CSE 421 Lecture 2

January 5, 2022

1 Big O inequalities

Lemma 1. *$a_0 + a_1 n + \cdots + a_d n^d = O(n^d)$.*

Proof. For any positive integer n, we have

$$\begin{aligned}&a_0 + a_1 n + \cdots + a_d n^d\\ \le& |a_0| n^d + |a_1| n^d + \cdots + |a_d| n^d\\ =& C n^d\end{aligned}$$

where $C = \sum_{i=0}^{d} |a_i|$. □

Lemma 2. *$\log_a n = O(\log_b n)$.*

Proof. Note that $\log_b n = \frac{\log_a n}{\log_a b}$. Hence, we have $\log_a n \le C \log_b n$ for $C = \log_a b$. □

Lemma 3. *$\log^k n = O(n)$ for all $k \ge 0$. (Corollary: $n \log^3 n = O(n^2)$ and $\log^{10} n = O(\sqrt{n})$)*

Remark. We will not ask difficult math questions in HW/exam like this. You only need to understand the statement, but not the proof.

Proof. We claim that $x \le 2^x$ for all $x \ge 0$. Using this, for any $n \ge 1$, we have

$$\begin{aligned}\log^k n &= k^k \cdot (\frac{\log n}{k})^k\\ &\le k^k \cdot (e^{\frac{\log n}{k}})^k\\ &= k^k \cdot e^{\log n}\\ &= k^k \cdot n.\end{aligned}$$

To prove the claim, we define $f(x) = 2^x - x$. For $0 \le x \le 1$, we have $f(x) \ge 2^0 - 1 \ge 0$. For $1 \le x \le 2$, we note that $f(x) \ge 2^1 - 2 \ge 0$. For $x \ge 2$, we have

$$f'(x) = 2^x \log 2 - 1 \ge 2^2 \log 2 - 1 \ge 0.$$

Hence, $f(x) \ge 0$ for all $0 \le x \le 2$ and f is increasing for $x \ge 2$. Thus, $f(x) \ge 0$ for all $x \ge 0$. Hence, $x \le 2^x$. □

Fact 4. $3^n \ne O(2^n)$

Example 5. $n \log^{11} n \lesssim n 2^{\sqrt{\log n / \log\log n}} \lesssim n^{3/2} \lesssim n^{5/3} \lesssim n^3 \lesssim n^{\sqrt{n} \log n}$.

Proof. To prove $n \log^{11} n = O(n 2^{\sqrt{\log n / \log\log n}})$, it suffices to show that

$$\log^{11} n = O(2^{\sqrt{\log n / \log\log n}}).$$

Taking log on both sides (and using log is increasing), it suffices to show that

$$\log\log^{11} n = O(\sqrt{\frac{\log n}{\log\log n}}).$$

To see this, we note that

$$\begin{aligned}\log\log^{11} n\sqrt{\log\log n} &= O(\log^{3/2}\log n)\\ &= O(\sqrt{\log n})\end{aligned}$$

because $\log^3 \log n = O(\log n)$ (Lemma 3 with $k = 3$ and n replaced by $\log n$). □

2 Bounding m for General Graphs

Theorem 6. *Every undirected graph with n vertices has at most $\frac{n(n-1)}{2} = O(n^2)$ edges.*

Proof. Note that

- every edge connects two distinct vertices (no loops)
- no two edges connect to the same pair (no multiedges)

Hence, $m \leq \binom{n}{2} = \frac{n(n-1)}{2}$. □

3 Bounding m for Trees

Lemma 7. *If G has no cycle, then it has a vertex of degree at most 1.*

Proof. (Prove by contradiction.)

Suppose every vertex has degree at least 2.

Start from some vertex v_1 and follow a path $v_1, v_2, \cdots$.

Say we are at v_i. Since degree ≥ 2, we can always find $v_{i+1} \neq v_{i-1}$.

Now keeping going until we see a repeated vertex $v_j = v_i$. (We must see a repeated vertex because there are finitely many vertices)

Note that $v_i, v_{i+1}, \cdots, v_j$ forms a cycle. This is a contradiction. □