CSE 421

LP Duality

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Linear Programming and Approximation Algorithms

Integer Program for Vertex Cover

Given a graph G=(V,E) with costs c_v on the vertices. Find a vertex cover of G with minimum cost, i.e., $\min \sum_{v \in S} c_v$

Write LP with Integrality Constraint:

- Variables: One variable x_v for each vertex v
- Bound: $x_v \in \{0,1\}$
- Edge cover Constraints: $x_u + x_v \geq 1$ for every edge $(u, v) \in E$
- Obj: $\min \sum_v c_v x_v$

IP for Vertex Cover

$$\min \quad \sum_v c_v x_v$$

$$s.t., \quad x_v + x_u \geq 1 \quad \forall (u,v) \in E$$

$$x_v \in \{0,1\} \quad \forall v \in V$$

IP is NP-complete general!
But there are fast algorithms in practice that often work

Fact:
Pf:

- For min vertex cover S, $x_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{o.w.} \end{cases}$ is feasible, so

OPT-IP ≤ Min Vertex Cover

- For optimum solution x, the $S = \{v: x_v = 1\}$ is a vertex cover

Min Vertex Cover ≤ OPT-IP

LP Relaxation Vertex Cover

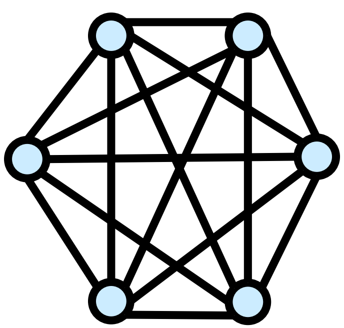
$$\begin{aligned} \min \quad & \sum_v c_v x_v \\ s.t., \quad & x_v + x_u \geq 1 \quad \forall (u,v) \in E \\ & 0 \leq x_v \leq 1 \quad \forall v \in V \end{aligned}$$

Fact: OPT-LP ≤ Min Vertex Cover
Pf: Min vertex cover is a feasible solution of the LP

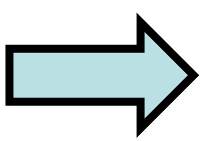
Q: Can we hope to get an integer solution?

Bad Optimum solutions

$$\begin{aligned} &min && \sum_v c_v x_v \\ &s.t., && x_v + x_u \geq 1 \quad \forall (u,v) \in E \\ & && 0 \leq x_v \leq 1 \quad \forall v \in V \end{aligned}$$



K_n complete graph



A feasible solution:
Set $x_v = 0.5$ for all v
in the complete graph

If $c_v = 1$ for all v, then
Min vertex cover=$n - 1$
But OPT LP=n/2.

Approximation Alg for Vertex Cover

Given a graph G=(V,E) with costs c_v on the edges. Find a vertex cover of G with minimum cost, i.e., $\min \sum_{v \in S} c_v$

Thm: There is a 2-approximation Alg for weighted vertex cover.

ALG: Solve LP. Let $S = \{v: x_v \geq 0.5\}$. Output S.

Pf: First, for every edge (u, v), $x_u + x_v \geq 1$ So at least one is in S. So, S is a vertex cover.

Second,

$$\sum_{v \in S} c_v \leq \sum_{v \in S} c_v (2x_v) \leq 2\text{OPTLP} \leq 2\text{Min Vertex Cov}$$

Intro to Duality

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.,} \quad & x_1 + 3x_2 \leq 2 \\ & 2x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimum solution: $x_1 = 5/4$ and $x_2 = 1/4$ with value $x_1 + 2x_2 = 7/4$
How can you prove an upper-bound on the optimum?

First attempt: Since $x_1, x_2 \geq 0$

$$x_1 + 2y_2 \leq x_1 + 3x_2 \leq 2$$

Second attempt:

$$x_1 + 2x_2 \leq \frac{2}{3}(x_1 + 3x_2) + \frac{1}{3}(2x_1 + 2x_2) \leq \frac{2}{3}(2) + \frac{1}{3}(3) = \frac{7}{3}$$

Third attempt:

$$x_1 + 2x_2 \leq \frac{1}{2}(x_1 + 3x_2) + \frac{1}{4}(2x_1 + 2x_2) \leq \frac{1}{2}(2) + \frac{1}{4}(3) = \frac{7}{4}$$

Dual Certificate

$$
\begin{aligned}
max \quad & x_1 + 2x_2 \\
s.t., \quad & x_1 + 3x_2 \leq 2 & \quad y_1 \\
& 2x_1 + 2x_2 \leq 3 & \quad y_2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Goal: Minimize $2y_1 + 3y_2$

But, we must make sure the sum of the LHS is at most objective, i.e.,

$$x_1 + 2x_2 \leq y_1(x_1 + 3x_2) + y_2(2x_1 + 2x_2)$$

In other words,

$$1 \leq 1 \cdot y_1 + 2 \cdot y_2$$
$$2 \leq 3 \cdot y_1 + 2 \cdot y_2 \geq$$

Finally, $y_1, y_2 \geq 0$ (else the direction of inequalities change)

Dual Program

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.,} \quad & x_1 + 3x_2 \leq 2 \\ & 2x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

OPT: $x_1 = 5/4$ and $x_2 = 1/4$
Value $7/4$

$$\begin{aligned} \min \quad & 2y_1 + 3y_2 \\ \text{s.t.,} \quad & y_1 + 2y_2 \geq 1 \\ & 3y_1 + 2y_2 \geq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

OPT: $y_1 = 1/2$ and $y_2 = 1/4$
Value $7/4$

Dual of Standard LP

$$
\begin{aligned}
\max \quad & \langle c, x\rangle \\
\text{s.t.,} \quad & \langle a_1, x\rangle \le b_1 \qquad y_1 \\
& \langle a_2, x\rangle \le b_2 \qquad y_2 \\
& \vdots \\
& \langle a_m, x\rangle \le b_m \qquad y_m \\
& x_1, \dots, x_n \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \langle b, y\rangle \\
\text{s.t.,} \quad & a_{1,1}y_1 + \cdots + a_{m,1}y_m \ge c_1 \\
& a_{1,2}y_1 + \cdots + a_{m,2}y_m \ge c_2 \\
& \vdots \\
& a_{1,n}y_1 + \cdots + a_{m,n}y_m \ge c_n \\
& y_1, \dots, y_m \ge 0
\end{aligned}
$$

$$
\begin{aligned}
max \quad & \langle c, x\rangle \\
\text{s.t.,} \quad & Ax \le b \\
& x \ge 0
\end{aligned}
$$

Primal

$$
\begin{aligned}
min \quad & \langle b, y\rangle \\
\text{s.t.,} \quad & A^T y \ge b \\
& y \ge 0
\end{aligned}
$$

Dual

Facts About Linear Programs

Lem: Dual of Dual = Primal

Thm (weak duality): Every solution to the primal is at most every solution to the dual

$$\langle c, x \rangle \leq \langle b, y \rangle$$

Thm (strong duality): If primal has a solution and dual has a solution then optimum of primal is equal to optimum of dual

Dual of Max-Flow

$$\max \quad \sum_{e \text{ out of } s} x_e$$

$$s.t. \quad \sum_{e\ out\ of\ v} x_e = \sum_{e\ in\ to\ v} x_e \qquad \forall v \neq s,t \qquad \color{red}{b_v}$$

$$x_e \leq c(e) \qquad \forall e \qquad \color{red}{a_e}$$

$$x_e \geq 0 \qquad \forall e$$

$$\min \quad \langle c, a \rangle$$

$$\text{s.t.}, \quad a_e + b_v \geq 1 \qquad e = (s,v)$$

$$a_e - b_v \geq 0 \qquad e = (v,t)$$

$$a_e + b_u - b_v \geq 0 \qquad \text{other } e = (u,v)$$

$$a_e \geq 0 \qquad \forall e$$

$$
\begin{array}{lll}
\min & \langle c, a \rangle & \\
\text{s.t.,} & a_e + b_v \geq 1 & e = (s, v) \\
 & a_e - b_v \geq 0 & e = (v, t) \\
 & a_e + b_u - b_v \geq 0 & \text{other } e = (u, v) \\
 & a_e \geq 0 & \forall e
\end{array}
$$

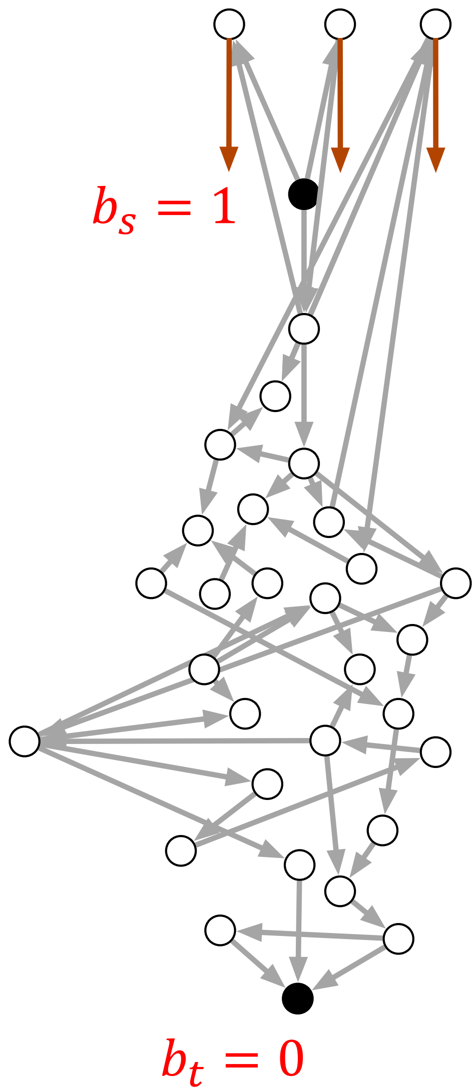
⇩

$$
\begin{array}{lll}
\min & \langle c, a \rangle & \\
\text{s.t.,} & \color{red}{a_e = \max(0, 1 - b_v)} & \color{red}{e = (s, v)} \\
 & a_e = \max(0, b_v) & e = (v, t) \\
 & a_e = \max(0, b_v - b_u) & \text{other } e = (u, v)
\end{array}
$$

$$
\begin{aligned}
\min \quad & \langle c, a \rangle & \\
\text{s.t.}, \quad & a_e = \max(0, 1 - b_v) & e = (s, v) \\
& a_e = \max(0, b_v) & e = (v, t) \\
& a_e = \max(0, b_v - b_u) & \text{other } e = (u, v)
\end{aligned}
$$

Lem: In OPT $0 \leq b_v \leq 1$ for all v

Pf: If not, move up/down the value only decreases

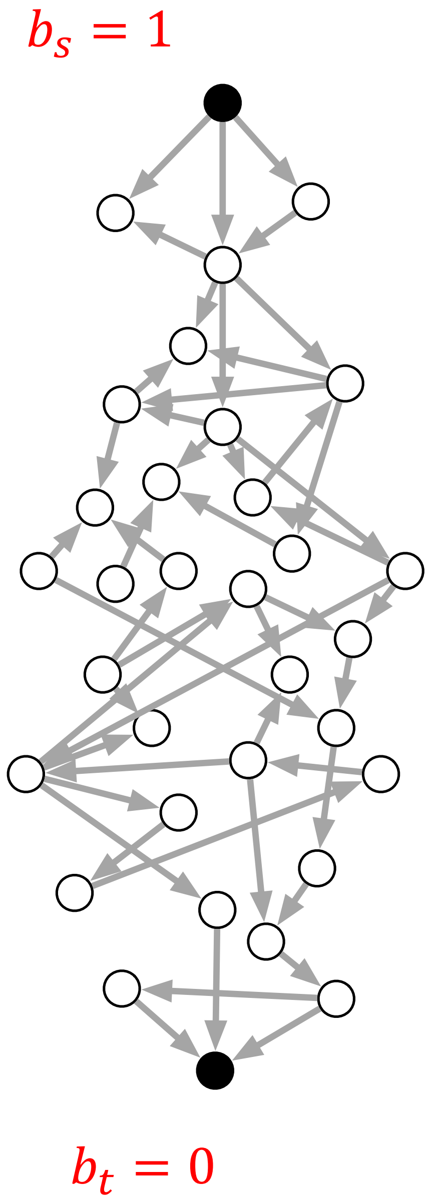


$$\begin{aligned} \min \quad & \langle c, a \rangle \\ \text{s.t.,} \quad & b_s = 1, b_t = 0 \\ & 0 \le b_v \le 1 \\ & a_e = \max(0, b_v - b_u) \quad e = (u, v) \end{aligned}$$

Lem: In OPT $0 \le b_v \le 1$ for all v

Pf: If not, move up/down the value only decreases

Lem: In OPT $b_v \in \{0,1\}$ for all v
Pf: If not, choose a u.r. $0 \le t \le 1$
If $b_v \ge t$ set $b_v = 1$ else set $b_v = 0$.
Then, the expected value of resulting solution sames as OPT.



$$\min \quad \langle c, a \rangle$$
$$\text{s.t.,} \quad b_s = 1, b_t = 0$$
$$0 \le b_v \le 1$$
$$a_e = \max(0, b_v - b_u) \quad \text{other } e = (u, v)$$

Lem: In OPT $0 \le b_v \le 1$ for all v

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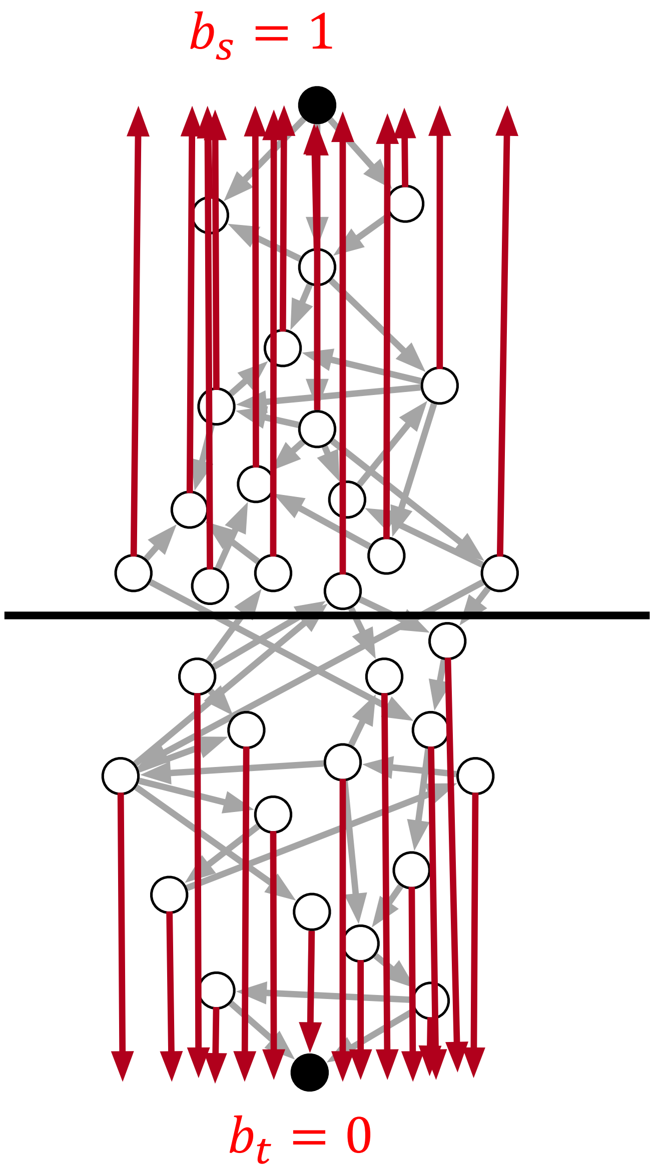


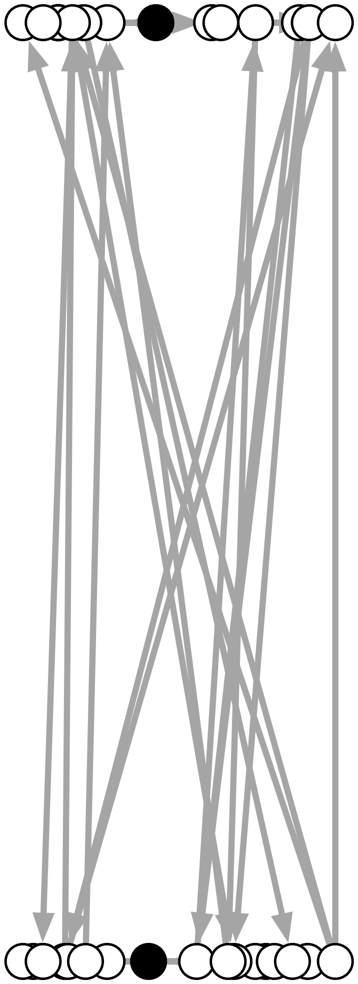
$$\min \quad \langle c, a \rangle$$

$$\text{s.t.,} \quad b_s = 1, b_t = 0$$

$$b_v \in \{0,1\}$$

$$a_e = \max(0, b_v - b_u) \quad \text{other } e = (u, v)$$

Min Cut!



Beyond LP: Convex Programming

A function $f: \mathbb{R} \to \mathbb{R}$ is convex if $f'' \geq 0$.

e.g., $f(x) = x^2$.

A function $f: \mathbb{R}^d \to \mathbb{R}$ is convex if $\nabla^2 f \succcurlyeq 0$

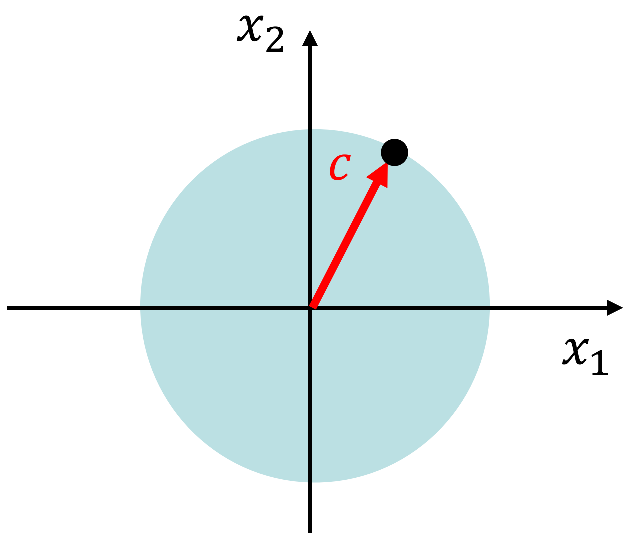
Convex Program

$$
\begin{array}{ll}
\min & f(x) \\
\text{s.t.}, & g_1(x) \leq b_1 \\
 & g_2(x) \leq b_2 \\
 & \vdots \\
 & g_m(x) \leq b_m
\end{array}
$$

f and $g_1, \dots, g_m$ must be convex.
$\geq$ and $=$ are not allows!

Example

$$\begin{aligned} \max \quad & c_1 x_1 + c_2 x_2 \\ \text{s.t.,} \quad & x_1^2 + x_2^2 \leq 1 \end{aligned}$$



Summary (Linear Programming)

- Linear programming is one of the biggest advances in 20th century
- It is being used in many areas of science: Mechanics, Physics, Operations Research, and in CS: AI, Machine Learning, Theory, …
- Almost all problems that we talked can be solved with LPs, Why not use LPs?
  - Combinatorial algorithms are typically faster
  - They exhibit a better understanding of worst case instances of a problem
  - They give certain structural properties, e.g., Integrality of Max-flow when capacities are integral
- There is rich theory of LP-duality which generalizes max-flow min-cut theorem