Lecture 4: BFS, DFS Properties/Applications, Topological Sort
Undirected Graph Search Application: Connected Components

Want to answer questions of the form:

**Given**: vertices \( u \) and \( v \) in \( G \)
Is there a path from \( u \) to \( v \)?

**Idea**: create array \( A \) s.t

\[ A[u] = \text{smallest numbered vertex connected to } u \]

Answer is yes iff \( A[u] = A[v] \)

Q: Why is this better than an array \( \text{Path}[u, v] \)?
Undirected Graph Search Application: Connected Components

Initial state: all $v$ unvisited
for $s \leftarrow 1$ to $n$ do
  if $\text{state}(s) \neq \text{fully-explored}$ then
    $\text{BFS}(s)$: setting $A[u] \leftarrow s$ for each $u$ found
    (and marking $u$ visited/fully-explored)
  endfor
endfor

Total cost: $O(n + m)$

• Each vertex is touched once in outer procedure and edges examined in different BFS runs are disjoint
• Works also with Depth First Search ...
**DFS(u) – Recursive Procedure**

Global Initialization: mark all vertices "unvisited"

\[
\text{DFS}(u) \\
\text{mark } u \text{ “visited” and add } u \text{ to } R \\
\text{for each edge } (u, v) \\
\text{if (v is “unvisited”) } \\
\text{DFS}(v) \\
\text{end for} \\
\text{mark } u \text{ “fully-explored”}
\]
Properties of DFS($s$)

Like BFS($s$):

• DFS($s$) visits $x$ iff there is a path in $G$ from $s$ to $x$
• Edges into undiscovered vertices define depth-first spanning tree of $G$

Unlike the BFS tree:

• the DFS spanning tree isn’t minimum depth
• its levels don’t reflect min distance from the root
• non-tree edges never join vertices on the same or adjacent levels

BUT...
Non-tree edges in DFS tree of undirected graphs

Claim: All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

• In other words ... No “cross edges”.
No cross edges in DFS on undirected graphs

Claim: During DFS(\(x\)) every vertex marked “visited” is a descendant of \(x\) in the DFS tree \(T\)

Claim: For every \(x, y\) in the DFS tree \(T\), if \((x, y)\) is an edge not in \(T\) then one of \(x\) or \(y\) is an ancestor of the other in \(T\)

Proof:
• One of DFS(\(x\)) or DFS(\(y\)) is called first, suppose WLOG that DFS(\(x\)) was called before DFS(\(y\))
• During DFS(\(x\)), the edge \((x, y)\) is examined
• Since \((x, y)\) is a not an edge of \(T\), \(y\) was already visited when edge \((x, y)\) was examined during DFS(\(x\))
• Therefore \(y\) was visited during the call to DFS(\(x\)) so \(y\) is a descendant of \(x\). ■
Applications of Graph Traversal: Bipartiteness Testing

**Definition:** An undirected graph $G$ is **bipartite** iff we can color its vertices **red** and **green** so each edge has different color endpoints.

**Input:** Undirected graph $G$

**Goal:** If $G$ is bipartite, output a coloring; otherwise, output “NOT Bipartite”.

**Fact:** Graph $G$ contains an odd-length cycle $\Rightarrow$ it is not bipartite.

Just coloring the cycle part of $G$ is impossible.

On a cycle the two colors must alternate, so:

- **green** every 2\(^{nd}\) vertex
- **red** every 2\(^{nd}\) vertex

Can’t have either if length is not divisible by 2.
Applications of Graph Traversal: Bipartiteness Testing

**WLOG** ("without loss of generality"): Can assume that $G$ is connected

- Otherwise run on each component

**Simple idea:** start coloring nodes starting at a given node $s$

- Color $s$ red
- Color all neighbors of $s$ green
- Color all their neighbors red, etc.
- If you ever hit a node that was already colored
  - the **same** color as you want to color it, ignore it
  - the **opposite** color, output “NOT Bipartite” and halt
BFS gives Bipartiteness

Run BFS assigning all vertices from layer $L_i$ the color $i \mod 2$
  • i.e., red if they are in an even layer, green if in an odd layer

  • if no edge joining two vertices of the same color
    • then it is a good coloring
  • otherwise
    • there is a bad edge; output “Not Bipartite”

Why is that “Not Bipartite” output correct?
Why does BFS work for Bipartiteness?

Recall: All edges join vertices on the same or adjacent BFS layers
⇒ Any bad edge must join two vertices \( u \) and \( v \) in the same layer

Say the layer with \( u \) and \( v \) is \( L_j \)

\( u \) and \( v \) have common ancestor at some level \( L_i \) for \( i < j \)

Odd cycle of length \( 2(j - i) + 1 \)
⇒ Not Bipartite
DFS($v$) for a directed graph
**DFS(ν)**

- **Tree edges**
- **Forward edges**
- **Back edges**
- **Cross edges**

The diagram illustrates a directed graph with nodes labeled from 1 to 13, highlighting different types of edges as defined in depth-first search (DFS) traversals.
Properties of Directed DFS

• Before $\text{DFS}(s)$ returns, it visits all previously unvisited vertices reachable via directed paths from $s$

• Every cycle contains a back edge in the DFS tree
Strongly Connected Components of Directed Graphs

**Defn:** Vertices $u$ and $v$ are strongly connected iff they are on a directed cycle (there are paths from $u$ to $v$ and from $v$ to $u$).

**Defn:** Can partition vertices of any directed graph into strongly connected components:
1. all pairs of vertices in the same component are strongly connected
2. can’t merge components and keep property 1

- Strongly connected components can be stored efficiently just like connected components
- Can be found by extending DFS algorithm in $O(n + m)$ time using extra bookkeeping
  - We won’t cover the details
Strongly Connected Components

diagram of a directed graph with labeled nodes and arrows indicating back edges, forward edges, tree edges, and cross edges.
Strongly Connected Components
Strongly Connected Components
Directed Acyclic Graphs

A directed graph $G = (V, E)$ is acyclic iff it has no directed cycles.

Terminology: A directed acyclic graph is also called a DAG.

After shrinking the strongly connected components of a directed graph to single vertices, the result is a DAG.
Topological Sort

**Given:** a directed acyclic graph (DAG) \( G = (V, E) \)

**Output:** numbering of the vertices of \( G \) with distinct numbers from 1 to \( n \) so that edges only go from lower numbered to higher numbered vertices

**Applications:**
- nodes represent tasks
- edges represent precedence between tasks
- topological sort gives a sequential schedule for solving them

**Nice algorithmic paradigm for general directed graphs:**
- Process strongly connected components one-by-one in the order given by topological sort of the DAG you get from shrinking them.
Directed Acyclic Graph
In-degree 0 vertices

Claim: Every DAG has a vertex of in-degree 0

Proof: By contradiction
  Suppose every vertex has some incoming edge
  Consider following procedure:
  \[\text{while (true) do}\
  \quad v \leftarrow \text{some predecessor of } v\]

  • After \(n + 1\) steps where \(n = |V|\) there will be a repeated vertex
  • This yields a cycle, contradicting that it is a DAG.
Topological Sort

• Can do using DFS

• Alternative simpler idea:
  • Any vertex of in-degree 0 can be given number 1 to start
  • Remove it from the graph
  • Then give a vertex of in-degree 0 number 2
  • Etc.
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Implementing Topological Sort

• Go through all edges, computing array with in-degree for each vertex \( O(m + n) \)

• Maintain a list of vertices of in-degree 0

• Remove any vertex in list and number it

• When a vertex is removed, decrease in-degree of each neighbor by 1 and add them to the list if their degree drops to 0

Total cost: \( O(m + n) \)