CSE421: Design and Analysis of Algorithms April 6, 2022

Homework 2

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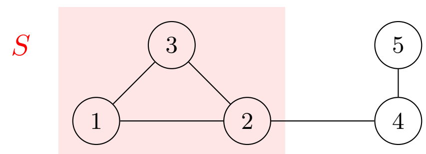
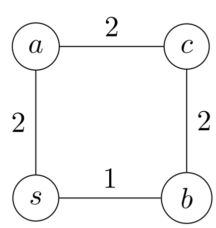
P1) (20 points) Let $G = (V, E)$ a be an undirected graph. For a set $S \subset V$, write $E(S, \overline{S})$ to denote the set of edges with exactly one endpoint in S and one endpoint in $\overline{S}$. Suppose that $|E(S, \overline{S})|$ is odd for some $S \subset V$. Prove that G has an odd degree vertex. For example, in the above picture, $S = \{1, 2, 3\}, \overline{S} = \{4, 5\}$ and $E(S, \overline{S})$ has only the edge $(2, 4)$ so it is odd. The graph has an odd degree vertex, e.g., vertex 2.



P2) (20 points) Let G be a tree. Use induction to prove that we can color vertices of G with 2 colors such that the endpoints of every edge have opposite colors. For example, in the following picture we give a coloring of the given tree with black and white.

P3) (20 points) Given a connected undirected graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges, design an $O(m + n)$-time algorithm to find a vertex in G whose removal does not disconnect G. Note that as a consequence this algorithm shows that every connected graph contains such a vertex.



P4) (20 points) Given a weighted connected graph $G = (V, E)$ with $m = |E|$ edges and $n = |V|$ vertices where every edge $e \in E$ has a weight $w_e \in \{1, 2\}$ and a vertex $s \in V$. For a path $s = v_0, v_1, \ldots, v_k = t$ from s to a vertex t define its weight to be $w_{(v_0,v_1)} \cdot w_{(v_1,v_2)} \ldots w_{(v_{k-1},v_k)}$, i.e., instead of taking the sum of weights we take their product. Design an $O(m + n)$ time algorithm to output the length of the shortest path from s to all vertices of G. You will receive partial credit if your algorithm runs in polynomial time (as opposed to $O(m+n)$). For example, in the above picture the length of the shortest path from s to b is 1 and the length of the shortest paths from s to each of a and c is 2.

P5) **Extra Credit:** Prove that we can color the edges of every graph G with two colors (red and blue) such that, for every vertex v, the number of red edges touching v and the number of blue edges touching v differ by at most 2.