1 Dijkstra’s Algorithm

**Theorem 1.** Let $T$ be the spanning tree found by Dijkstra$(s)$. Then, $d_G(s, u) = d_T(s, u)$ for all $u$.

**Proof:**
- Let $S_k$ be the set $S$ in the algorithm before step $k$.
- Let induction statement $P(k)$ be “$d_T(s, u) = d_G(s, u)$ for all $u \in S_k$”

**Base case** $k = 1$:
- $S_1 = \{s\}$. $d_T(s, s) = 0 = d_G(s, s)$.

**Induction step:**
- Let $v$ be the new vertex in $S_k$.
- Let $P$ be the path from $s$ to $v$ using the tree and the addition edge.

// The idea: Consider a shortest path $P^*$ from $s$ to $v$. By the choice of the algorithm, $P$ is the shortest path exiting the set $S_{k-1}$. So, $c(P^*) \geq c(P)$.
- Let $P^*$ be some shortest path from $s$ to $v$.
- Let $(u, v)$ be the edge that $P$ exit $S_{k-1}$.
- Let $(x, y)$ be the first edge that $P^*$ exit $S_{k-1}$.
- Note that

\[
\begin{align*}
  c(P^*) &\geq d_G(s, x) + c(x, y) \quad \text{(it is a subpath of $P^*$)} \\
  &= d_T(s, x) + c(x, y) \quad (x \in S_{k-1}) \\
  &\geq d_T(s, u) + c(u, v) \quad \text{(by the choice of algo)} \\
  &= c(P).
\end{align*}
\]

2 Quiz

**Algorithm:**
- Run dijkstra to find a shortest path from $s$ to $t$ with the new length $\tilde{l}_e = l_e + \frac{1}{n}$.

**Output the shortest path dijkstra gives.**

**Runtime:**
- $O(m + n \log n)$ due to dijkstra.

**Correctness:**
- Claim: Any shortest path for the distance $\tilde{l}$ is a shortest path for the distance $l$.
- Proof: Any shortest path has length at most $n - 1$. So, we at most add $\frac{n-1}{n} < 1$. Since all the costs are integer, any shortest path for the distance $\tilde{l}$ is a shortest path for the distance $l$.

Now, note that any shortest path in $l$ with less length is shorter in $\tilde{l}$. So, the algo correctly outputs a shortest path with minimum length.