CSE 421

Divide and Conquer

Yin Tat Lee

Announcements

- From Jan 31 (next mon), all lectures and OH will be in person.
- There will be recording (Panopto or zoom).
- If you feel sick, don’t go to the class.
- Midterm is in person.
  - Feb 4 (next Friday)
  - Open book and notes (hard copies only)
  - Coverage: All topics through divide and conquer
- If you cannot attend the midterm, please contact me ASAP.
- HW4 is out!

HW2 Comments

- What are **not** considered as a proof?
  - You just describe what your algorithm does.
  - Bad: We explores edges in alphabetical order, so the output is correct.
  - You can always look at lecture notes to see how things are proved.
  - Techniques:
    - Induction (Key: Come up with a good hypothesis)
    - Contradiction (If the output is wrong, what do we contradicts to?)

- I changed the guideline to
  - Discuss runtime
  - Prove correctness

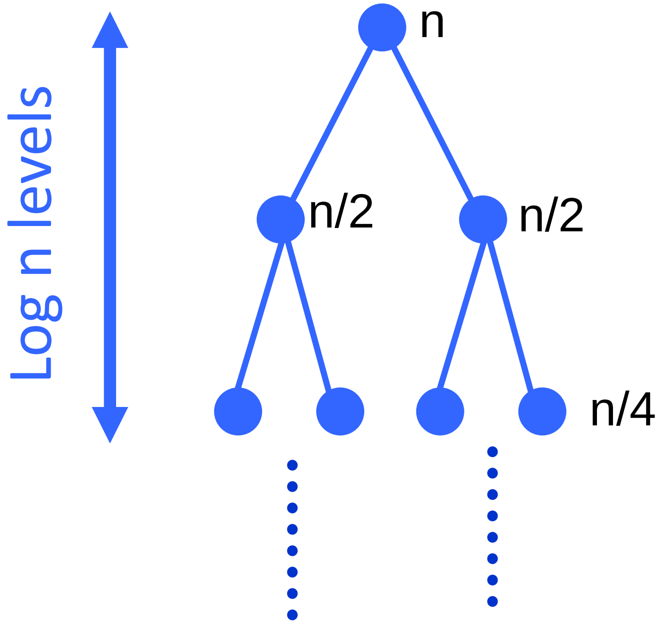
- Still, the most important thing is to come up with a right algorithm. You are doing a great job here.

Divide and Conquer Approach

Divide and Conquer

We reduce a problem to several subproblems.

Typically, each sub-problem is
at most a constant fraction of
the size of the original problem



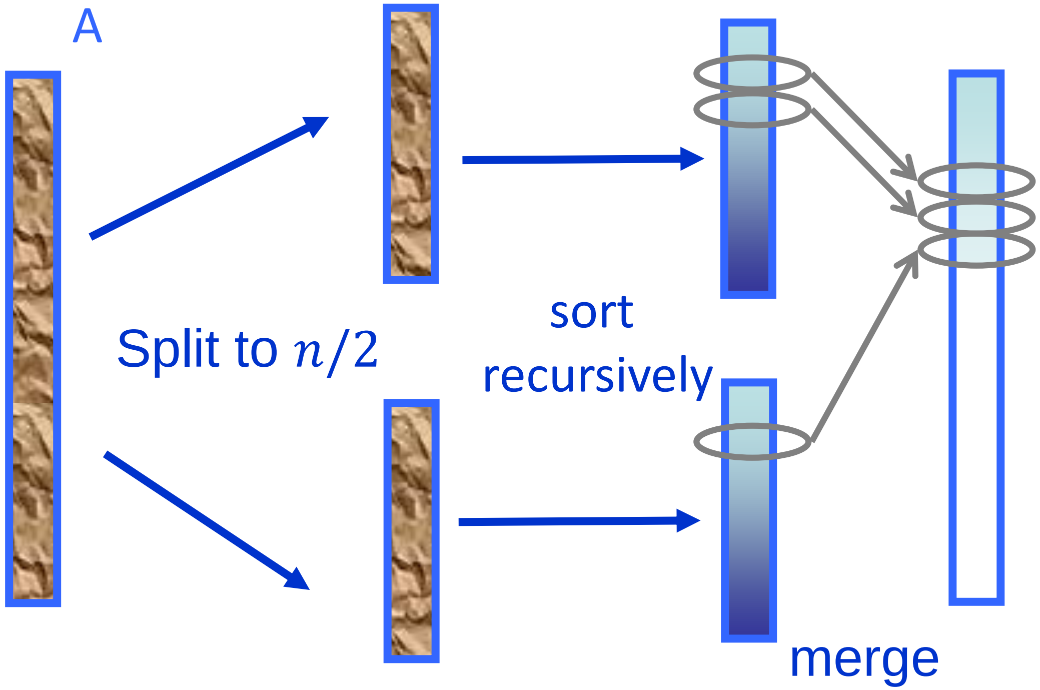
Recursively solve each subproblem

Merge the solutions

Examples:

- Mergesort, Binary Search, Strassen's Algorithm,

A Classical Example: Merge Sort



Why Balanced Partitioning?

An alternative "divide & conquer" algorithm:

- Split into n-1 and 1
- Sort each sub problem
- Merge them

Runtime

$$T(n) = T(n-1) + T(1) + n$$

Solution:

$$\begin{aligned} T(n) &= n + T(n-1) + T(1) \\ &= n + n - 1 + \ T(n-2) \\ &= n + n - 1 + n - 2 + T(n-3) \\ &= n + n - 1 + n - 2 + \cdots + 1 = O(n^2) \end{aligned}$$

Reinventing Mergesort

Suppose we've already invented Bubble-Sort, and we know it takes n^2

Try just one level of divide & conquer:

Bubble-Sort (first n/2 elements)

Bubble-Sort (last n/2 elements)

Merge results

D&C in a nutshell

Time: $2\,T(n/2) + n = n^2/2 + n \ll n^2$

Almost twice as fast!

Why quick sort is better?

Reinventing Mergesort

- “the more dividing and conquering, the better”
  - Two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing.
  - Best is usually full recursion down to a small constant size (balancing "work" vs "overhead").

  In the limit: you’ve just rediscovered mergesort!
- Even unbalanced partitioning is good, but less good
  - Bubble-sort improved with a 0.1/0.9 split:

    $$(.1n)^2 + (.9n)^2 + n = .82n^2 + n$$

    The 18% savings compounds significantly if you carry recursion to more levels, actually giving $O(n \log n)$, but with a bigger constant.
- This is why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.

In C++, stdlib do quick sort for $n > 16$ and insertion sort for $n \leq 16$.

See https://www.youtube.com/watch?v=FJJTYQYB1JQ

Finding the Root of a Function

Finding the Root of a Function

Given a continuous function f and two points $a < b$ such that

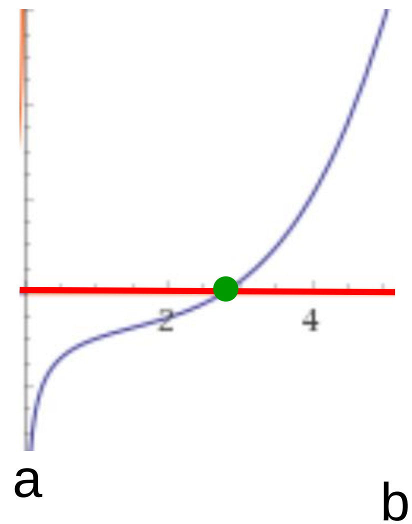
$$f(a) \leq 0$$
$$f(b) \geq 0$$

Goal: Find a point c where $f(c)$ is close to 0.

f has a root in $[a, b]$ by intermediate value theorem

Note that roots of f may be irrational,
So, we want to approximate
the root with an arbitrary precision!

$$f(x) = \sin(x) - \frac{100}{\sqrt{x}} + x^4$$



A Naive Approach

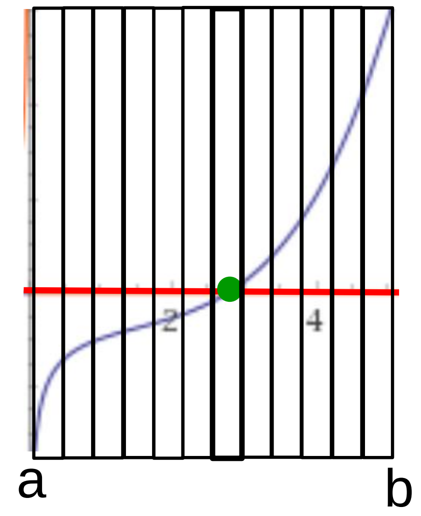
Suppose we want ϵ approximation to a root.

Divide $[a, b]$ into $n = \frac{b-a}{\epsilon}$ intervals. For each interval check

$$f(x) \leq 0, f(x+\epsilon) \geq 0$$

This runs in time $O(n) = O(\frac{b-a}{\epsilon})$

Can we do faster?



Divide & Conquer (Binary Search)

Bisection (a, b, ε)

if $(b - a) < \varepsilon$ then

return a;

else

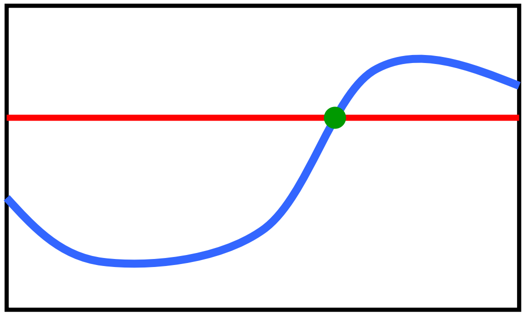
$m \leftarrow (a + b)/2$;

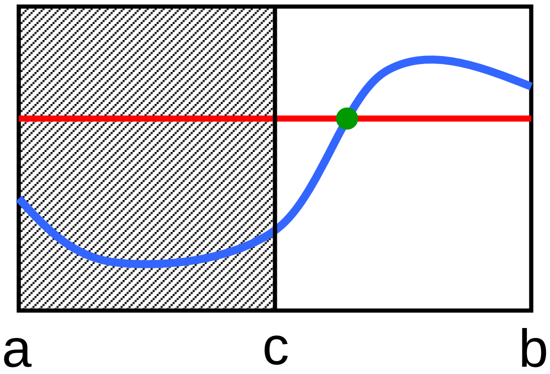
if $f(m) \leq 0$ then

return Bisection(m, b, ε);

else

return Bisection(a, m, ε);

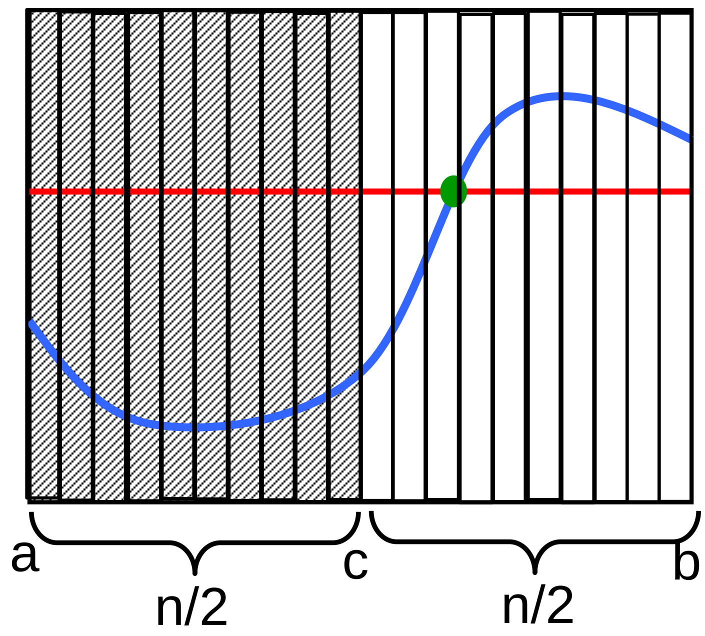




Time Analysis

Let $n = \frac{b-a}{\epsilon}$ be the # of intervals and $c = (a+b)/2$

Always half of the intervals lie to the left and half lie to the right of c



So,

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

i.e., $T(n) = O(\log n) = O(\log(\frac{b-a}{\epsilon}))$

For d dimension ,
"Binary search" can be used to minimize convex functions.
The current best algorithms take $O(d^3 \log^{O(1)}(d/\epsilon))$ time.

Fast Exponentiation

Fast Exponentiation

- Power(a, n)

  Input: integer $\boldsymbol{n} \geq \mathbf{0}$ and number $\boldsymbol{a}$

  Output: $\boldsymbol{a^n}$

- Obvious algorithm

  $\boldsymbol{n} - \mathbf{1}$ multiplications

- Observation:

  if $\boldsymbol{n}$ is even, then $a^n = a^{n/2} \cdot a^{n/2}$.

Divide & Conquer (Repeated Squaring)

```
Power(a,n) {
   if (n = 0)
       return 1
   else if (n is even)
       return Power(a,n/2)• Power(a,n/2)
   else
       return Power(a,(n − 1)/2)• Power(a,(n − 1)/2) • a
}
```

Is there any problem in the program?

$k = \text{Power}(a, n/2)$; return $k \bullet k$;

$k = \text{Power}(a, (n-1)/2)$; return $k \bullet k \bullet a$;

Time (# of multiplications):

$T(n) \leq T(\lfloor n/2 \rfloor) + 2$ for $n \geq 1$

$T(0) = 0$

Solving it, we have

$$T(n) \leq T(\lfloor n/2 \rfloor) + 2 \leq T(\lfloor n/4 \rfloor) + 2 + 2$$
$$\leq \cdots \leq T(1) + \underbrace{2 + \cdots + 2}_{\log_2(n) \text{ copies}} \leq 2 \log_2 n.$$

Quiz

Problem 4 (20 points).
Given an array of positive numbers $a = [a_1, a_2, \cdots, a_n]$. Give an $O(n \log n)$ time algorithm that find i and j (with $i \leq j$) that maximize the subarray product $\prod_{k=i}^{j} a_k$. Prove the correctness and the runtime of the algorithm.

For example, in the array $a = [3, 0.2, 5, 7, 0.4, 4, 0.01]$, the sub-array from $i = 3$ to $j = 6$ has the product $5 \times 7 \times 0.4 \times 4 = 56$ and no other sub-array contains elements that product to a value greater than 56. So, the answer for this input is $i = 3, j = 6$.

Hints: Divide and Conquer.

Quiz

Algorithm

function $(i, j) = \texttt{MAXSUB}(a_1, a_2, \cdots, a_n)$

- *If* $n = 1$
  - *Output* $i = j = 1$.
- *Else*
  - $(i_1, j_1) = \texttt{MAXSUB}(a_1, \cdots, a_{\lfloor n/2 \rfloor})$.
  - $(i_2, j_2) = \texttt{MAXSUB}(a_{\lfloor n/2 \rfloor + 1}, \cdots, a_n)$.
  - *Find* $i_3 \leq \lfloor n/2 \rfloor$ *that maximize* $\prod_{k=i_3}^{\lfloor n/2 \rfloor} a_k$.
  - *Find* $j_3 > \lfloor n/2 \rfloor$ *that maximize* $\prod_{k=\lfloor n/2 \rfloor + 1}^{j_3} a_k$.
  - *Compare the subarray product for* (i_1, j_1), (i_2, j_2) *and* (i_3, j_3) *and output the one with the largest subarray product.*

Runtime

The runtime satisfies $T(n) = 2T(n/2) + O(n)$. *So, we have* $T(n) = O(n \log n)$.

Quiz

Correctness

Induction statement: "The algorithm is correct for input size $\leq n$*"*

Base case $n = 1$*: The algorithm is correct because* $i = j = 1$ *is the only possible output.*

Inductive step:

Case 1: $j \leq \lfloor n/2 \rfloor$.

The algorithm finds the solution from the first sub-problem (due to the induction hypothesis).

Case 2: $i > \lfloor n/2 \rfloor$.

The algorithm finds the solution from the second sub-problem (due to the induction hypothesis).

Case 3: $i \leq \lfloor n/2 \rfloor$ *and* $j > \lfloor n/2 \rfloor$.

Note that $\prod_{k=i}^{j} a_k = \prod_{k=i}^{\lfloor n/2 \rfloor} a_k \times \prod_{k=\lfloor n/2 \rfloor}^{j} a_k$*. Since* i *and* j *maximize the left hand side,* i *must be the maximizer of* $\prod_{k=i}^{\lfloor n/2 \rfloor} a_k$ *and* j *must be the maximizer of* $\prod_{k=\lfloor n/2 \rfloor}^{j} a_k$.

Therefore, the algorithm correctly finds it in this case.

Master Theorem

Master Theorem

Suppose $T(n) = a\,T\left(\frac{n}{b}\right) + cn^k$ for all $n > b$. Then,

- If $a < b^k$ then $T(n) = \Theta\left(n^k\right)$
- If $a = b^k$ then $T(n) = \Theta\left(n^k \log n\right)$
- If $a > b^k$ then $T(n) = \Theta\left(n^{\log_b a}\right)$

Works even if it is $\left\lceil\frac{n}{b}\right\rceil$ instead of $\frac{n}{b}$.

We also need $a \geq 1, b > 1\,, k \geq 0$.

Master Theorem

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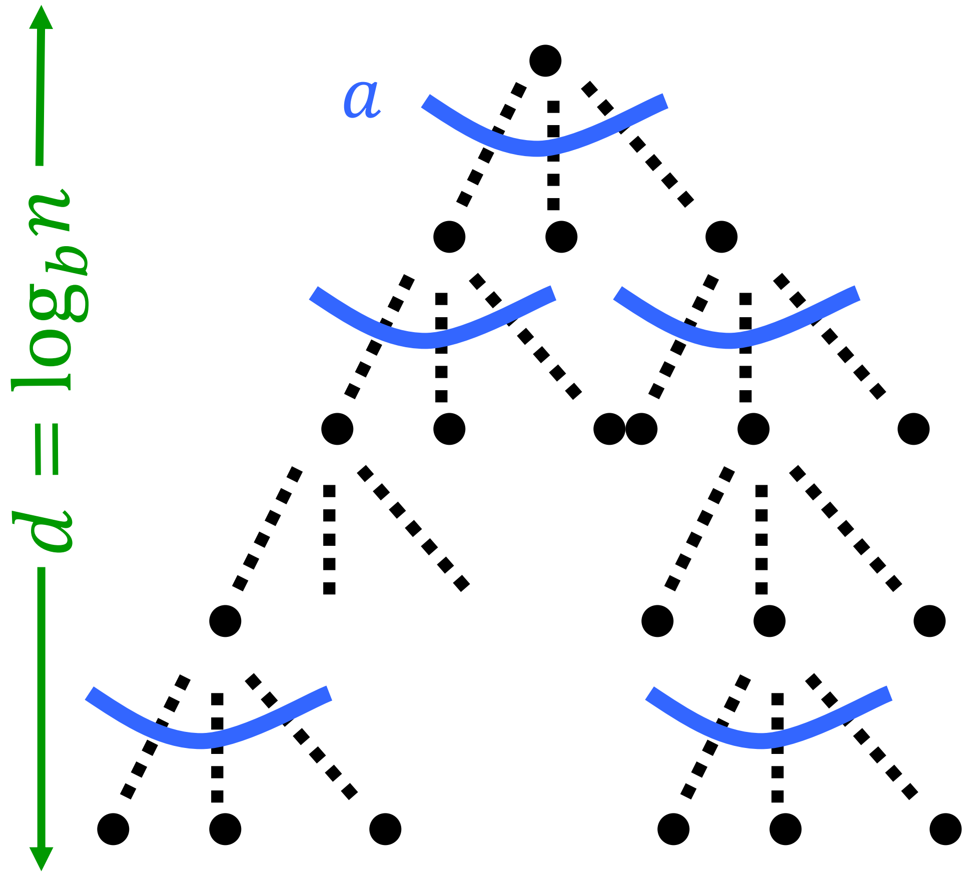
Example: For mergesort algorithm we have

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n).$$

So, $k = 1, a = b^k$ and $T(n) = \Theta(n \log n)$

Proving Master Theorem

$T(n) = aT(n/b) + cn^k$



Problem size	# probs	cost
n	1	cn^k
n/b	a	$a \cdot c(n/b)^k$
n/b^2	a^2	$a^2 \cdot c(n/b^2)^k$
1	a^d	$a^d \cdot c\left(n/b^d\right)^k$

$$T(n) = \sum_{i=0}^{d=\log_b n} a^i c \left(\frac{n}{b^i}\right)^k$$

Master Theorem

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- If $a < b^k$ then $T(n) = \Theta\left(n^k\right)$

# of problems increases slower than the decreases of cost.
First term dominates.

- If $a = b^k$ then $T(n) = \Theta\left(n^k \log n\right)$

- If $a > b^k$ then $T(n) = \Theta\left(n^{\log_b a}\right)$

# of problems increases faster than the decreases of cost
Last term dominates.