CSE 421: Introduction to Algorithms

Greedy Algorithms
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Directed Acyclic Graphs (DAG)

A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

**Def:** A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge $(v_i, v_j)$ we have $i < j$. 

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a DAG → a topological ordering of that DAG—all edges left-to-right
Lemma: If G is a DAG, then G has a topological order

Pf. (by induction on n)
Base case: true if n = 1.

IH: Every DAG with n-1 vertices has a topological ordering.

IS: Given DAG with $n > 1$ nodes, find a source node v. $G - \{v\}$ is a DAG, since deleting v cannot create cycles.

By IH, $G - \{v\}$ has a topological ordering. Place v first in topological ordering; then append nodes of G - {v} in topological order. This is valid since v has no incoming edges.

Reminder: Always remove vertices/edges to use IH
A Characterization of DAGs

G has a topological order \iff G is a DAG
Topological Order Algorithm: Example
Topological Order Algorithm: Example

Topological order: 1, 2, 3, 4, 5, 6, 7
Topological Sorting Algorithm

Maintain the following:

- \( \text{count}[w] = \) (remaining) number of incoming edges to node \( w \)
- \( S = \) set of (remaining) nodes with no incoming edges

Initialization:

- \( \text{count}[w] = 0 \) for all \( w \)
- \( \text{count}[w]++ \) for all edges \((v,w)\) \( \quad \text{O}(m + n) \)
- \( S = S \cup \{w\} \) for all \( w \) with \( \text{count}[w]=0 \)

Main loop:

- while \( S \) not empty
  - remove some \( v \) from \( S \)
  - make \( v \) next in topo order \( \quad \text{O}(1) \) per node
  - for all edges from \( v \) to some \( w \) \( \quad \text{O}(1) \) per edge
    - decrement \( \text{count}[w] \)
    - add \( w \) to \( S \) if \( \text{count}[w] \) hits 0

Correctness: clear, I hope

Time: \( \text{O}(m + n) \) (assuming edge-list representation of graph)
DFS on Directed Graphs

• Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s

• Every cycle contains a back edge in the DFS tree
Summary

• Graphs: abstract relationships among pairs of objects

• Terminology: node/vertex/vertices, edges, paths, multi-edges, self-loops, connected

• Representation: Adjacency list, adjacency matrix

• Nodes vs Edges: \( m = O(n^2) \), often less

• BFS: Layers, queue, shortest paths, all edges go to same or adjacent layer

• DFS: recursion/stack; all edges ancestor/descendant

• Algorithms: Connected Comp, bipartiteness, topological sort
Greedy Algorithms
Greedy Strategy

**Goal**: Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

**Ex**: 34¢.

**Cashier's algorithm**: At each iteration, give the *largest* coin valued ≤ the amount to be paid.

**Ex**: $2.89.
Greedy is not always Optimal

Observation: Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.
Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
Optimal: 70, 70.

Lesson: Greedy is short-sighted. Always chooses the most attractive choice at the moment. But this may lead to a dead-end later.
Greedy Algorithms Outline

Pros
• Intuitive
• Often simple to design (and to implement)
• Often fast

Cons
• Often incorrect!

Proof techniques:
• Stay ahead
• Structural
• Exchange arguments
Interval Scheduling
Interval Scheduling

- Job \( j \) starts at \( s(j) \) and finishes at \( f(j) \).
- Two jobs \textbf{compatible} if they don’t overlap.
- Goal: find maximum subset of mutually compatible jobs.
Greedy Strategy

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

Main question:

• What order?

• Does it give the Optimum answer?

• Why?
Possible Approaches for Inter Sched

Sort the jobs in some order. Go over the jobs and take as much as possible provided it is compatible with the jobs already taken.

[Earliest start time] Consider jobs in ascending order of start time $s_j$.

[Earliest finish time] Consider jobs in ascending order of finish time $f_j$.

[Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.

[Fewest conflicts] For each job, count the number of conflicting jobs $c_j$. Schedule in ascending order of conflicts $c_j$. 
Greedy Alg: Earliest Finish Time

Consider jobs in increasing order of finish time. Take each job provided it’s compatible with the ones already taken.

\[
\text{Sort jobs by finish times so that } f(1) \leq f(2) \leq \ldots \leq f(n). \\
A \leftarrow \emptyset \\
\text{for } j = 1 \text{ to } n \{ \\
\quad \text{if (job } j \text{ compatible with } A) \\
\quad \quad A \leftarrow A \cup \{j\} \\
\} \\
\text{return } A
\]

**Implementation.** $O(n \log n)$.
- Remember job $j^*$ that was added last to $A$.
- Job $j$ is compatible with $A$ if $s(j) \geq f(j^*)$.*
Greedy Alg: Example
Correctness

**Theorem:** Greedy algorithm is optimal.

**Pf:** (technique: “Greedy stays ahead”)

Let $i_1, i_2, \ldots, i_k$ be jobs picked by greedy, $j_1, j_2, \ldots, j_m$ those in some optimal solution in order.

We show $f(i_r) \leq f(j_r)$ for all $r$, by induction on $r$.

**Base Case:** $i_1$ chosen to have min finish time, so $f(i_1) \leq f(j_1)$.

**IH:** $f(i_r) \leq f(j_r)$ for some $r$

**IS:** Since $f(i_r) \leq f(j_r) \leq s(j_{r+1})$, $j_{r+1}$ is among the candidates considered by greedy when it picked $i_{r+1}$, & it picks min finish, so $f(i_{r+1}) \leq f(j_{r+1})$

Observe that we must have $k \geq m$, else $j_{k+1}$ is among (nonempty) set of candidates for $i_{k+1}$
Interval Partitioning Technique: Structural
Interval Partitioning

Lecture $j$ starts at $s(j)$ and finishes at $f(j)$.

**Goal:** find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.
Interval Partitioning

Note: graph coloring is very hard in general, but graphs corresponding to interval intersections are simpler.
A Better Schedule

This one uses only 3 classrooms
A Structural Lower-Bound on OPT

Def. The depth of a set of open intervals is the maximum number that contain any given time.
A Structural Lower-Bound on OPT

**Def.** The depth of a set of open intervals is the maximum number that contain any given time.

**Key observation.** Number of classrooms needed \( \geq \) depth.

**Ex:** Depth of schedule below = 3 \( \Rightarrow \) schedule below is optimal.

**Q.** Does there always exist a schedule equal to depth of intervals?
A Greedy Algorithm

**Greedy algorithm**: Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

Sort intervals by starting time so that \( s_1 \leq s_2 \leq \ldots \leq s_n \).

\[
d \leftarrow 0
\]

\[
\text{for } j = 1 \text{ to } n \{ \\
\quad \text{if (lect } j \text{ is compatible with some classroom } k, 1 \leq k \leq d) \\
\quad \quad \text{schedule lecture } j \text{ in classroom } k \\
\quad \text{else} \\
\quad \quad \text{allocate a new classroom } d + 1 \\
\quad \quad \text{schedule lecture } j \text{ in classroom } d + 1 \\
\quad d \leftarrow d + 1
\}
\]

**Implementation**: Exercise!
Correctness

**Observation:** Greedy algorithm never schedules two incompatible lectures in the same classroom.

**Theorem:** Greedy algorithm is optimal.

*Pf (exploit structural property).*

Let \( d \) = number of classrooms that the greedy algorithm allocates. Classroom \( d \) is opened because we needed to schedule a job, say \( j \), that is incompatible with all \( d-1 \) previously used classrooms. Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than \( s(j) \).

Thus, we have \( d \) lectures overlapping at time \( s(j) + \epsilon \), i.e. \( \text{depth} \geq d \)

“OPT Observation” \( \Rightarrow \) all schedules use \( \geq \text{depth} \) classrooms, so \( d = \text{depth} \) and greedy is optimal \( \blacksquare \).