CSE 421: Introduction to Algorithms

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Guest Lecturer: Jeremy Lin (TA)

Lecture Outline:

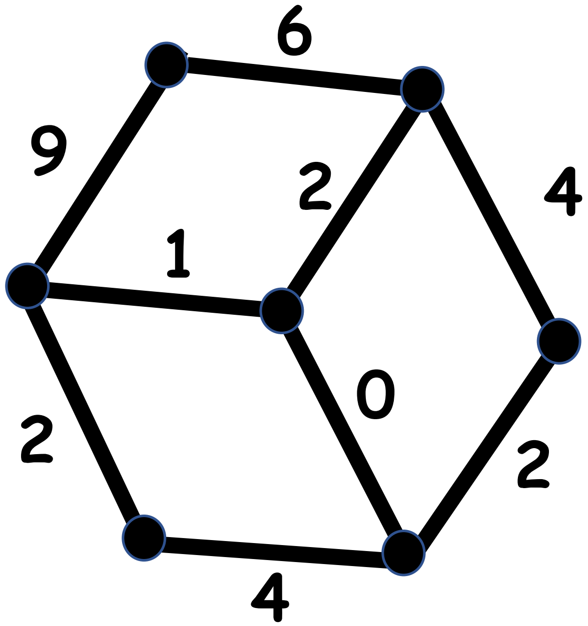
- Spanning Trees / Minimum Spanning Trees
- Cut Property
- Cycle Property
- Kruskal's Algorithm
  - Union Find Data Structure
- (Briefly) talk about Prim's and Reverse-Delete algorithms

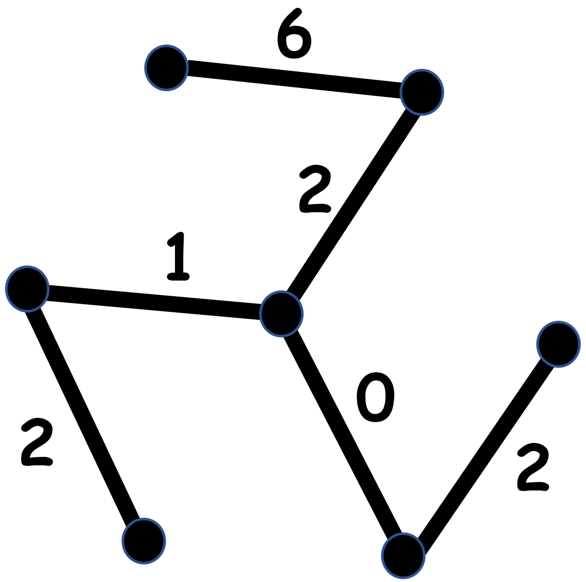
Spanning Trees

- A tree T is a **spanning tree** of a graph G if:
  - T is a valid tree (obviously)
  - T includes all vertices in G
  - T includes only edges in G (but possibly not all edges in G)
- More formally, if $G = (V, E)$ and $T = (V', E')$ then:
  - $V' = V$
  - $|E'| = |V'| - 1$
  - $E' \in E$

Minimum Spanning Trees (MST)

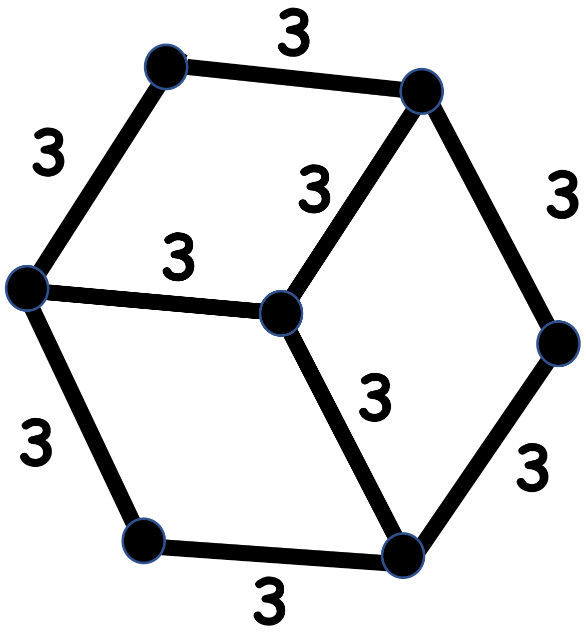
- An MST is the **lowest-cost spanning tree** of a graph

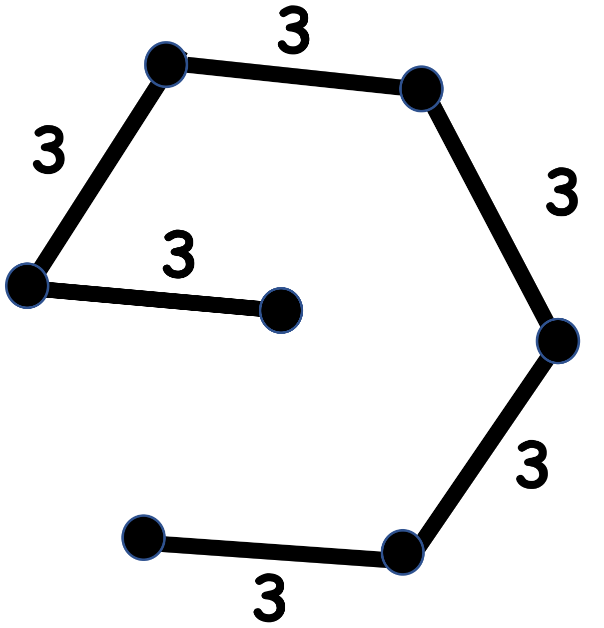


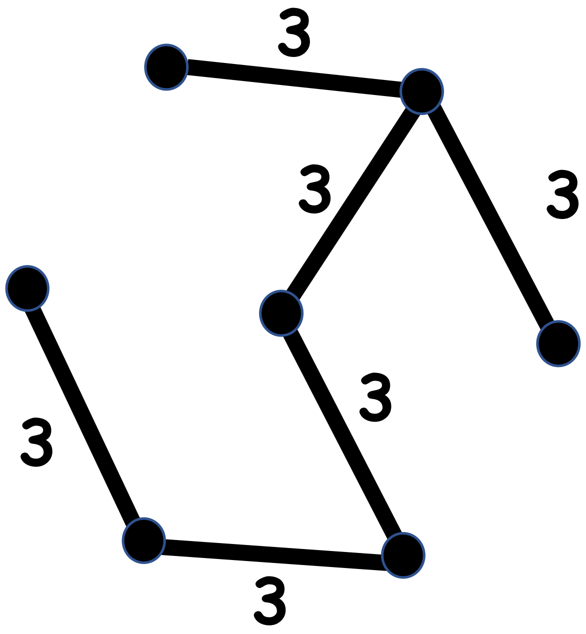


Minimum Spanning Trees (MST)

- A graph may have multiple possible MSTs!
- Trivial example:







Yesterday: Dijkstra's Algorithm

- Find the shortest path from vertex S to all other vertices in G
  - Guaranteed non-negative edges, etc.
- If you draw out all the shortest paths calculated on G, do they always form a (spanning) tree?
  - (Assume no two paths from S to T "tied" for shortest)

Yesterday: Dijkstra's Algorithm

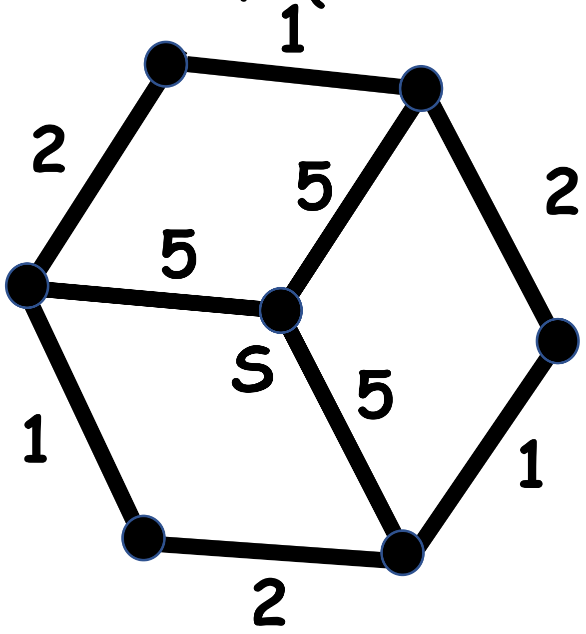
- Find the shortest path from vertex S to all other vertices in G
  - Guaranteed non-negative edges, etc.
- If you draw out all the shortest paths calculated on G, do they always form a spanning tree? **Yes!**
- **Proof Sketch: (Contradiction)**
  - Suppose that the graph G' formed by connecting the shortest paths as described above is not a tree -> it must have a cycle by definition)
  - Let T be a vertex in a cycle. Therefore, there must be two paths from S to T
    - One of them is not used for any shortest paths, since for any of T's neighbors T' we will always path from S to T' by taking the shorter path from S to T first, then the path from T to T'
    - But then this contradicts how we constructed G'!

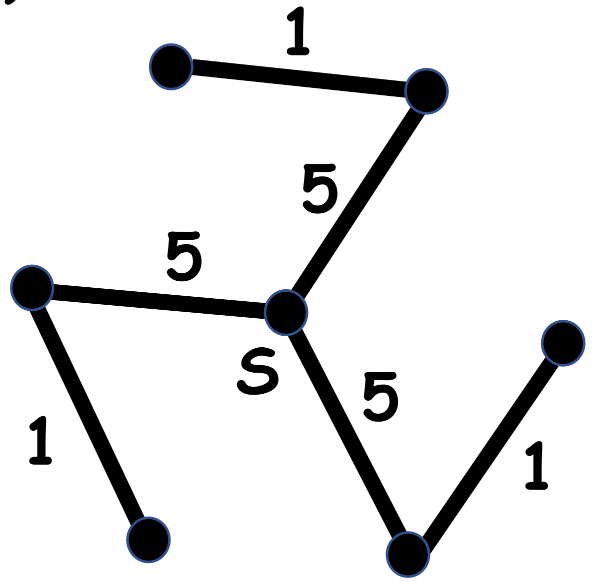
Yesterday: Dijkstra's Algorithm

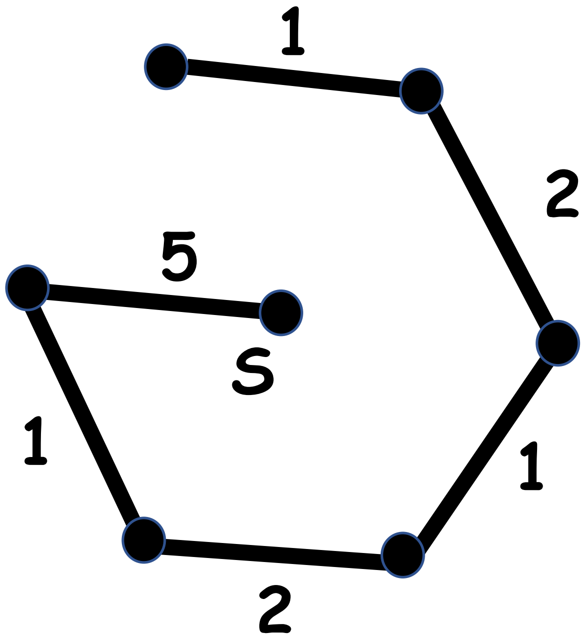
- Find the shortest path from vertex S to all other vertices in G
  - Guaranteed non-negative edges, etc.
- If you draw out all the shortest paths calculated:
  - Do they form a (spanning) tree? **Yes!**
  - Do they form a **minimum** spanning tree?

Yesterday: Dijkstra's Algorithm

- Find the shortest path from vertex S to all other vertices in G
  - Guaranteed non-negative edges, etc.
- If you draw out all the shortest paths calculated:
  - Do they form a **minimum** spanning tree? **No!**
  - **Proof: (counterexample)**







Why MSTs?

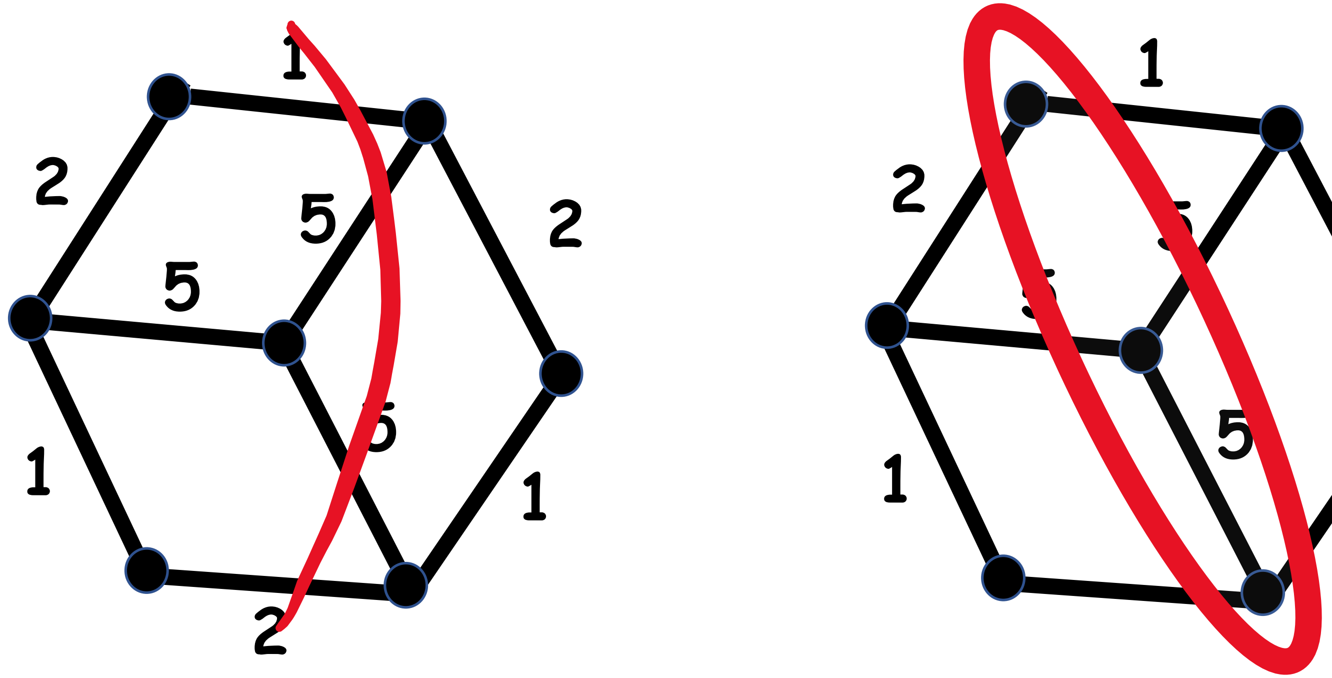
- **LOTS** of applications
  - **Network Design:**
    - Roads, TV cables, Electrical wires, etc.
  - **Approximations for (NP-) hard problems**
    - Travelling Salesperson
  - **And many more!**

Properties of MSTs

- Cut Property
- Cycle Property

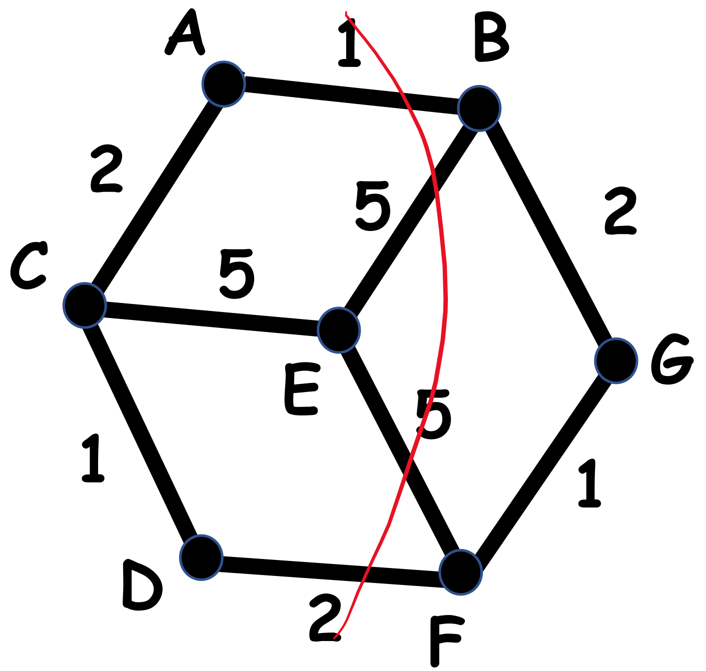
Cuts

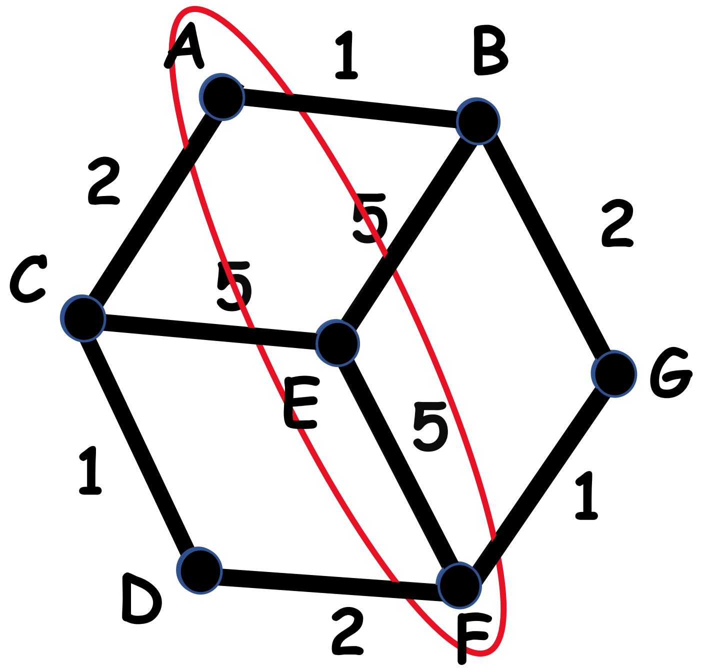
- A cut is any partition of the vertices in G into two disjoint sets of vertices (denoted by (A, B))
  - The vertices in each set don't need to be connected to each other
  - This will come up again later! (~Lecture 18 on flows and cuts)



Cut Property

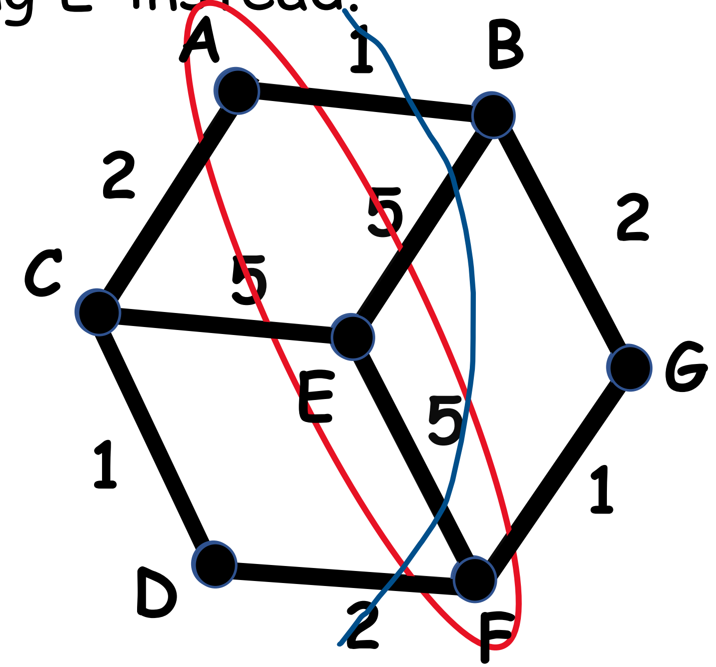
- The **lightest** (least weight) edge connecting the two sets of vertices in each cut must be in **every** MST
  - If there are multiple edges tied for the lowest weight then all MSTs must contain at least one of them





Cut Property

- **General Proof: (contradiction)**
  - Say a cut (A, B) results in the two sets of vertices A and B. Say an MST includes an edge E going across the cut (connecting A and B).
  - If there exists a *lighter* edge E' going across the cut, then we would get a "better" MST by removing E and adding E' instead.
  - But this is a contradiction!



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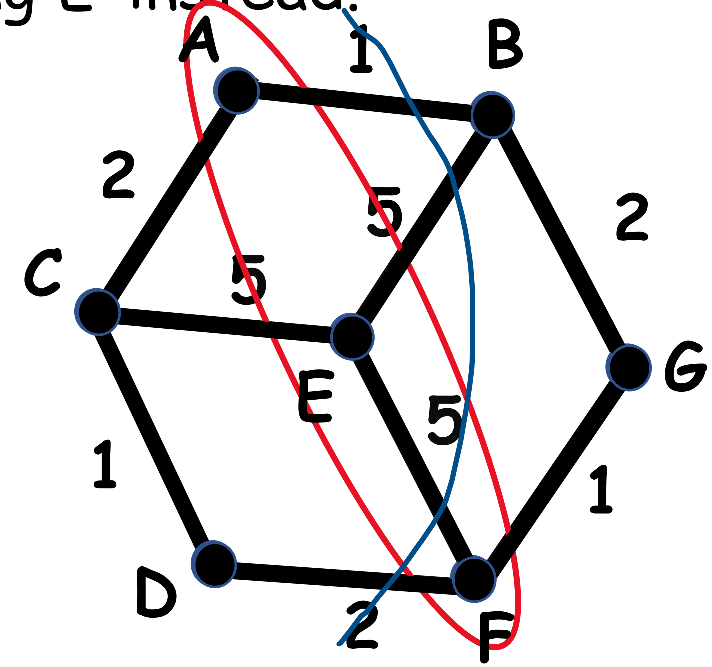
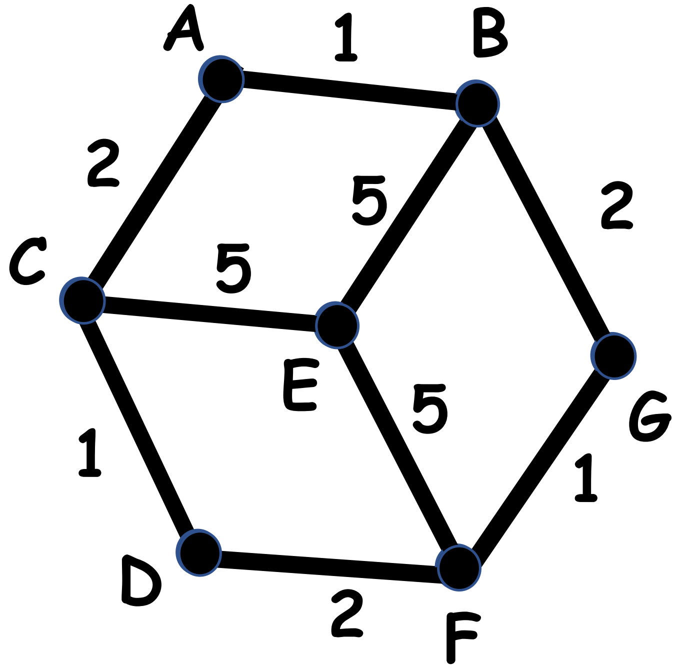
  - If there exists a *lighter* edge E' going across the cut, then we would get a "better" MST by removing E and adding E' instead.
  - But this is a contradiction!
  - Minor details to think about:
    - Prove that replacing E with E' creates a valid tree
      - (Hint: prove it doesn't create a cycle first)



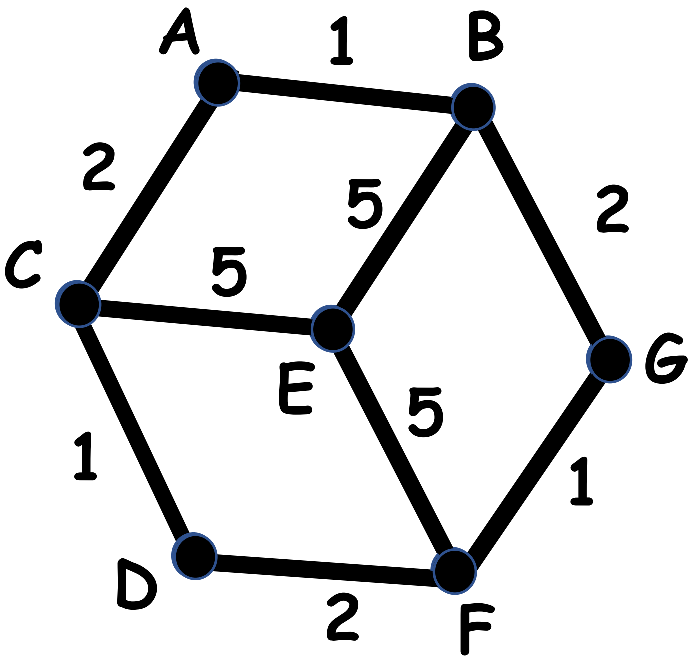
Cycle Property

- The **heaviest** (most weight) edge in every cycle in G cannot be in **any** MST
  - If there are multiple edges tied for the highest weight then all MSTs can contain at most all but one of them



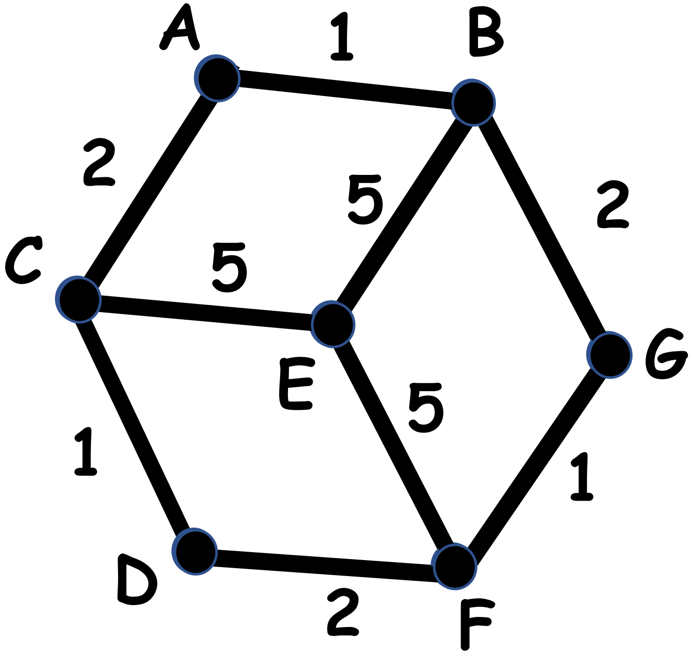
Cycle Property

- **General Proof: (contradiction)**
  - Say that while constructing the MST we keep (all of the) heaviest edges in a cycle C and remove one of the lighter edges E' instead
  - But then we will be able to construct a "better" MST by removing one of the heaviest edges and adding E' back in!
  - But this is a contradiction!



Cycle Property

- **General Proof: (contradiction)**
  - Say that while constructing the MST we keep (all of the) heaviest edges in a cycle C and remove one of the lighter edges E' instead
  - But then we will be able to construct a "better" MST by removing one of the heaviest edges and adding E' back in!
  - But this is a contradiction!
  - Minor detail:
    - Prove that replacing the heaviest edge forms a tree
    - General sketch:
      - Doing this doesn't create a cycle
      - Keep number of edges the same
      - -> by Pigeonhole Principle we form a valid tree still

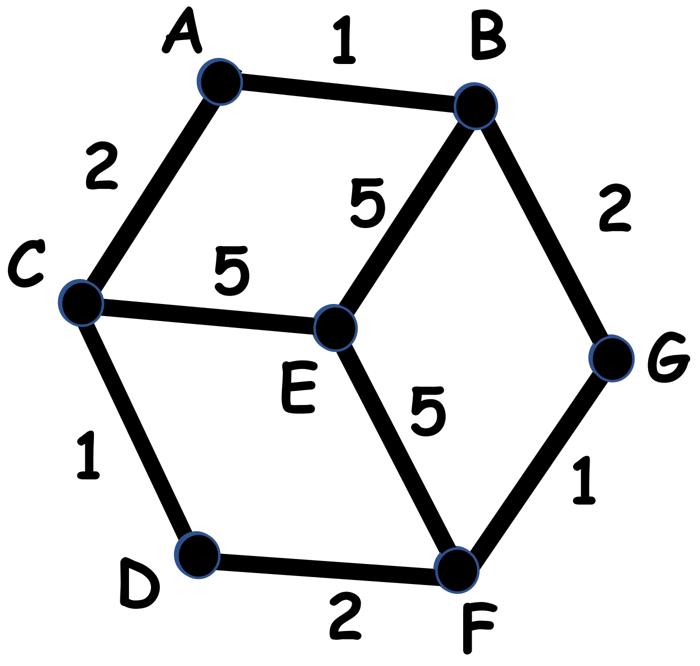


Using Cuts and Cycles to build MSTs

- Kruskal's Algorithm
- Prim's Algorithm
- Reverse-Delete Algorithm
- (and more!)

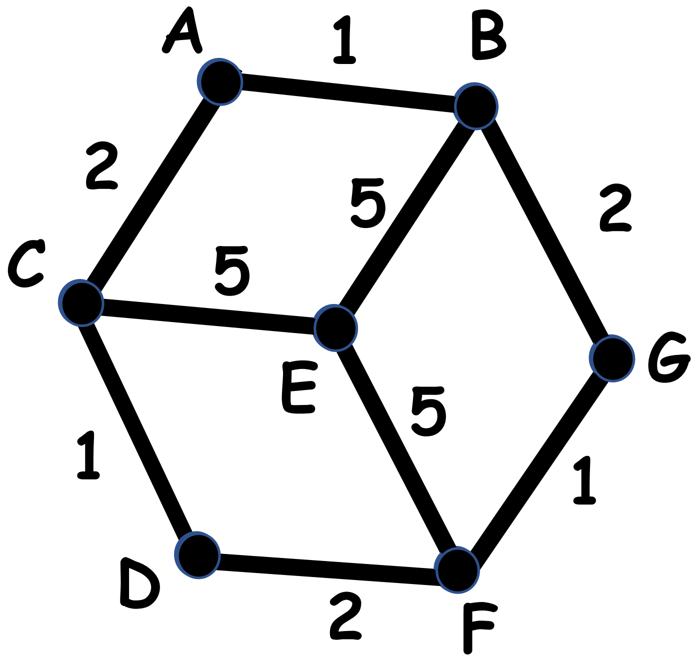
Kruskal's Algorithm:

- Greedy Algorithm!
  - Greedy Rule: Add the lowest-cost edge that doesn't create a cycle
  - **Which property discussed previously does Kruskal's use?**

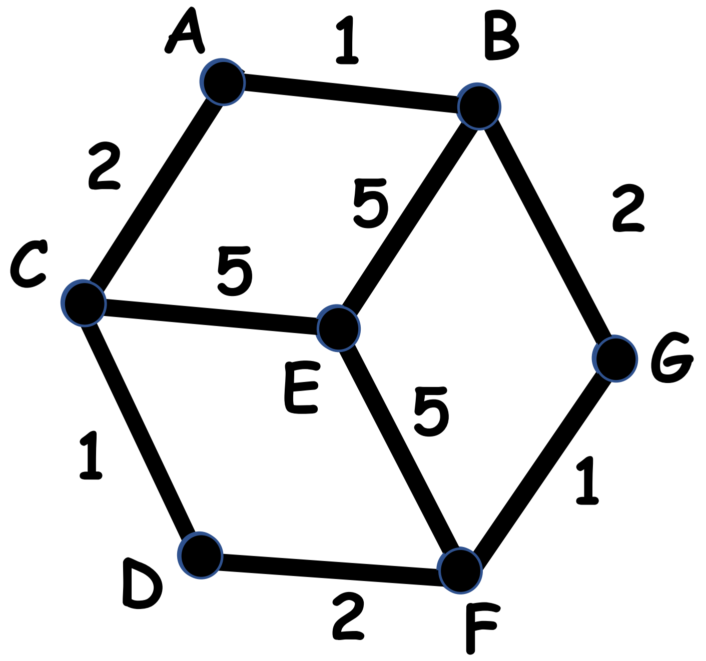


Kruskal's Algorithm:

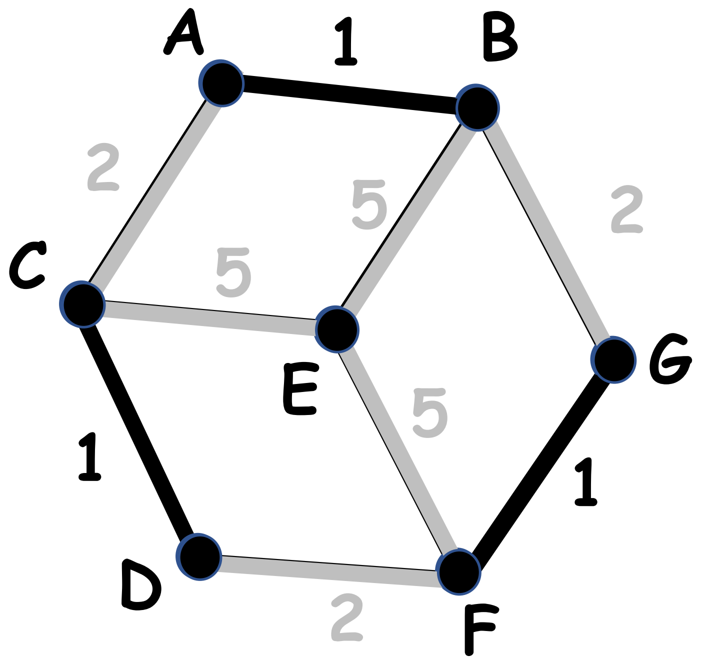
- Greedy Algorithm!
  - Greedy Rule: Add the lowest-cost edge that doesn't create a cycle
  - **Which property discussed previously does Kruskal's use?**
    - **Uses both Cut and Cycle Properties!**



Kruskal's Execution:

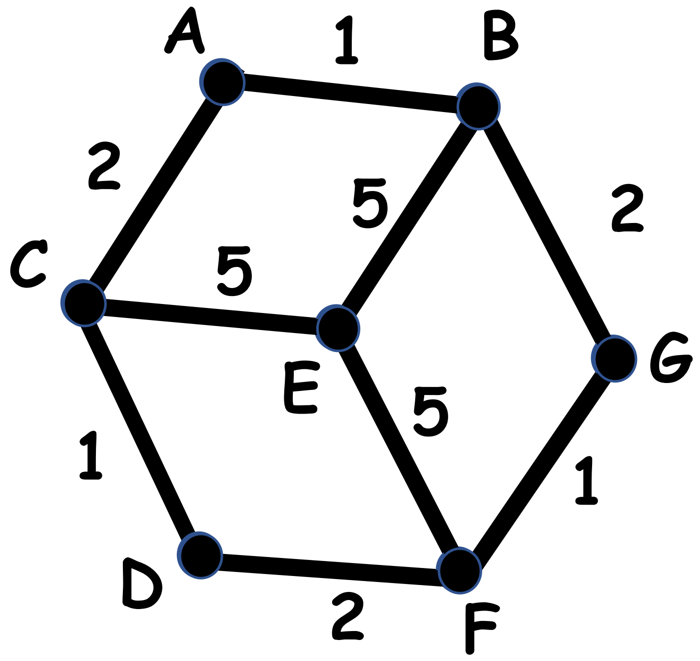


Original Graph



Minimum Spanning Tree

Kruskal's Execution:



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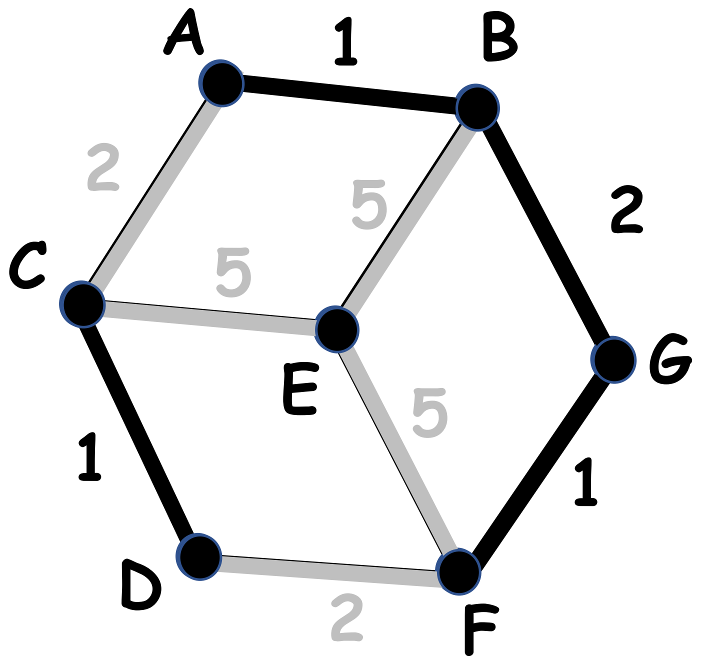

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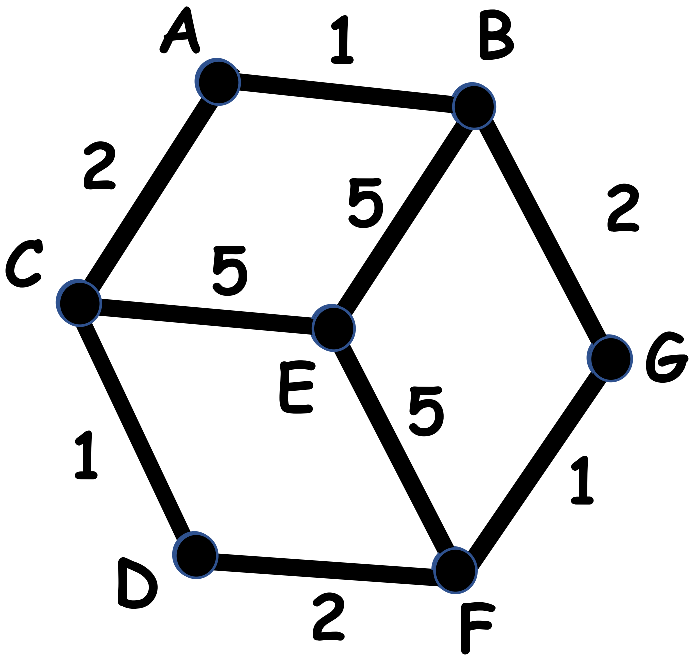

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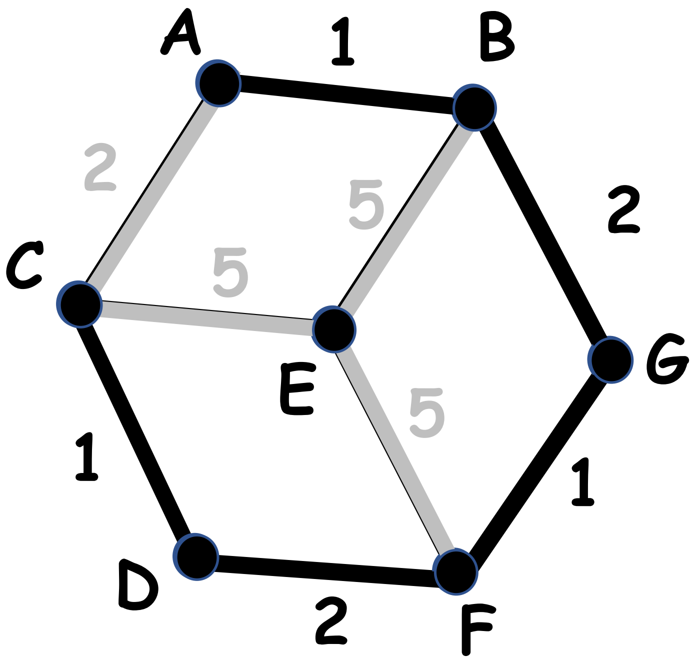

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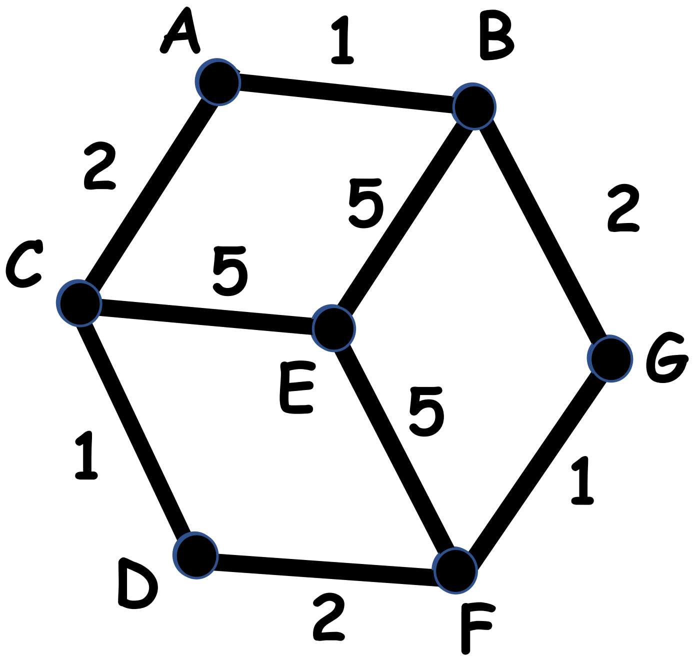

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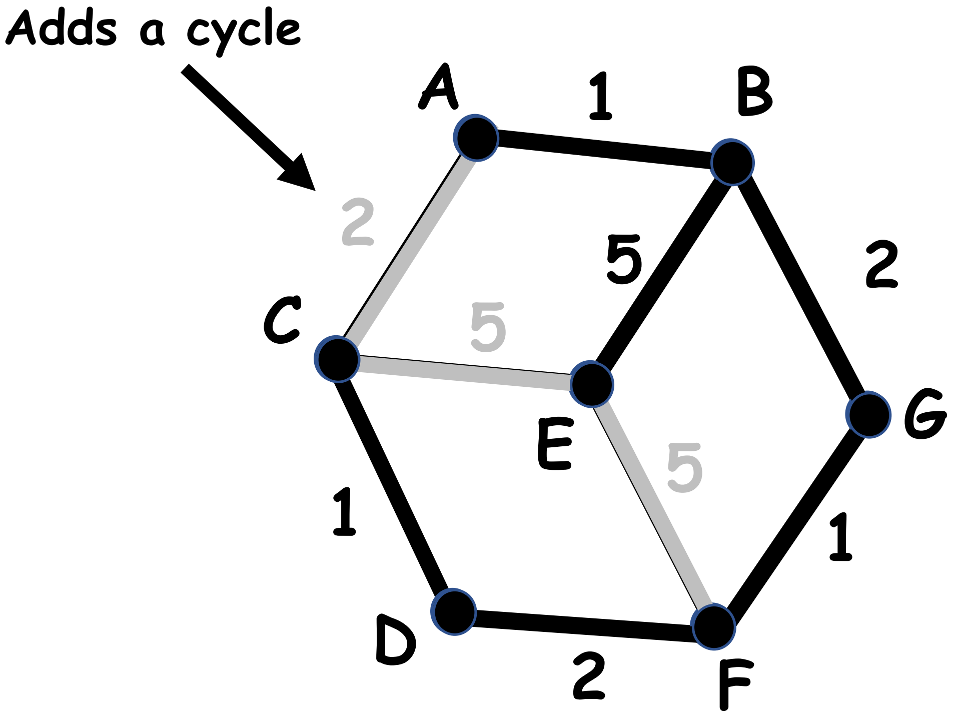
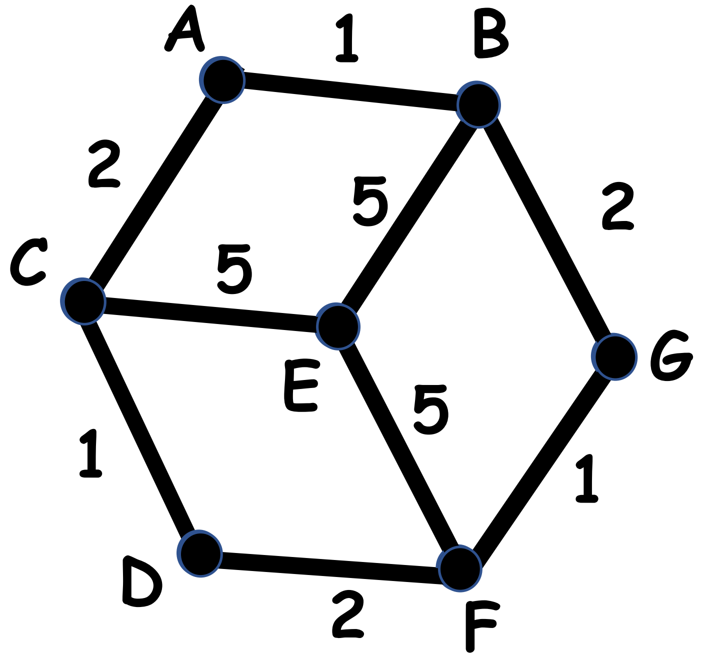


Minimum Spanning Tree

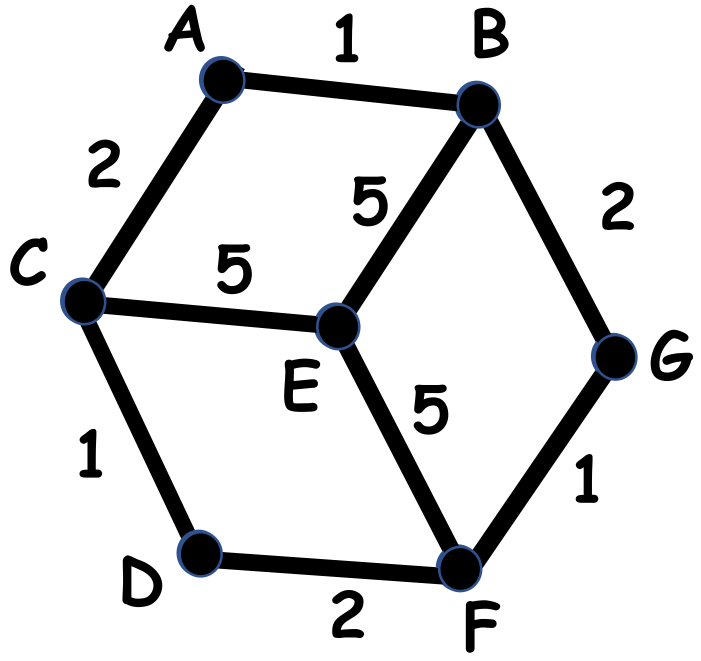
Kruskal's Pseudocode:



- Let w_e denote the weight of edge e.

```
Kruskals(V, E):
   sort E in non-decreasing order (w_0 ≤ w_1 … ≤ w_m)
   Initialize each vertex in its own "island"
   for i = 1 … m:
        let e_i = (u,v)
        if u and v are in different connected components:
             add e_i into the MST
             merge the connected components containing u,v
return the MST
```

Kruskal's Pseudocode:



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    Initialize each vertex in its own "island"
    for i = 1 … m:
        let e_i = (u,v)
        if u and v are in different "islands":
            add e_i into the MST
            merge the "islands" containing u and v
    return the MST
```

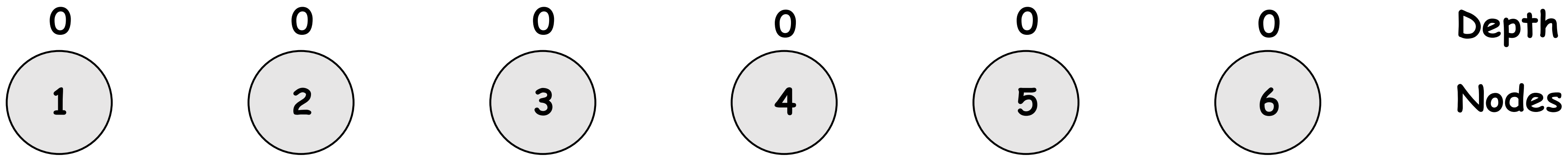
How to do this **efficiently**?
(easy O(n log n) implementation)

Union Find!!!

- Both a data structure and an algorithm
- Runtime:
  - O(log n) for checking if two nodes are in the same group ☺
  - O(log n) for merging two groups ☺

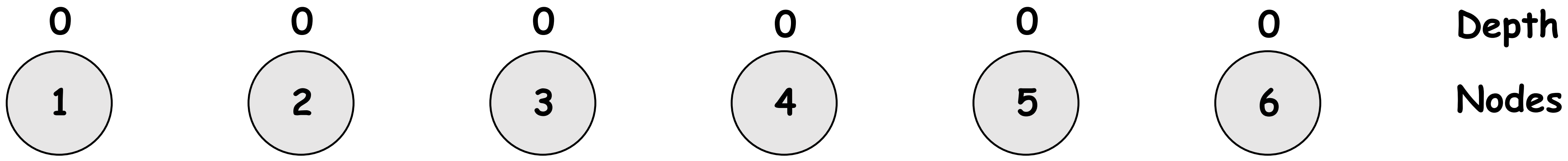
Union Find

- For each node, keep track of two things:
  - Pointer to its "parent"
  - "Depth" of its tree (length of longest path ending at that node)
- All pointers initially uninitialized, "depth" = 0



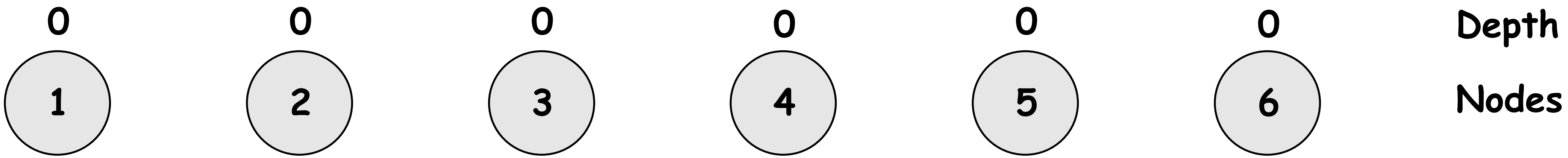
Union Find

- To check whether A and B are part of the same “island”:
  - Follow the pointers up to the root of the tree, check if identical



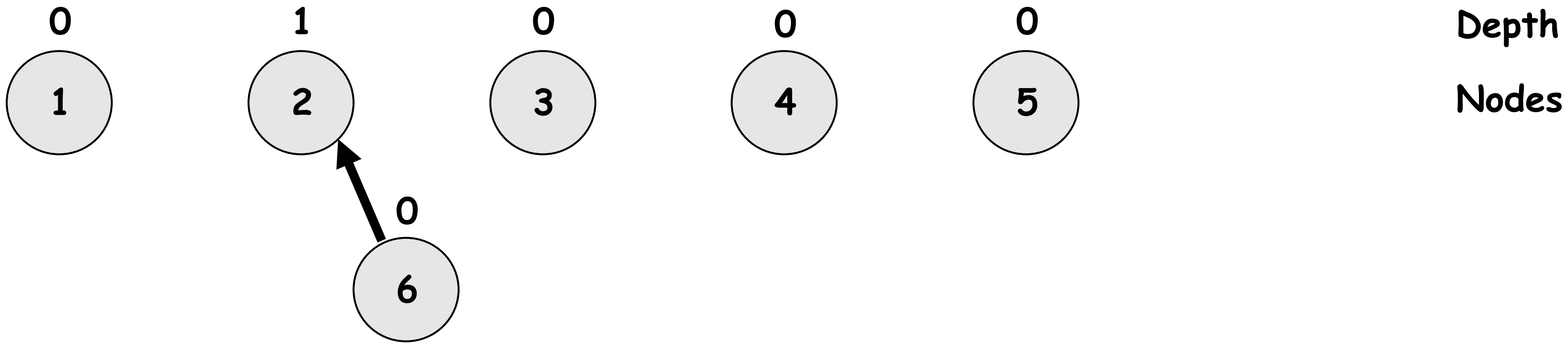
Union Find

- To merge two "islands":
  - First find the root of each tree
  - Assign the **lower-depth** root to point to the **higher-depth** root
    - If roots are the same depth tiebreak arbitrarily
  - Adjust the depths if necessary



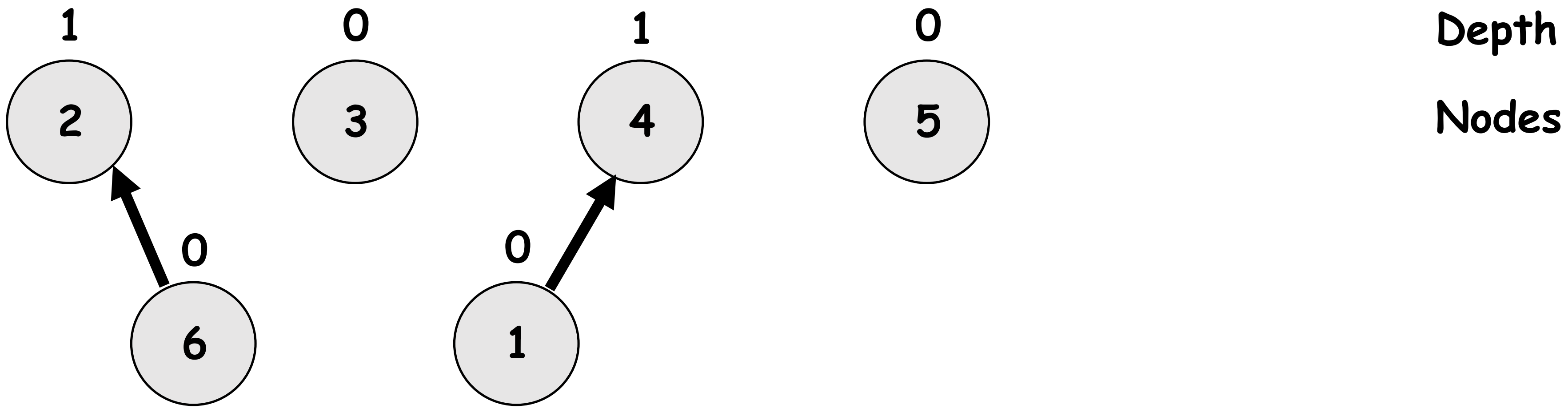
Union Find Example

- Merge(2, 6)



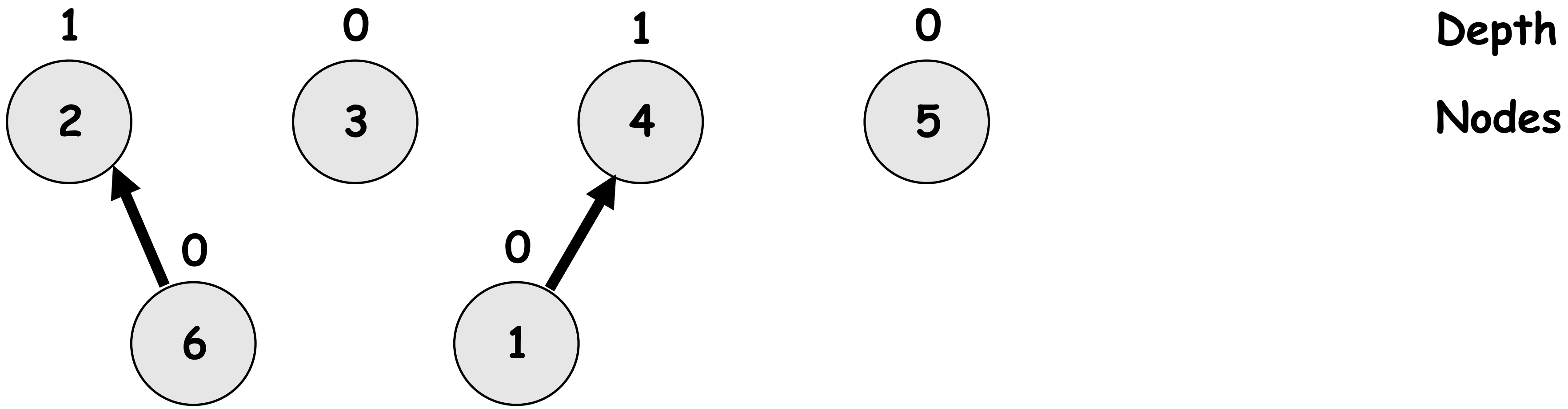
Union Find Example

- Merge(4, 1)



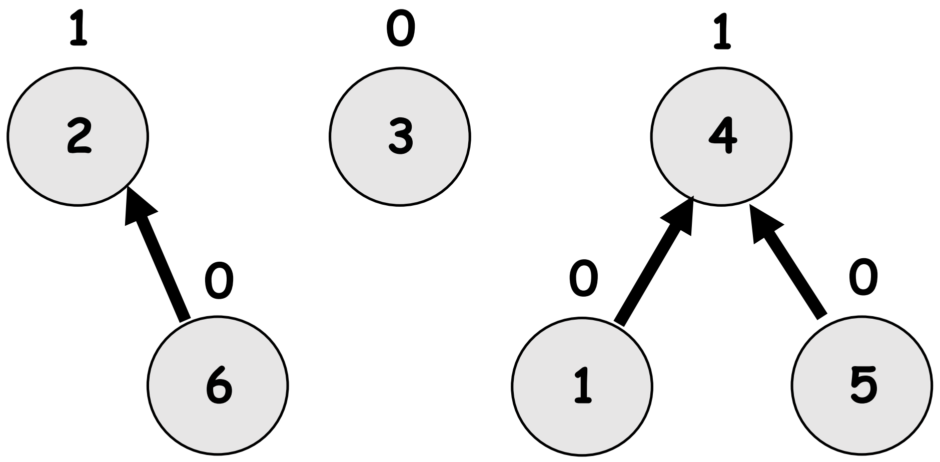
Union Find Example

- CheckSame(1,2)
- CheckSame(6,2)



Union Find Example

- Merge(5, 4)

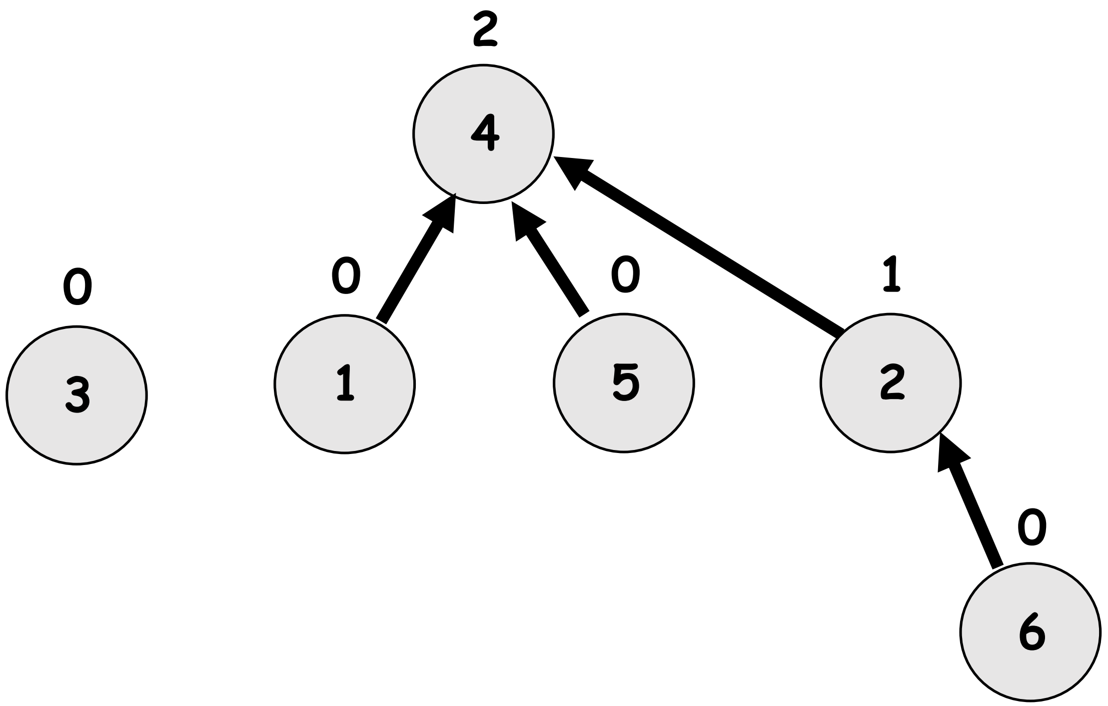


Depth

Nodes

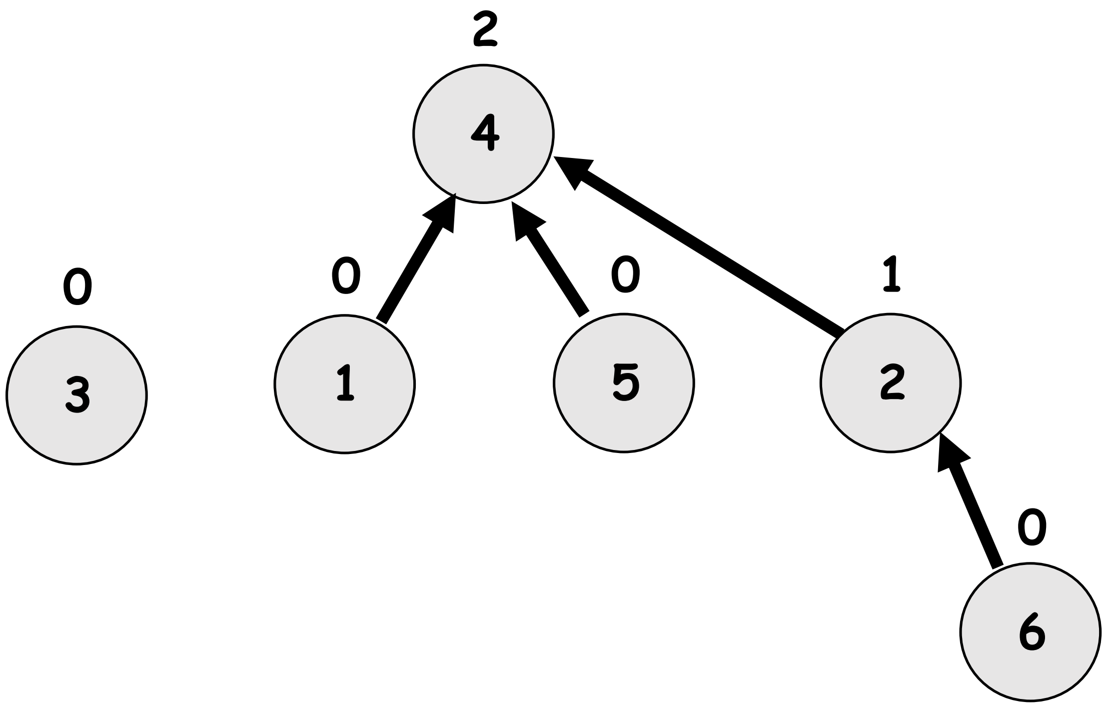
Union Find Example

- Merge(2, 4)



Union Find Example

- CheckSame(5, 1)
- CheckSame(6, 2)

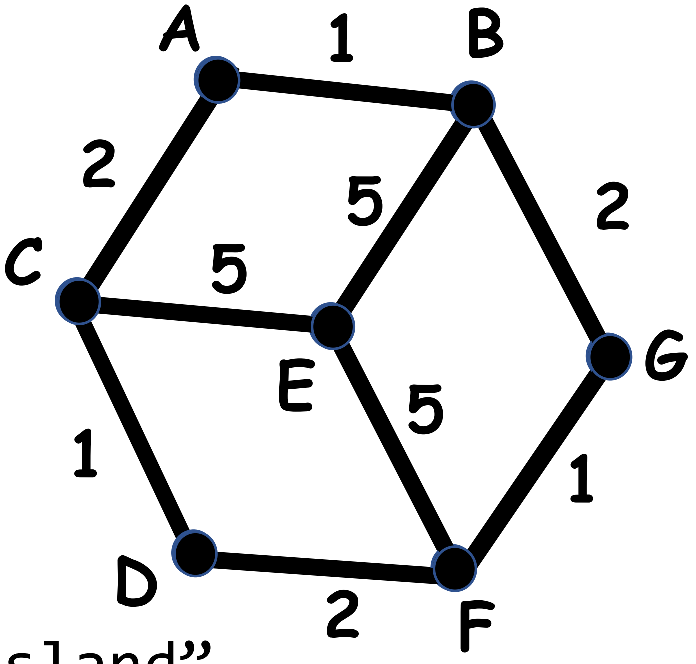


Union Find Runtime Proof:

- **Claim:** If the label of a node is k, then there must be $\geq 2^k$ elements in the tree
  - Equivalently, if there are n nodes in a tree, the depth of the tree is at most log(n)
- **General Proof: (Induction)**
  - <u>Base case:</u> True initially
  - <u>Inductive step:</u> Each step we merge a tree of depth at most log(n)
    - From inductive hypothesis it also must contain at most n elements
    - Depth increases by at most 1, number of elements can double

      -> Inductive hypothesis holds!
- As a consequence, union find is guaranteed to be log(n)
  - Or better! (See Tarjan's 1975 paper for details if you want)

Kruskal's Runtime:




```
        Kruskals(V, E):
O(M log M)      sort E in non-decreasing order
O(N)            Initialize each vertex in its own "island"
O(M log N)      for i = 1 … m:
                     let e_i = (u,v)
     O(log N)        if u and v are in different "islands":
                          add e_i into the MST
     O(log N)             merge the "islands" containing u and v
                return the MST
```

Overall: O(M log M) = O(M log N)

Kruskal's Proof of Correctness:

- Add the lowest-cost edge that doesn't create a cycle

-> Equivalently:

  - If adding e to T creates a cycle, then delete it according to the **cycle property**
  - Otherwise, add it according to the **cut property**

Other MST greedy algorithms:

- Prim's Algorithm: Similar to Dijkstra's Algorithm
  - Start from an arbitrary vertex
  - At each step add the lowest-weight edge coming out of the tree
  - Straightforward application of cut property
- Reverse-Delete:
  - Keep deleting the highest-weight edge unless it disconnects the graph
  - (Somewhat) straightforward application of cycle property