Algorithmic Paradigms

**Greedy:** Build up a solution incrementally, myopically optimizing some local criterion.

**Divide-and-conquer:** Break up a problem into sub-problems (each typically a constant factor smaller), solve each sub-problem *independently*, and combine solution to sub-problems to form solution to original problem.

**Dynamic programming:** Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.
Algorithm Design Techniques

Dynamic Programming:

• Technique for making building solution to a problem based on solutions to smaller subproblems (recursive ideas).
  • The subproblems just have to be smaller, but don’t need to be a constant-factor smaller like divide and conquer.

• Useful when the same subproblems show up over and over again

• The final solution is simple iterative code when the following holds:
  • The parameters to all the subproblems are predictable in advance
Dynamic Programming History

Bellman. [1950s]  Pioneered the systematic study of dynamic programming.

Etymology

• Dynamic programming = planning over time.
• Secretary of Defense was hostile to mathematical research.
• Bellman sought an impressive name to avoid confrontation.

"it's impossible to use dynamic in a pejorative sense"
"something not even a Congressman could object to"

Dynamic Programming Applications

Areas.

• Bioinformatics.
• Control theory.
• Information theory.
• Operations research.
• Computer science: theory, graphics, AI, compilers, systems, ....

Some famous dynamic programming algorithms.

• Unix **diff** for comparing two files.
• Viterbi for hidden Markov models.
• Smith-Waterman for genetic sequence alignment.
• Bellman-Ford for shortest path routing in networks.
• Cocke-Kasami-Younger for parsing context free grammars.
Three Steps to Dynamic Programming

1. Formulate the answer as a recurrence relation or recursive algorithm

2. Show that the number of different values of parameters in the recursive calls is “small”, e.g., bounded by a low-degree polynomial
   • Can use memoization to store a cache of previously computing values

3. Specify an order of evaluation for the recurrence so that you already have the partial results stored in memory when you need them.
A Simple Case: Computing Fibonacci Numbers

Recall $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0, F_1 = 1$

The obvious recursive algorithm direct from this recurrence is

\[
F(n) \{
    \text{if } n=0 \text{ return (0)} \\
    \text{else if } n=1 \text{ return (1)} \\
    \text{else return (F(n-1)+F(n-2))}
\}
\]
Let’s start tracking the call tree...
The full call tree has $> F_n$ leaves (exponential in $n$)
Memoization (Caching)

Remember all values from previous recursive calls in a cache
  • the parameters and
  • The values returned on those parameters

Before each recursive call, test to see if value has already been computed for those parameters
Memoization
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Fibonacci: Dynamic Programming Version

FiboDP(n):
    F[0] ← 0
    F[1] ← 1
    for i ← 2 to n {
        F[i] ← F[i-1] + F[i-2]
    }
    return (F[n])
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Once you have an iterative DP solution: see if you can save space...
Fibonacci: Space-Saving Dynamic Programming

FiboDP(n):
    prev ← 0
    curr ← 1
    for i ← 2 to n {
        temp ← curr
        curr ← curr + prev
        prev ← temp
    }
    return (curr)
Dynamic Programming

When is dynamic programming useful?

• For optimization problems this typically requires that the “Principle of optimality” hold for the problem

  “Optimal solutions to the sub-problems suffice for optimal solution to the whole problem”
Weighted Interval Scheduling

Input: Like interval scheduling each request $i$ has start and finish times $s_i$ and $f_i$. Each request $i$ also has an associated value or weight $v_i$.

$v_i$ might be
- the amount of money we get from renting out the resource
- the amount of time the resource is being used ($v_i = f_i - s_i$)

Find: A maximum-weight compatible subset of requests.
Weighted Interval Scheduling

**Input**: Set of jobs with start times, finish times, and weights

**Goal**: Find maximum weight subset of mutually compatible jobs.
**Weighted Interval Scheduling**

**Input:** Set of jobs with start times, finish times, and **weights**

**Goal:** Find **maximum weight** subset of mutually compatible jobs.

![Diagram of weighted interval scheduling with jobs a, b, c, d, e, f, g, h, and the time axis from 0 to 10 with greedy by finish times giving a result of 9]
Weighted Interval Scheduling

**Input:** Set of jobs with start times, finish times, and **weights**

**Goal:** Find **maximum weight** subset of mutually compatible jobs.

![Diagram showing the scheduling of jobs over time](image)

Optimal yields 10
Greedy Algorithms for Weighted Interval Scheduling?

• What criterion should we try?
  • Earliest start time $s_i$
    • Doesn’t work
  • Shortest request time $f_i - s_i$
    • Doesn’t work
  • Fewest conflicts
    • Doesn’t work
  • Earliest finish time $f_i$
    • Doesn’t work
  • Largest value/weight $v_i$
    • Doesn’t work
Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$. 
Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$. 
Towards Dynamic Programming: Step 1 – Recursive Algorithm

Suppose that we first sort the requests by finish time $f_i$ so $f_1 \leq f_2 \leq \ldots \leq f_n$.

We now want

• a recursive solution that makes calls to smaller problems and
• the indices for those smaller problems to be convenient,

so we first focus on the options for the last request, request $n$. 
Weighted Interval Scheduling

Notation: Label jobs by finishing time: \( f_1 \leq f_2 \leq \ldots \leq f_n \).

There are two cases we need to compare:

- When we don’t include request \( n \).
  In this case all the other requests are still fair game.
- When we do include request \( n \).
Weighted Interval Scheduling

Notation: Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

There are two cases we need to compare:

- When we don’t include request $n$.
  - In this case all the other requests are still fair game

- When we do include request $n$.
  - In this case we need to rule out some incompatible requests.

It will be convenient to be able to prune incompatible requests quickly...
Weighted Interval Scheduling

**Notation:** Label jobs by finishing time: $f_1 \leq f_2 \leq \cdots \leq f_n$.

**Defn:** $p(j) = \text{largest index } i < j \text{ s.t. job } i \text{ is compatible with } j$.

**Example:** $p(8) = 5$, $p(7) = 3$, $p(2) = 0$

![Diagram showing job intervals and corresponding $p(j)$ values]
Structure of the subproblems

Notation: \( \text{OPT}(j) = \) value of optimal solution to the problem consisting of job requests \(1, 2, \ldots, j\).

Case 1: \( \text{OPT} \) selects job \( j \)
- It can’t use incompatible jobs \( p(j) + 1, \ldots, j - 1 \)
- It must include an optimal solution to problem consisting of remaining compatible jobs \(1, \ldots, p(j)\).

Case 2: \( \text{OPT} \) doesn’t select job
- It must include an optimal solution to problem consisting of remaining compatible jobs \(1, \ldots, j - 1\)

\[
\text{OPT}(j) = \begin{cases} 
0 & \text{if } j = 0 \\
\max\{v_j + \text{OPT}(p(j)), \text{OPT}(j - 1)\} & \text{otherwise}
\end{cases}
\]
Weighted Interval Scheduling: Recursive Solution

Input: $n, s_1, ..., s_n, f_1, ..., f_n, v_1, ..., v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq ... \leq f_n$.

Compute $p(1), p(2), ..., p(n)$

Compute-Opt($j$) {
    if ($j = 0$)
        return 0
    else
        return max($v_j + $Compute-Opt($p(j)$), Compute-Opt($j-1$))
}
This recursive algorithm can be very bad...

Suppose that $p(j) = j - 2$ for every $j \geq 2$.

- Then $\text{Compute-Opt}(j)$ calls $\text{Compute-Opt}(j - 1)$ and $\text{Compute-Opt}(j - 2)$
- This is the same exponential run-time as the recursive Fibonacci code!
Weighted Interval Scheduling: Step 2 Memoization

**Memoization:** Store results of each sub-problem in a cache; lookup as needed.

**Input:** \( n, s_1, \ldots, s_n, f_1, \ldots, f_n, v_1, \ldots, v_n \)

Sort jobs by finish times so that \( f_1 \leq f_2 \leq \ldots \leq f_n \).

Compute \( p(1), p(2), \ldots, p(n) \)

For \( j = 1 \) to \( n \)

- \( M[j] = \text{empty} \)
- \( M[0] = 0 \)

\[ 
M-\text{Compute-Opt}(j) \{
\text{if } (M[j] \text{ is empty})
\quad M[j] = \max(v_j + M-\text{Compute-Opt}(p(j)), M-\text{Compute-Opt}(j-1))
\text{return } M[j]
\}
\]
Weighted Interval Scheduling: Step 3

1. Formulate the answer as a recurrence relation or recursive algorithm.

2. Show that the number of different values of parameters in the recursive calls is “small”, e.g., bounded by a low-degree polynomial.
   • Can use memoization to store a cache of previously computing values.

3. Specify an order of evaluation for the recurrence so that you already have the partial results stored in memory when you need them.
   • Produces iterative code.

Recursion for $\text{OPT}[j]$ only needs values of $\text{OPT}[i]$ for $0 \leq i < j$.
• So we can evaluate them in order $j = 0, 1, 2, ..., n$. 
Weighted Interval Scheduling: Iterative Solution

Input: \( n, s_1, \ldots, s_n, f_1, \ldots, f_n, v_1, \ldots, v_n \)

Sort jobs by finish times so that \( f_1 \leq f_2 \leq \ldots \leq f_n \).

Compute \( p(1), p(2), \ldots, p(n) \)

Iterative-Compute-Opt {
    OPT[0] = 0
    for \( j = 1 \) to \( n \)
        OPT[j] = max(\( v_j + \text{OPT}[p(j)] \), \( \text{OPT}[j-1] \))
}

Time complexity:
- \( O(n \log n) \)
- \( O(n) \)
Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: \( f_1 \leq f_2 \leq \cdots \leq f_n \).

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</tr>
<tr>
<td>8</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>
Weighted Interval Scheduling: Iterative Solution

Notation: Label jobs by finishing time: \( f_1 \leq f_2 \leq \cdots \leq f_n \).

Defn: \( p(j) \) = largest index \( i < j \) s.t. job \( i \) is compatible with \( j \).

Example: \( p(8) = 5, \ p(7) = 3, \ p(2) = 0 \)
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<table>
<thead>
<tr>
<th>$j$</th>
<th>$v_j$</th>
<th>$p(j)$</th>
<th>OPT[$j$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
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<td>6</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
<td>6</td>
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<td>5</td>
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<td>0</td>
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<tr>
<td>6</td>
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<td>3</td>
<td>10</td>
</tr>
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<td>8</td>
<td>3</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>
Weighted Interval Scheduling: Finding the Solution

So far we have computed the value $\text{OPT}(n)$ but we probably want to know what that solution $\text{OPT}$ actually is!

We can do this, too, by keeping track of which option was better at each step.

Define $\text{Used}[j] = \begin{cases} 1 & \text{solution with value } \text{OPT}(j) \text{ includes request } j \\ 0 & \text{otherwise} \end{cases}$

This gives a “pointer” that leads the way along a path to the optimal solution...
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Example: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$
Weighted Interval Scheduling: Finding the Solution

Input: \( n, s_1, \ldots, s_n, f_1, \ldots, f_n, v_1, \ldots, v_n \)

Sort jobs by finish times so that \( f_1 \leq f_2 \leq \ldots \leq f_n \).

Compute \( p(1), p(2), \ldots, p(n) \)

Iterative-Compute-Opt {
    \( \text{OPT}[0] = 0 \)
    for \( j = 1 \) to \( n \)
        if \( v_j + \text{OPT}[p(j)] > \text{OPT}[j-1] \) {
            \( \text{OPT}[j] = v_j + \text{OPT}[p(j)] \)
            \( \text{Used}[j] = 1 \)
        } else {
            \( \text{OPT}[j] = \text{OPT}[j-1] \)
            \( \text{Used}[j] = 0 \)
        }
    \}

Find-Opt {
    \( j = n \)
    \( \text{OPTSol} = \emptyset \)
    while \( j > 0 \)
        if \( \text{Used}[j] = 0 \) {
            \( j = j-1 \)
        } else {
            \( \text{OPTSol} = \text{OPTSol} \cup \{j\} \)
            \( j = p(j) \)
        }
    \}

Three Steps to Dynamic Programming

1. Formulate the answer as a recurrence relation or recursive algorithm

2. Show that the number of different values of parameters in the recursive calls is “small”, e.g., bounded by a low-degree polynomial
   • Can use memoization to store a cache of previously computing values

3. Specify an order of evaluation for the recurrence so that you already have the partial results stored in memory when you need them.
   • Produces iterative code

Once you have an iterative DP solution: see if you can save space...
Dynamic Programming Patterns

Fibonacci pattern:
• 1-dimensional, $O(1)$ values immediately prior
• Space saving possible

Weighted interval scheduling pattern:
• 1-dimensional, $O(1)$ values arbitrarily far back
• No space saving possible