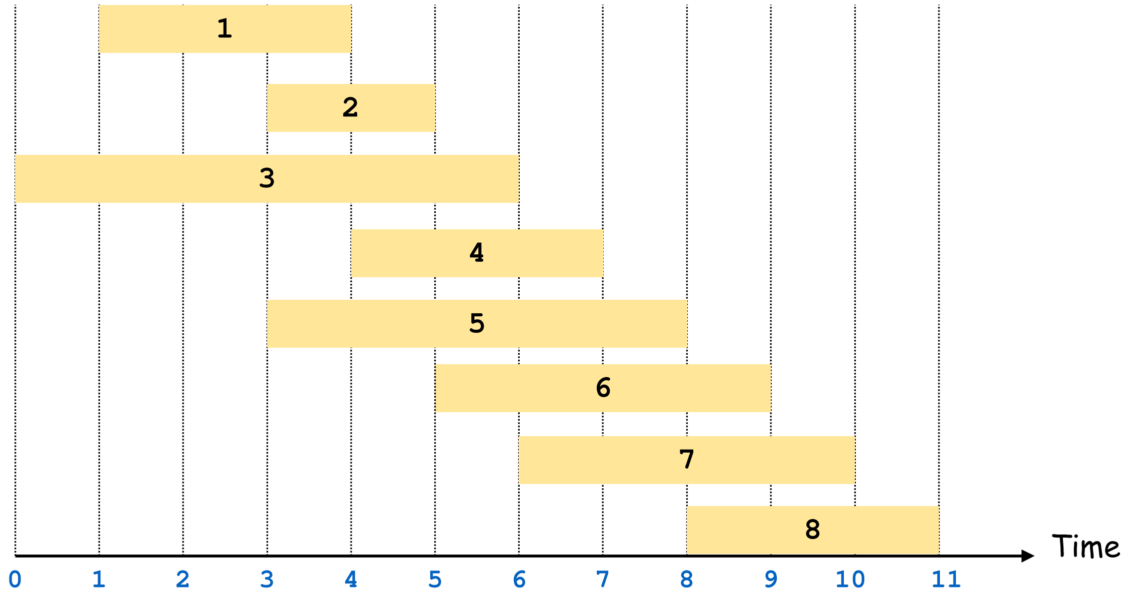

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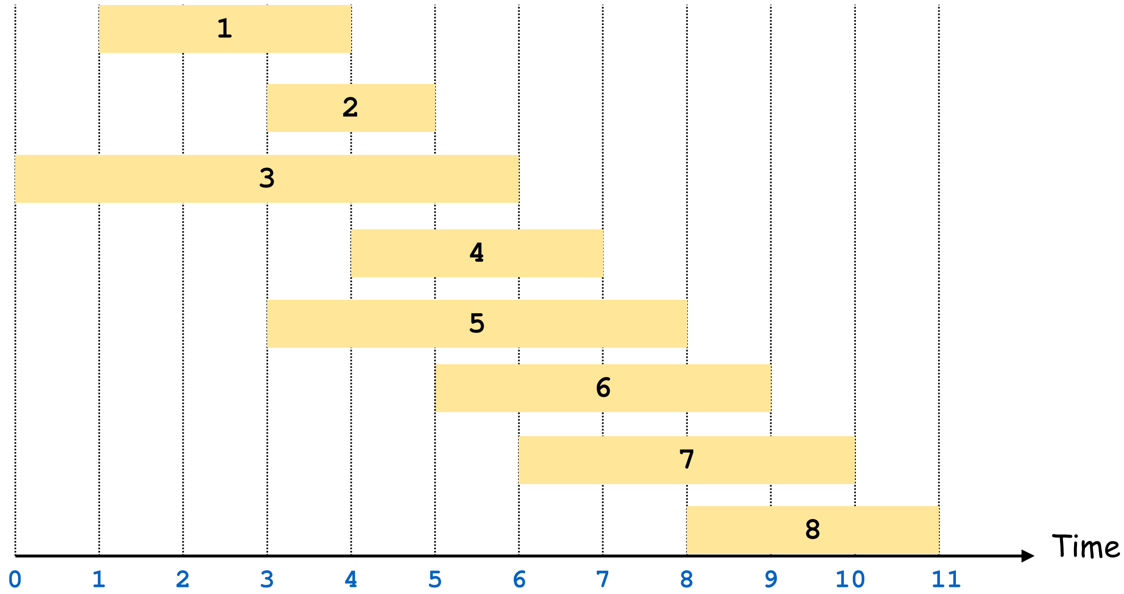

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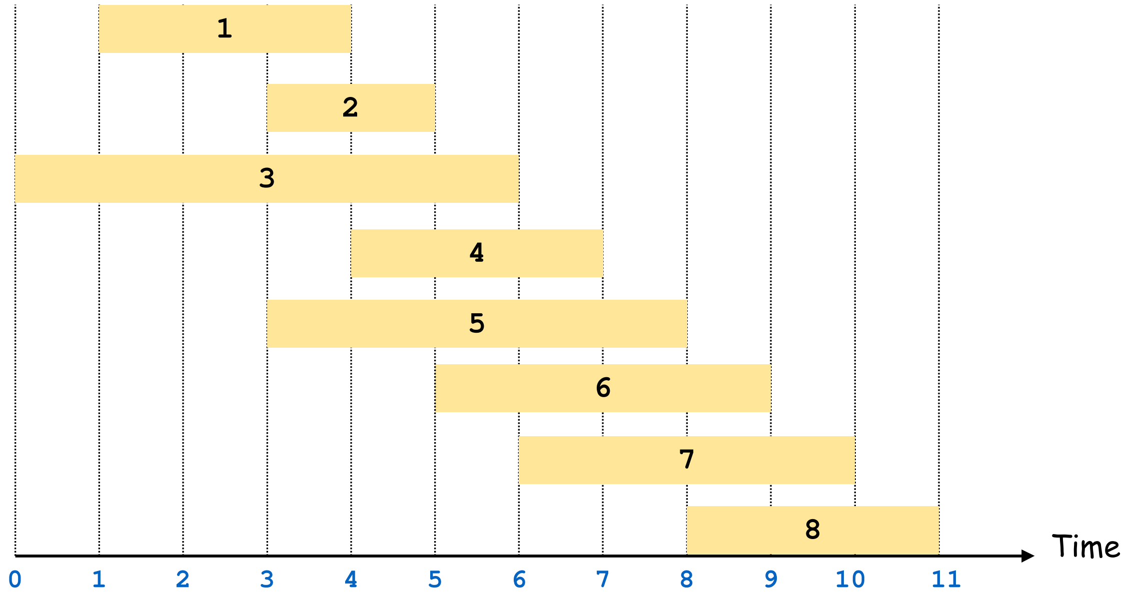

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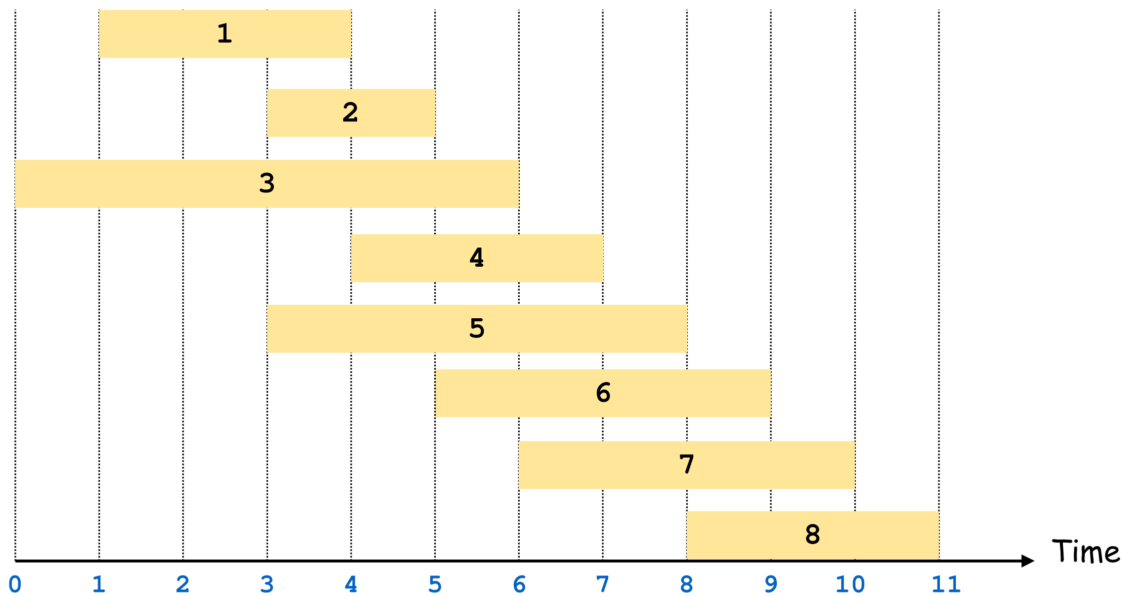

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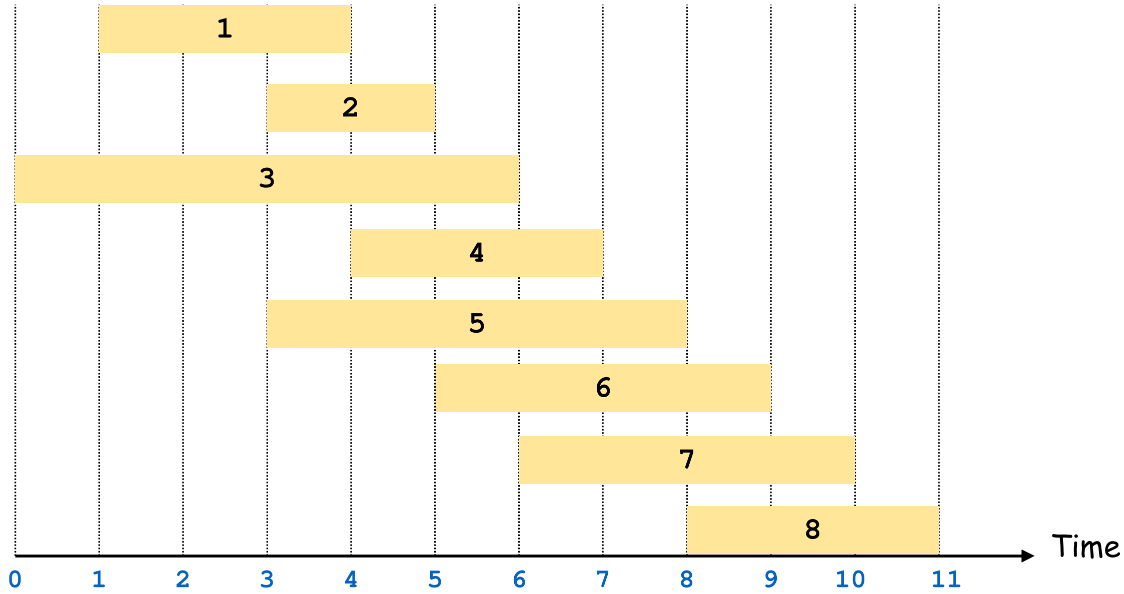

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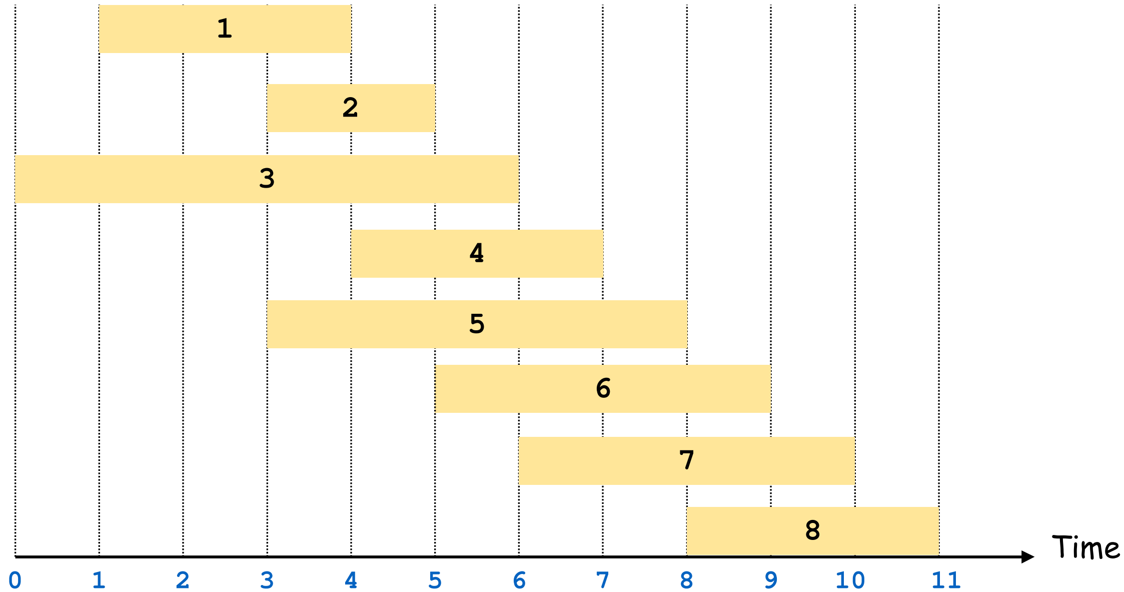

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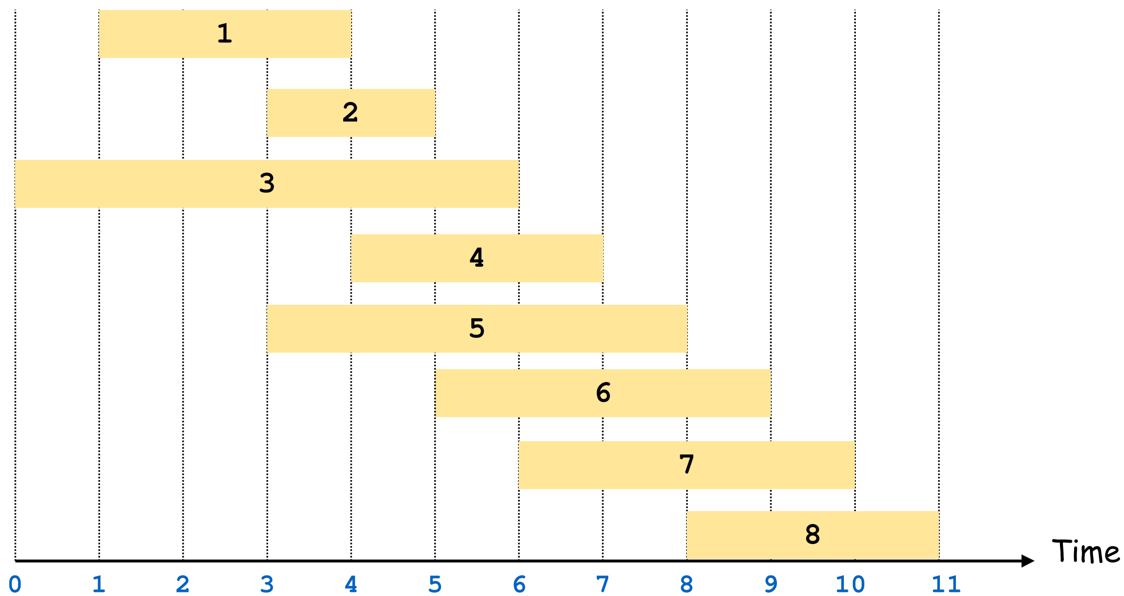

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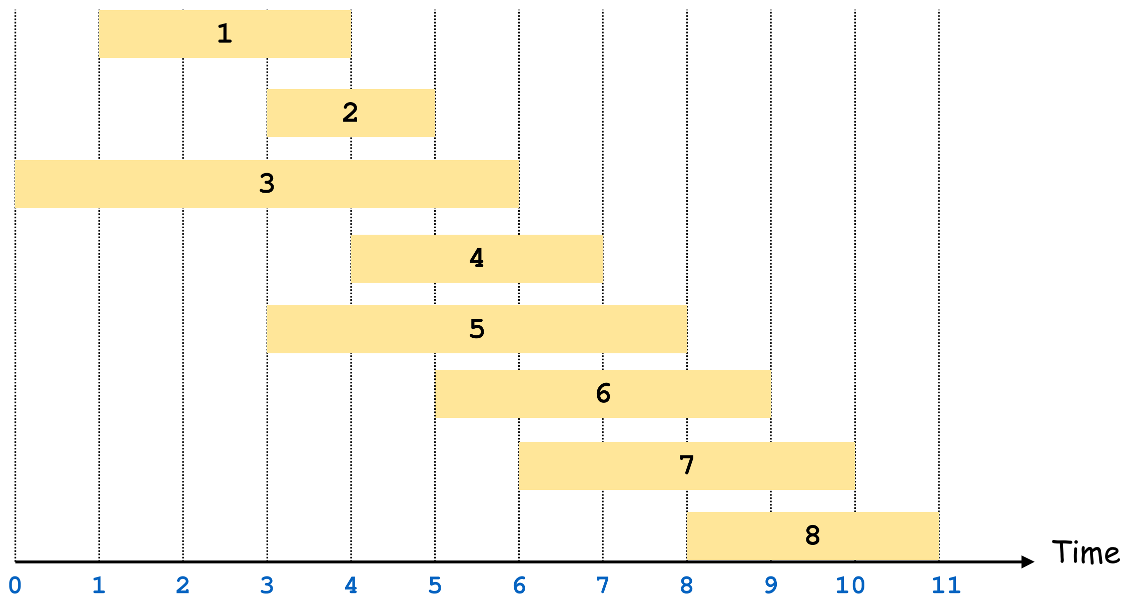
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	
8	3	5	

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

Example: $p(8) = 5, p(7) = 3, p(2) = 0$



j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	10
8	3	5	

Weighted Interval Scheduling: Iterative Solution

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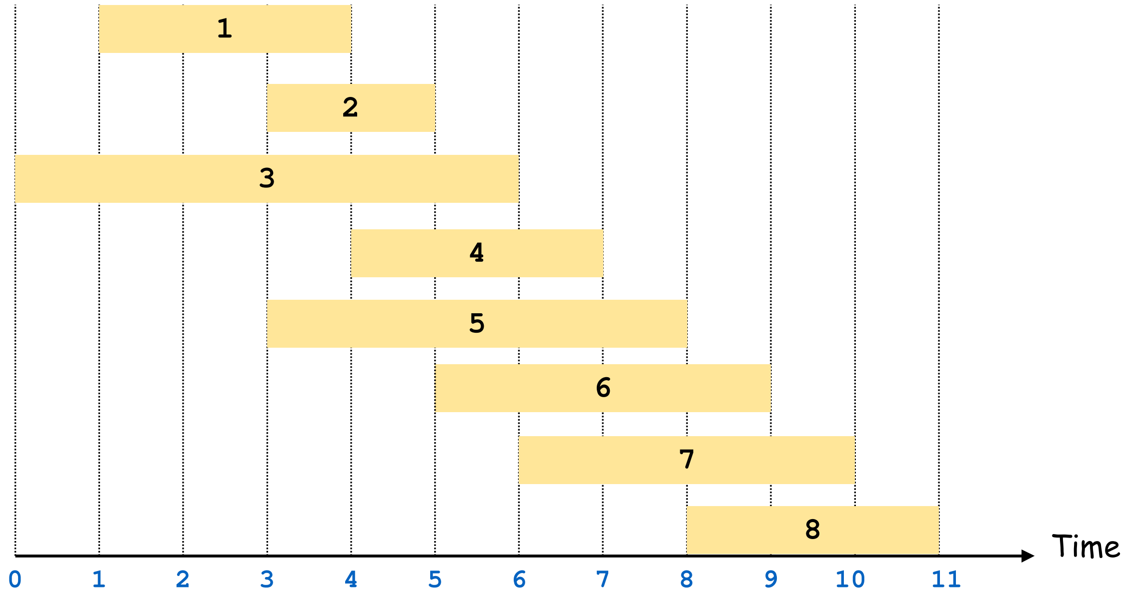

j	v_j	$p(j)$	OPT$[j]$
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	10
8	3	5	

Weighted Interval Scheduling: Iterative Solution

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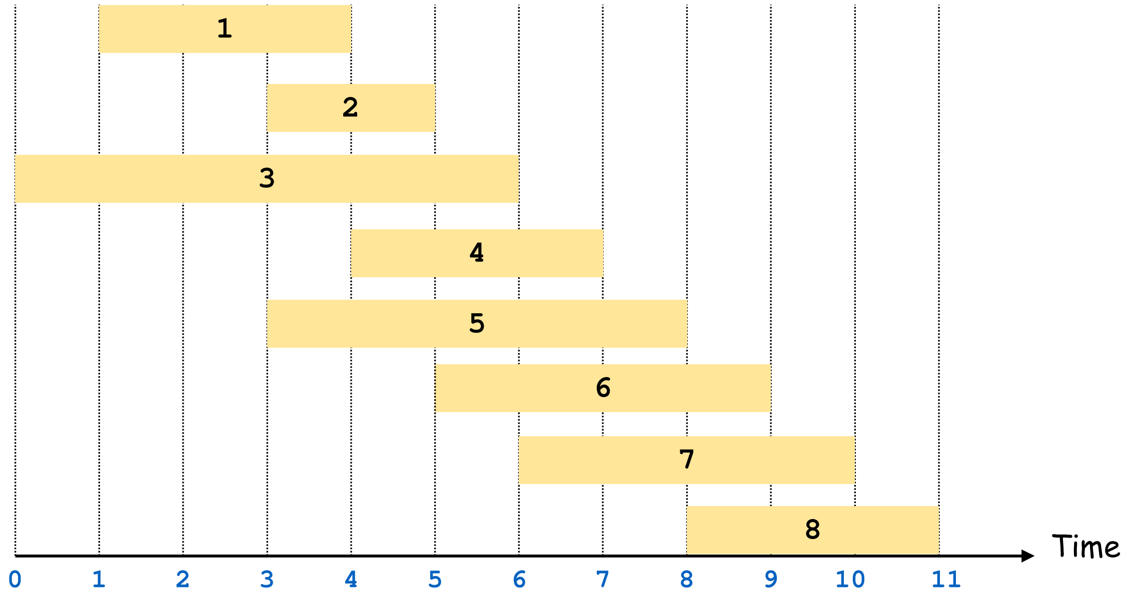


j	v_j	$p(j)$	OPT[j]
0	-	-	0
1	3	0	3
2	2	0	3
3	6	0	6
4	3	1	6
5	5	0	6
6	4	2	7
7	4	3	10
8	3	5	10

Weighted Interval Scheduling: Finding the Solution

So far we have computed the value **OPT**(n) but we probably want to know what that solution **OPT** actually is!

We can do this, too, by keeping track of which option was better at each step.

Define **Used**$[j]$ = $\begin{cases} \mathbf{1} & \text{solution with value } \mathbf{OPT}(j) \text{ includes request } j \\ \mathbf{0} & \text{otherwise} \end{cases}$

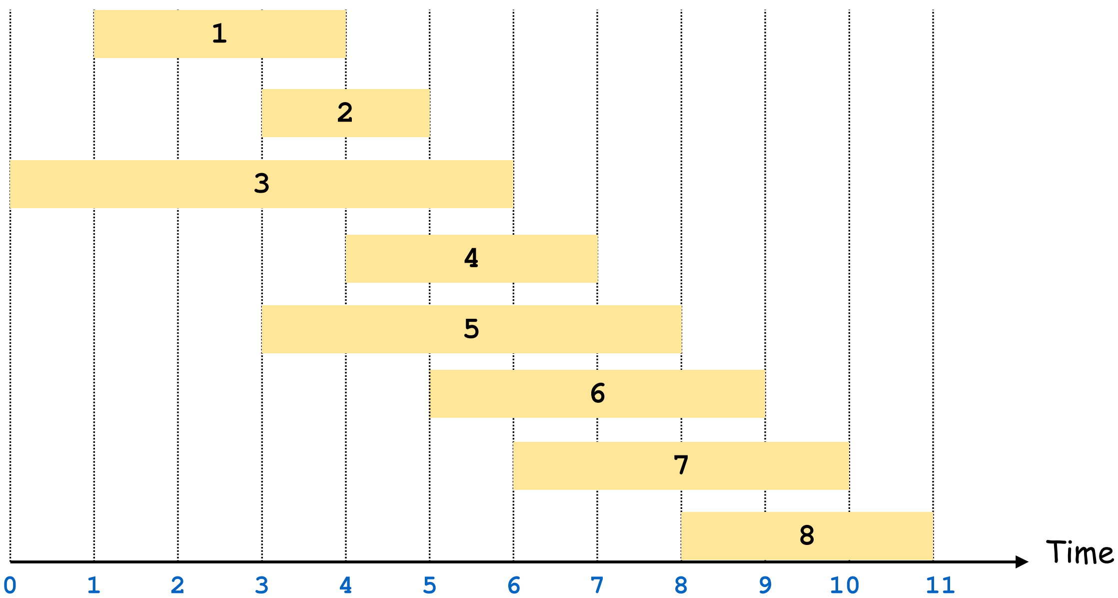
This gives a "pointer" that leads the way along a path to the optimal solution…

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \le f_2 \le \cdots \le f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

Example: $p(8) = 5, p(7) = 3, p(2) = 0$



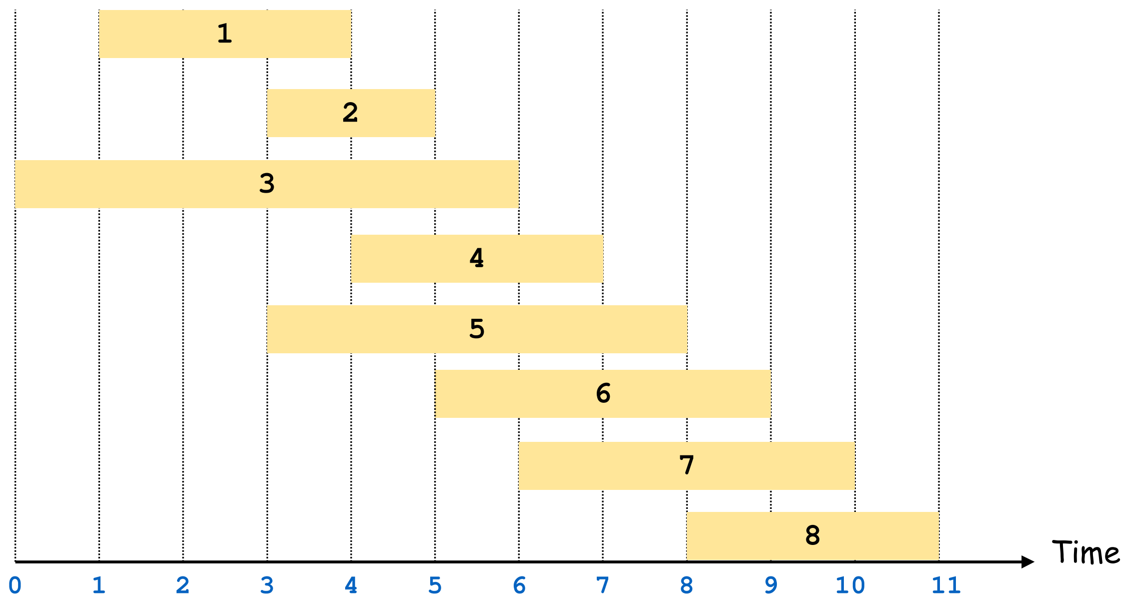
j	v_j	$p(j)$	OPT[j]	Used[j]
0	-	-	0	-
1	3	0	3	1
2	2	0	3	0
3	6	0	6	1
4	3	1	6	1
5	5	0	6	0
6	4	2	7	1
7	4	3	10	1
8	3	5	10	0

Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

Defn: $p(j)$ = largest index $i < j$ s.t. job i is compatible with j.

Example: $p(8) = 5, p(7) = 3, p(2) = 0$



j	v_j	$p(j)$	OPT[j]	Used[j]
0	-	-	0	-
1	3	0	3	1
2	2	0	3	0
3	6	0	6	1
4	3	1	6	1
5	5	0	6	0
6	4	2	7	1
7	4	3	10	1
8	3	5	10	0

Weighted Interval Scheduling: Finding the Solution

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   OPT[0] = 0
   for j = 1 to n
      if vj + OPT[p(j)] > OPT[j-1] {
           OPT[j] = vj + OPT[p(j)]
           Used[j] = 1
      } else {
           OPT[j] = OPT[j-1]
           Used[j] = 0
      }
}
```

```
Find-Opt {
   j = n
   OPTSol = ∅
   while j > 0
      if Used[j] == 0 {
         j = j-1
      } else {
         OPTSol = OPTSol ∪{j}
         j = p(j)
      }
}
```

Three Steps to Dynamic Programming

1. Formulate the answer as a recurrence relation or recursive algorithm

2. Figure out the possible values of parameters in the recursive calls.
   - This should be "small", i.e., bounded by a low-degree polynomial
   - Can use memoization to store a cache of previously computing values

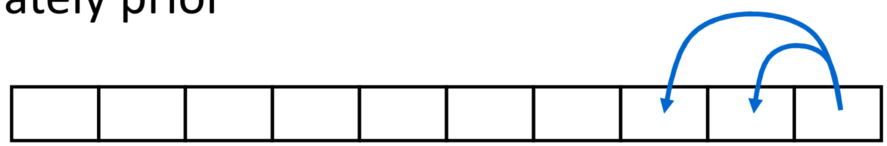
3. Specify an order of evaluation for the recurrence so that you already have the partial results stored in memory when you need them.
   - Produces iterative code

Once you have an iterative DP solution: see if you can save space...

Dynamic Programming Patterns

Fibonacci pattern:

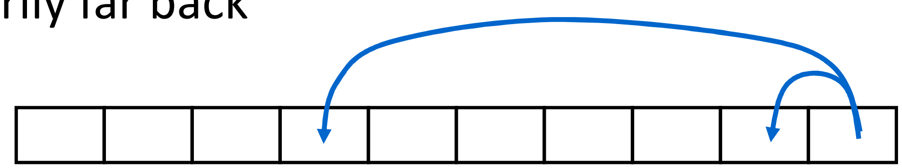
- 1-dimensional, $O(1)$ values immediately prior



- Space saving possible

Weighted interval scheduling pattern:

- 1-dimensional, $O(1)$ values arbitrarily far back



- No space saving possible