CSE 421 Lecture 2

January 5, 2022

1 Big $O$ inequalities

Lemma 1. $a_0 + a_1 n + \cdots + a_d n^d = O(n^d)$.

Proof. For any positive integer $n$, we have

$$a_0 + a_1 n + \cdots + a_d n^d \leq |a_0| n^d + |a_1| n^d + \cdots + |a_d| n^d \leq C n^d$$

where $C = \sum_{i=0}^{d} |a_i|$.

Lemma 2. $\log_a n = O(\log_b n)$.

Proof. Note that $\log_b n = \frac{\log_a n}{\log_a b}$. Hence, we have $\log_a n \leq C \log_a b$ for $C = \log_a b$.

Lemma 3. $\log^k n = O(n)$ for all $k \geq 0$. (Corollary: $n \log^3 n = O(n^2)$ and $\log^{10} n = O(\sqrt{n})$)

Remark. We will not ask difficult math questions in HW/exam like this. You only need to understand the statement, but not the proof.

Proof. We claim that $x \leq 2^x$ for all $x \geq 0$. Using this, for any $n \geq 1$, we have

$$\log^k n = k^k \cdot \left(\frac{\log n}{\log k}\right)^k$$

$$\leq k^k \cdot \left(\frac{\log n}{\log n}\right)^k$$

$$= k^k \cdot n.$$

To prove the claim, we define $f(x) = 2^x - x$. For $0 \leq x \leq 1$, we have $f(x) \geq 2^0 - 1 \geq 0$. For $1 \leq x \leq 2$, we note that $f(x) \geq 2^1 - 2 \geq 0$. For $x \geq 2$, we have

$$f'(x) = 2^x \log 2 - 1 \geq 2^2 \log 2 - 1 \geq 0.$$

Hence, $f(x) \geq 0$ for all $0 \leq x \leq 2$ and $f$ is increasing for $x \geq 2$. Thus, $f(x) \geq 0$ for all $x \geq 0$. Hence, $x \leq 2^x$.

Fact 4. $3^n \neq O(2^n)$

Example 5. $n \log^{11} n \leq n 2^{\sqrt{\log n / \log \log n}} \leq n^{3/2} \leq n^{5/3} \leq n^3 \leq n^{\sqrt{n \log n}}$.

Proof. To prove $n \log^{11} n = O(n 2^{\sqrt{\log n / \log \log n}})$, it suffices to show that

$$\log^{11} n = O(2^{\sqrt{\log n / \log \log n}}).$$

Taking log on both sides (and using log is increasing), it suffices to show that

$$\log \log^{11} n = O\left(\sqrt{\frac{\log n}{\log \log n}}\right).$$
To see this, we note that
\[
\log \log n \sqrt{\log \log n} = O\left(\log^{3/2} \log n\right)
= O\left(\sqrt{\log n}\right)
\]

because \(\log^3 \log n = O(\log n)\) (Lemma 3 with \(k = 3\) and \(n\) replaced by \(\log n\)).

\[ \square \]

2 Bounding \(m\) for General Graphs

**Theorem 6.** Every undirected graph with \(n\) vertices has at most \(\frac{n(n-1)}{2} = O(n^2)\) edges.

**Proof.** Note that
- every edge connects two distinct vertices (no loops)
- no two edges connect to the same pair (no multiedges)

Hence, \(m \leq \binom{n}{2} = \frac{n(n-1)}{2} \).

\[ \square \]

3 Bounding \(m\) for Trees

**Lemma 7.** If \(G\) has no cycle, then it has a vertex of degree at most 1.

**Proof.** (Prove by contradiction.)
Suppose every vertex has degree at least 2.
Start from some vertex \(v_1\) and follow a path \(v_1, v_2, \ldots\).
Say we are at \(v_i\). Since degree \(\geq 2\), we can always find \(v_{i+1} \neq v_{i-1}\).
Now keeping going until we see a repeated vertex \(v_j = v_i\). (We must see a repeated vertex because there are finitely many vertices)
Note that \(v_i, v_{i+1}, \ldots, v_j\) forms a cycle. This is a contradiction.

\[ \square \]