CSE421: Design and Analysis of Algorithms March 29th, 2022

Homework 1

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Please see https://courses.cs.washington.edu/courses/cse421/22sp/grading.html for general guidelines about Homework problems.

Most of the problems only require one or two key ideas for their solution. It will help you a lot to spell out these main ideas so that you can get most of the credit for a problem even if you err on the finer details. Please justify all answers.

P1) Construct an instance of the stable matching problem with 4 companies $c_1, \dots, c_4$ and 4 applicants $a_1, \dots, a_4$ with two stable matchings M_1, M_2 such that

- c_1 prefers its partner in M_1 to its partner in M_2.
- c_2 prefers its partner in M_2 to its partner in M_1.

Note that c_1 (and similarly c_2) must have different partners in M_1, M_2. You don't need to write a proof for this part. Just writing the instance and stable matchings M_1, M_2 suffices.

P2) Prove that in the output of the Gale-Shapley algorithm (when applicants propose) at most one applicant gets its last choice.

P3) a) Suppose we have $n \geq 1$ positive real numbers $x_1, \dots, x_n$ such that $x_1 x_2 \dots x_n = 1$. Show that for some i we must have $x_i \leq 1$ and for some j we must have $x_j \geq 1$.

b) Use induction to prove that for any n positive real numbers $x_1, \dots, x_n$ such that $x_1 \dots x_n = 1$ we have

$$(1 + x_1) \dots (1 + x_n) \geq 2^n.$$

P4) Arrange the following in increasing order of asymptotic growth rate. For full credit it is enough to just give the order. All logs are in base 2.

(a) $f_1(n) = \frac{n}{2^{\sqrt{\log \log n}}}$

(b) $f_2(n) = \frac{n (\log \log n)^{99}}{(\log n)^{101}}$

(c) $f_3(n) = n!^2$

(d) $f_4(n) = \left(3^2\right)^{\log n}$

(e) $f_5(n) = n^{n \log^2 n}$

(f) $f_6(n) = 2^{2\sqrt{\log n}}$

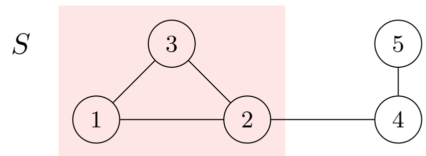
(g) $f_7(n) = 2^{\log^{1/3} n}$

(h) $f_8(n) = 2^{\frac{\log n}{\log \log n}}$

(i) $f_9(n) = 2^{\left(4^{\log n}\right)}$

(j) $f_{10}(n) = 2^{\log n + \log \log n}$

P5) Let $G = (V, E)$ a be an undirected graph. For a set $S \subset V$, write $E(S, \overline{S})$ to denote the set of edges with exactly one endpoint in S and one endpoint in $\overline{S}$. Suppose that $|E(S, \overline{S})|$ is odd for some $S \subset V$. Prove that G has an odd degree vertex. For example, in the above picture, $S = \{1, 2, 3\}, \overline{S} = \{4, 5\}$ and $E(S, \overline{S})$ has only the edge $(2, 4)$ so it is odd. The graph has an odd degree vertex, e.g., vertex 2.



P6) **Extra Credit:** Design a polynomial time algorithm that outputs a 100 stable matchings of a given instance with n companies and n applicants. If the instance does not have 100 stable matchings output "Impossible".