CSE 421 Lecture 11

1 Master Theorem

Lemma 1. *Let $S = 1 + x + x^2 + \cdots x^d$. Then, we have*

$$S = \begin{cases} \Theta_x(1) & \text{if } x < 1, \\ d+1 & \text{if } x = 1, \\ \Theta_x(x^d) & \text{if } x > 1 \end{cases}$$

where Θ_x means the constant depends on x.

Proof. Let $S = 1 + x + x^2 + \cdots + x^d$. Then, we have

$$\begin{aligned} xS &= x + x^2 + \cdots + x^{d+1}, \\ xS - S &= x^{d+1} - 1. \end{aligned}$$

Hence, we have

$$S = \frac{x^{d+1} - 1}{x - 1}.$$

If $x < 1$, we have $S = \frac{1-x^{d+1}}{1-x}$. Note that $1 \leq S \leq \frac{1}{1-x}$ where both left and right are independent to d, we have $S = \Theta_x(1)$.

If $x = 1$, we have $S = d + 1$ by the definition of S.

If $x > 1$, we have $x^d \leq S \leq \frac{x}{x-1} \cdot x^d$. Hence, we have $S = \Theta_x(x^d)$. □

Theorem 2. *Given $a \geq 1, b > 1, c > 0$ and $k \geq 0$. Suppose that $T(n) = aT(\frac{n}{b}) + cn^k$ for all $n \geq b$, then*

1. *If $a < b^k$, then $T(n) = \Theta(n^k)$.*
2. *If $a = b^k$, then $T(n) = \Theta(n^k \log n)$.*
3. *If $a > b^k$, then $T(n) = \Theta(n^{\log_b a})$.*

Proof. As we argued in the slide, we have

$$T(n) = cn^k \sum_{i=0}^{\log_b n} (\frac{a}{b^k})^i.$$

If $a < b^k$, the previous lemma shows

$$T(n) = cn^k \Theta(1) = \Theta(n^k).$$

If $a = b^k$, the previous lemma shows

$$T(n) = cn^k \Theta(\log_b n) = \Theta(n^k \log n).$$

If $a > b^k$, the previous lemma shows

$$T(n) = cn^k \Theta(\frac{a}{b^k})^{\log_b n}.$$

Note that $b^{\log_b n} = n$ and hence

$$T(n) = \Theta(n^k \frac{a^{\log_b n}}{n^k}) = \Theta(a^{\log_b n}) = \Theta((b^{\log_b a})^{\log_b n}) = \Theta((b^{\log_b n})^{\log_b a}) = \Theta(n^{\log_b a}).$$

□