Exercise 1

Algorithm.

- We define a new graph $\widetilde{G}$ as follows:
  - Add vertices s^* (super source) and t (sink)
  - For each $v \in V$.
    - Add vertices $v_1, v_2, \cdots, v_T$.
    - Add edges (v_i, v_{i+1}) with capacity $s(v)$ for all $i \in \{1, 2, \cdots, T-1\}$.
    - Add edges (v_i, t) with capacity $d(v, i)$ for all $i \in \{1, 2, \cdots, T\}$.
  - For each $i \in \{1, 2, \cdots, T\}$,
    - Add edges (s^*, s_i) with capacity C.
  - For each $(u, v) \in E$
    - Add edges (u_i, v_i) for all $i \in \{1, 2, \cdots, t\}$ with capacity $c(u, v)$
- Run maximum flow algorithm on $\widetilde{G}$ from s^* to t.
- Return true if the flow value is $\sum_{i=1}^{T} \sum_{v \in V} d(v, i)$.

Runtime. Since maximum flow problem can be solved in polynomial time, and the new graph has $O(nT)$ vertices and $O(mT)$ edges, our algorithm has runtime polynomial in n, m and T.

Correctness. Let K be the sum of the fulfilled demand. We claim that the maximum of K is exactly equals to the maximum flow value of the graph $\widetilde{G}$ from s^* to t. If the claim is true, the algorithm correctly outputs true if all demand can be satisfied.

Note that

- The flow on (s^*, s_i) denote the water produced on day i.
- The flow on (u_i, v_i) denote the flow on (u, v) on day i.
- The flow on (v_i, v_{i+1}) denote the amount of water stored on day i.
- The flow on (v_i, t) denote the fulfilled demand on v on day i.

Any $s^* - t$ flow on $\widetilde{G}$ corresponds to a water schedule on day 1 to day T. On the other hand, any water schedule on day 1 to day T can be represented by a $s^* - t$ flow on $\widetilde{G}$. Furthermore, the fulfilled demand is exactly the sum of flow to t, which equals to the flow value. This proves the claim.

Exercise 2

Algorithm.

- Let $f = 0$
- While s, t is connected in G_f
  - Run Sally's algorithm on G_f to obtain δ_f.
  - Set $f \leftarrow f + \delta_f$.
- Return f.

Runtime. Let $\mathrm{OPT}(G)$ be the maximum $s - t$ flow value in G. Let $\mathrm{val}(f)$ be the flow value of f. We first prove the following:

Lemma 1. *We have* $\mathrm{OPT}(G) = \mathrm{OPT}(G_f) + \mathrm{val}(f)$.

Proof. For any $s - t$ flow δ_f in G_f, $f + \delta_f$ is a $s - t$ flow in G. Hence, $\mathrm{OPT}(G) \geq \mathrm{val}(f + \delta_f) = \mathrm{val}(f) + \mathrm{val}(\delta_f)$. In particular, this shows

$$\mathrm{OPT}(G) \geq \mathrm{val}(f) + \mathrm{OPT}(G_f).$$

On the other hand, by maxflow mincut theorem, there is a $s - t$ cut (A, B) on G_f with $\mathrm{cap}_{G_f}(A, B) = \mathrm{OPT}(G_f)$. Using the definition of cap, the definition of G_f and the flow value lemma

$$\mathrm{cap}_G(A, B) = \mathrm{cap}_{G_f}(A, B) + \mathrm{val}(f).$$

Hence, we have

$$\begin{aligned} \mathrm{OPT}(G) &\leq \mathrm{cap}_G(A, B) \\ &= \mathrm{cap}_{G_f}(A, B) + \mathrm{val}(f) \\ &= \mathrm{OPT}(G_f) + \mathrm{val}(f). \end{aligned}$$

□

Let $f^{(k)}$ be the flow at the beginning of the k-th step. By the guarantee of the Sally's algorithm, we have

$$\mathrm{val}(\delta_f^{(k)}) \geq \frac{1}{2}\mathrm{OPT}(G_{f^{(k)}}).$$

Using this and the previous lemma, we have

$$\begin{aligned}\mathrm{OPT}(G_{f^{(k+1)}}) &= \mathrm{OPT}(G_{f^{(k)}}) - \mathrm{val}(\delta_f^{(k)}) \\ &\leq \frac{1}{2}\mathrm{OPT}(G_{f^{(k)}}).\end{aligned}$$

Hence, the optimum value of the residual graph halves every step. Initially, $G_{f^{(1)}} = G$ with optimum value F. Hence, in $O(\log F)$ iterations, the optimal value is < 1. Since the Sally's algorithm outputs an integer flow, we have $\mathrm{OPT}(G_{f^{(k)}})$ is integer for all k. Hence, in $O(\log F)$ iterations, $\mathrm{OPT}(G_{f^{(k)}})$ becomes 0 and maxflow mincut theorem shows that $s - t$ is disconnected in $G_{f^{(k)}}$.

Since Sally's algorithm takes $O(m)$ time and since there are $O(\log F)$ iterations, the total runtime is $O(m \log F)$.

Correctness. Since δ_f is $s - t$ flow on G_f, $f + \delta_f$ is $s - t$ flow on G. Hence, the algorithm outputs a $s - t$ flow. Since the algorithm only terminates when $s - t$ is disconnected in $G_{f^{(k)}}$, maxflow mincut theorem shows that $\mathrm{OPT}(G_f) = 0$. By the previous lemma, this shows the flow is a maxflow ($\mathrm{val}(f) = \mathrm{OPT}(G)$).