P1) (20 points) Let $G = (V, E)$ be an undirected graph. For a set $S \subseteq V$, write $E(S, \overline{S})$ to denote the set of edges with exactly one endpoint in $S$ and one endpoint in $\overline{S}$. Suppose that $|E(S, \overline{S})|$ is odd for some $S \subseteq V$. Prove that $G$ has an odd degree vertex. For example, in the above picture, $S = \{1, 2, 3\}$, $\overline{S} = \{4, 5\}$ and $E(S, \overline{S})$ has only the edge $(2, 4)$ so it is odd. The graph has an odd degree vertex, e.g., vertex $2$. 

P2) (20 points) Let $G$ be a tree. Use induction to prove that we can color vertices of $G$ with 2 colors such that the endpoints of every edge have opposite colors. For example, in the following picture we give a coloring of the given tree with black and white.

P3) (20 points) Given a connected undirected graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges, design an $O(m + n)$-time algorithm to find a vertex in $G$ whose removal does not disconnect $G$. Note that as a consequence this algorithm shows that every connected graph contains such a vertex.

P4) (20 points) Given a weighted connected graph $G = (V, E)$ with $m = |E|$ edges and $n = |V|$ vertices where every edge $e \in E$ has a weight $w_e \in \{1, 2\}$ and a vertex $s \in V$. For a path $s = v_0, v_1, \ldots, v_k = t$ from $s$ to a vertex $t$ define its weight to be $w_{(v_0,v_1)} \cdot w_{(v_1,v_2)} \cdots w_{(v_{k-1},v_k)}$, i.e., instead of taking the sum of weights we take their product. Design an $O(m + n)$ time algorithm to output the length of the shortest path from $s$ to all vertices of $G$. You will receive partial credit if your algorithm runs in polynomial time (as opposed to $O(m + n)$). For example, in the above picture the length of the shortest path from $s$ to $b$ is 1 and the length of the shortest paths from $s$ to each of $a$ and $c$ is 2.
P5) **Extra Credit:** Prove that we can color the edges of every graph $G$ with two colors (red and blue) such that, for every vertex $v$, the number of red edges touching $v$ and the number of blue edges touching $v$ differ by at most 2.