CSE 421 Section 2

Graph Search

Administrivia

Announcements & Reminders

- HW1
  - Was due yesterday, 10/5
  - Remember, this quarter we have a LATE PROBLEMS policy, instead of a late assignments policy
    - Can turn in up to **5 problems late**
    - Can turn in a late problem up to **48 hours late**

- HW2
  - Due Wednesday 10/12 @ 10pm

Big-O

Big-O Review

- Big-O lets us analyze the runtime of algorithms as a function of the size of the input, usually denoted as n

- Super important for understanding algorithms and comparing them!
  (so, you will be doing this analysis for every algorithm you write)

- Given two functions f and g:
  - $f(n)$ is $\boldsymbol{O(g(n))}$ iff there is a constant $c > 0$ so that $f(n)$ is eventually always $< c \cdot g(n)$
  - $f(n)$ is $\boldsymbol{\Omega(g(n))}$ iff there is a constant $c > 0$ so that $f(n)$ is eventually always $> c \cdot g(n)$
  - $f(n)$ is $\boldsymbol{\Theta(g(n))}$ iff there are constants $c_1, c_2 > 0$ so that eventually always $c_1 \cdot g(n) < f(n) < c_2 \cdot g(n)$

$\boldsymbol{O(g(n))}$ is fancy $\leq$

$\boldsymbol{\Omega(g(n))}$ is fancy $\geq$

$\boldsymbol{\Theta(g(n))}$ is fancy $\approx$

Big-O Tips for Comparing

- We're looking for asymptotic comparison, so just testing values won't necessarily give you a good idea
  - **Exponentials:** 2^n and 3^n are different, which means 2^n and $2^{n/2}$ are different!
    [constant factors IN EXPONENTS are not constant factors]
  - **Exponentials vs Polynomials:** for all $r > 1$ and all $d > 0, n^d = O(r^n)$
    [in other words, every exponential grows faster than every polynomial]
  - **Logs vs Polynomials:** $\log^a(n)$ is asymptotically less than n^b for any positive constants a, b
- Key strategy: rewriting functions as $2^{f(n)}$ or $\log(f(n))$ will often make it easier to find the correct order for functions

Problem 1 – Big-O-No

Put these functions in increasing order. That is, if f comes before g in the list, it must be the case that f(n) is O(g(n)). Additionally, if there are any pairs such that f(n) is Θ(g(n)), mark those pairs.

- $2^{\log(n)}$
- $2^{n\log(n)}$
- $\log(\log(n))$
- $2^{\sqrt{n}}$
- $3^{\sqrt{n}}$
- $\log(n)$
- $\log(n^2)$
- $\sqrt{n}$
- $(\log(n))^2$

Hint: A useful trick in these problems is to know that since $\log(\cdot)$ is an increasing function, if $f(n)$ is $O(g(n))$, then $\log(f(n))$ is $O(\log(g(n))$. But be careful! Since $\log(\cdot)$ makes functions much smaller it can obscure differences between functions. For example, even though n^3 is less than n^4 , $\log(n^3)$ and $\log(n^4)$ are big-Θ of each other.

Work on ordering the functions in this problem with the people around you, and then we'll go over it together!

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a) $\log(\log(n))$
b) $\log(n)$
c) $\log(n^2)$
this function is Θ(log(n)); they differ only by the constant factor 2

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note: this is just n

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e) $\sqrt{n}$
f) $2^{\log(n)}$
note: this is just n
g) $2^{\sqrt{n}}$
h) $3^{\sqrt{n}}$
i) $2^{n \log(n)}$

Depth First Search

DFS

- Very similar to how we completed the BFS algorithm, but here we trade the queue for the **call stack**!

- You may have learned an alternate version of this this concept originally, but the way we will do it in this class is with **recursion**

    - There are some differences because of this, so pay attention to the pseudocode!

DFS Algorithm – simple version

```
DFS(u)
      Mark u as “seen”
      For each edge (u,v) //leaving u
            If v is not “seen”
                  DFS(v)
            End If
      End For
```

DFS Algorithm – counter version with wrapper

```
DFS(u)
        Mark u as “seen”
        u.start = counter++
        For each edge (u,v) //leaving u
                If v is not “seen”
                        DFS(v)
                End If
        End For
        u.end = counter++
```

```
DFSWrapper(G)
        counter = 1
        For each vertex u of G
                If u is not “seen”
                        DFS(u)
                End If
        End For
```

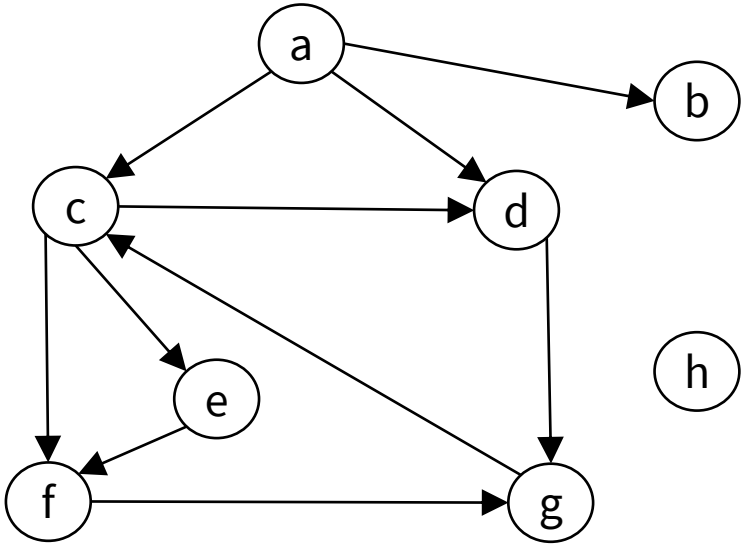
Tip: the counter just tracks when each node is first put on the stack, and when each node is removed

DFS and Edge Classification

- **Tree Edges**: edges where you discover a new vertex
- **Forward Edges**: edges that go from an ancestor to a descendant
- **Backward Edges**: edges that go from a descendant to an ancestor
- **Cross Edges**: edges going between vertices without an ancestor relationship

Problem 2 – Mechanical: DFS

Run Depth-First-Search on the graph below to classify the edges. Mark the start and end times for each vertex.



Reminder: Edge classifications are Tree Edge, Back Edge, Forward Edge, and Cross Edge

Work on running through the DFS of this graph with the people around you, and then we'll go over it together!

Problem 2 – Mechanical: DFS

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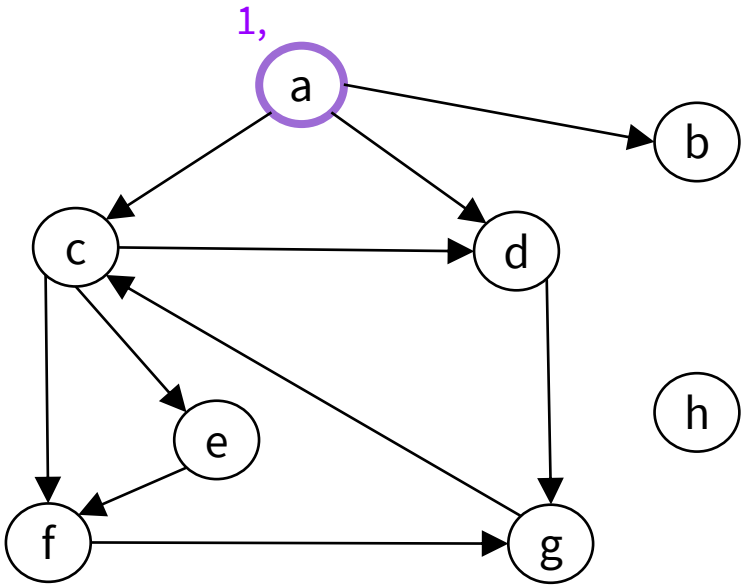

Seen: a

Stack: a

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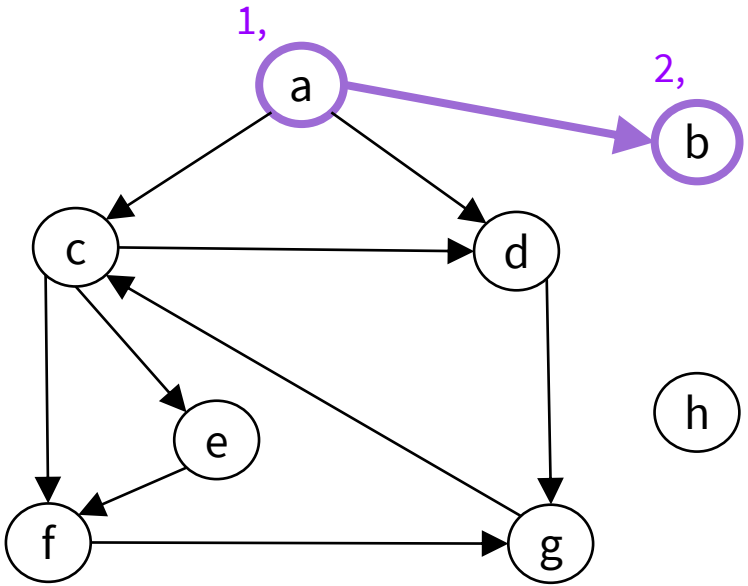

edge:
(a, b)

Seen:	Stack:
a	b
b	a

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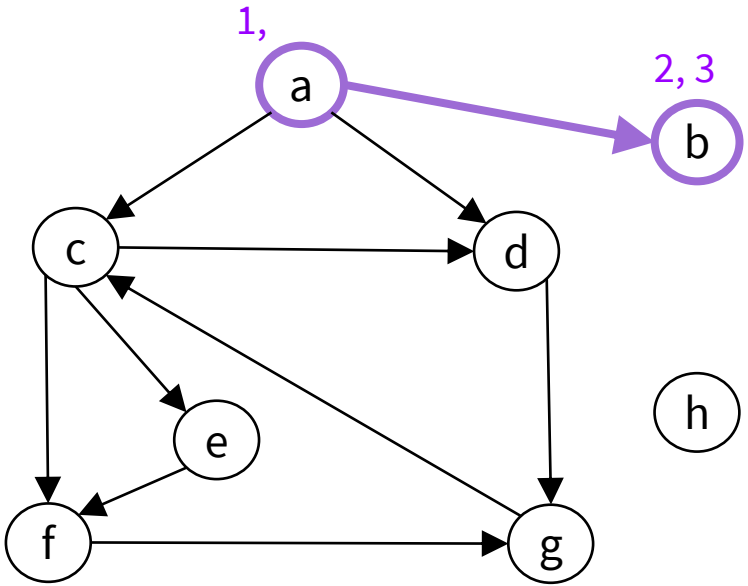

Seen:
a
b

Stack:
a

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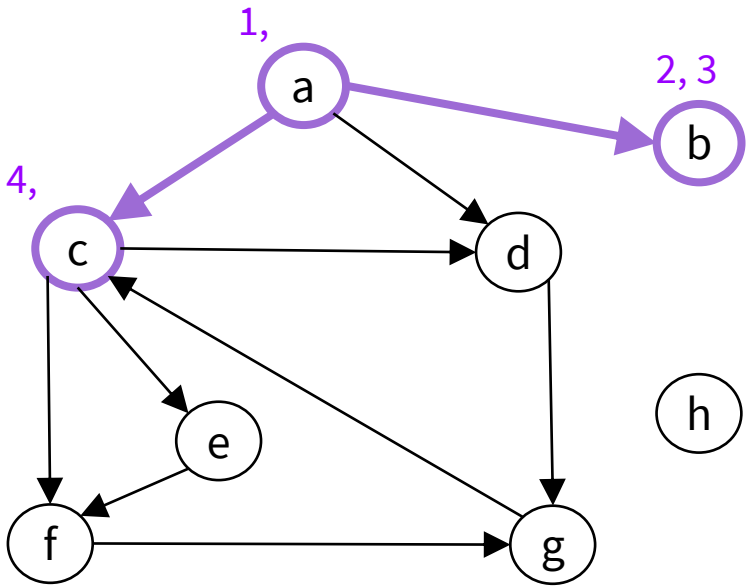

edge:
(a, c)

Seen:
a
b
c

Stack:
c
a

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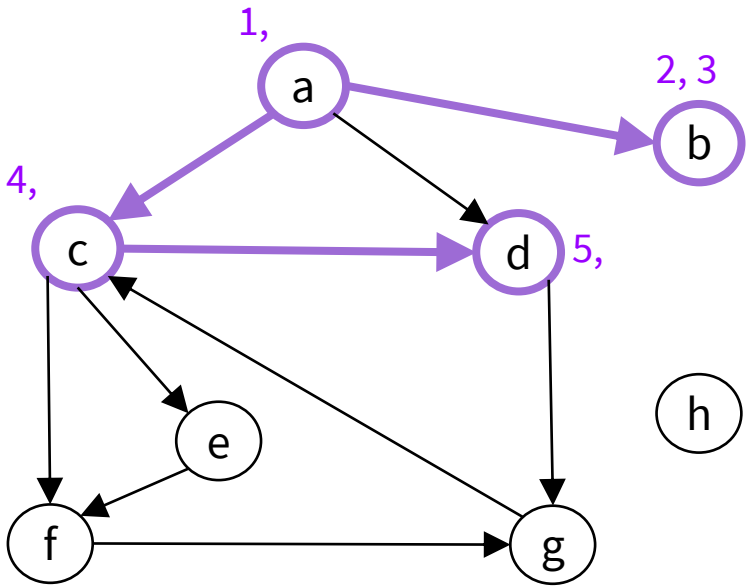

edge:
(c, d)

Seen:	Stack:
a	d
b	c
c	a
d	

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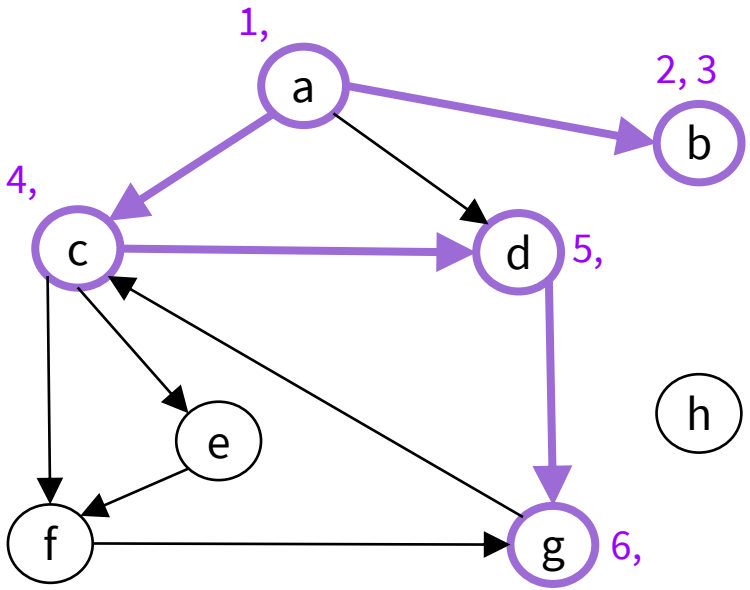

edge:
(d, g)

Seen:	Stack:
a	g
b	d
c	c
d	a
g	

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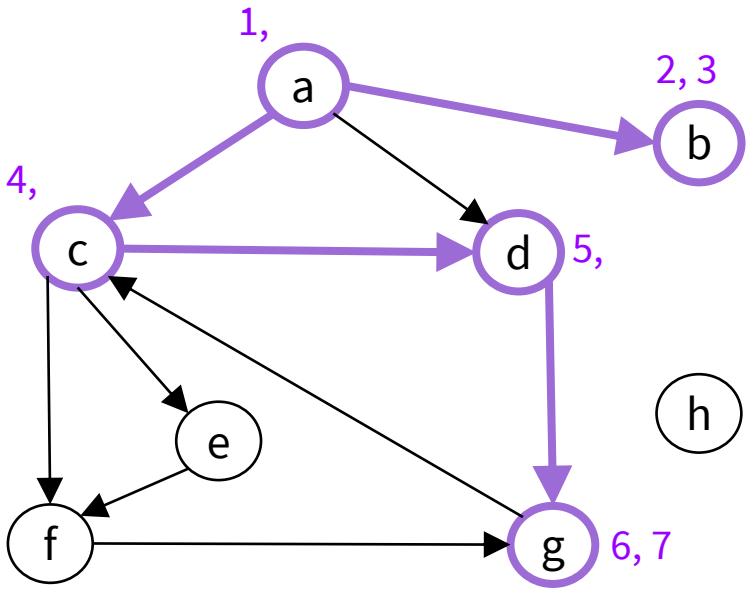

Seen:	Stack:
a	d
b	c
c	a
d	
g	

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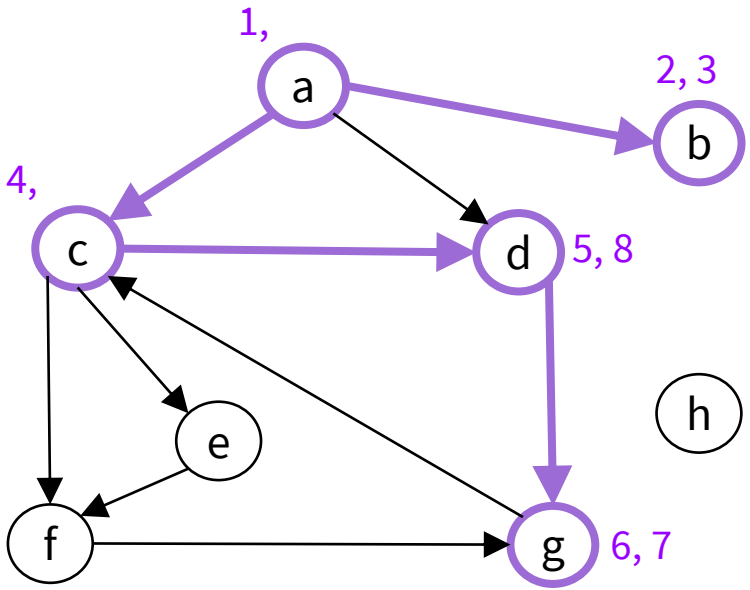

Seen:	Stack:
a	c
b	a
c	
d	
g	

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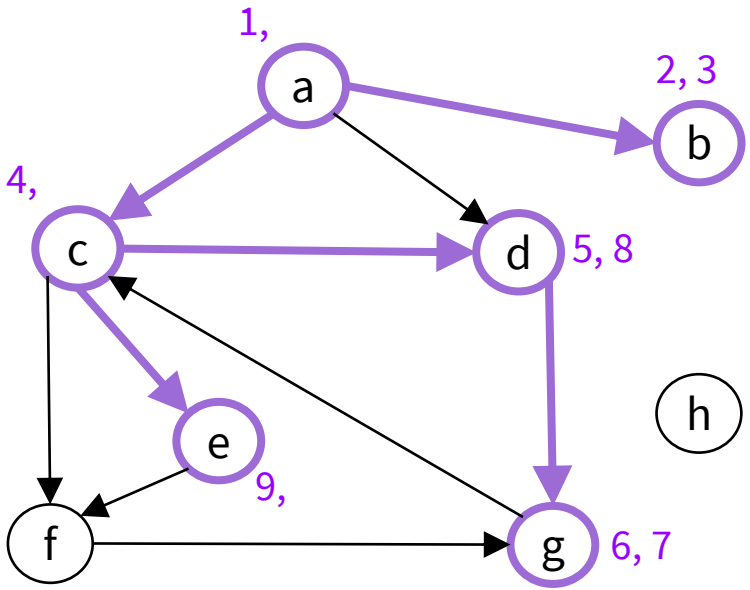

edge:
(c, e)

Seen:	Stack:
a	e
b	c
c	a
d	
g	
e	

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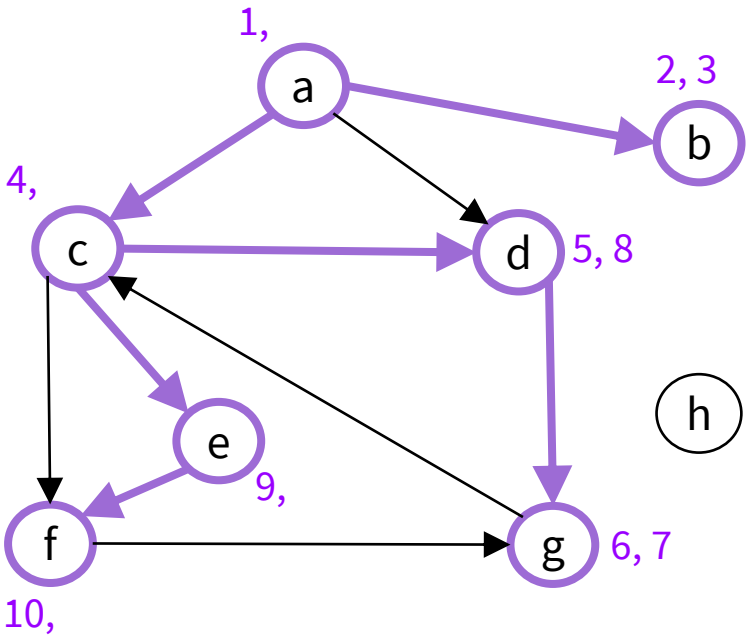

edge:
(e, f)

Seen:	Stack:
a	f
b	e
c	c
d	a
g	
e	
f	

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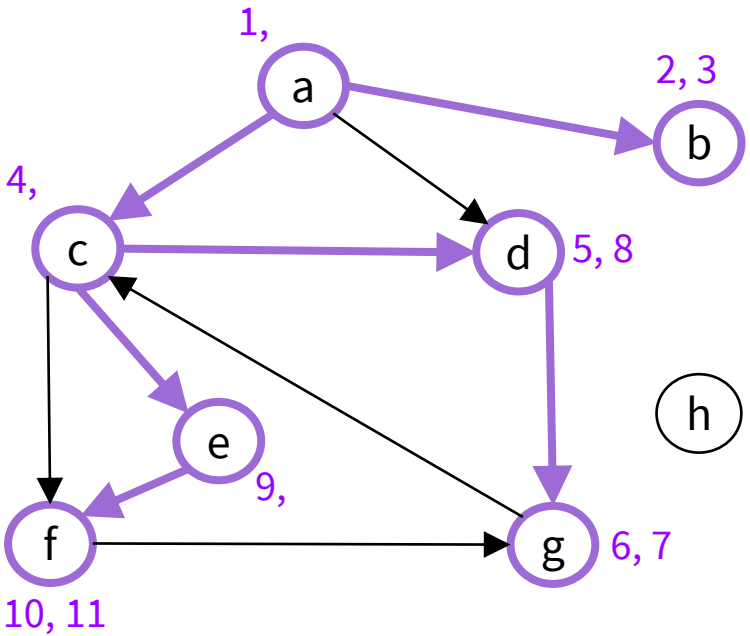

Seen:	Stack:
a	e
b	c
c	a
d	
g	
e	
f	

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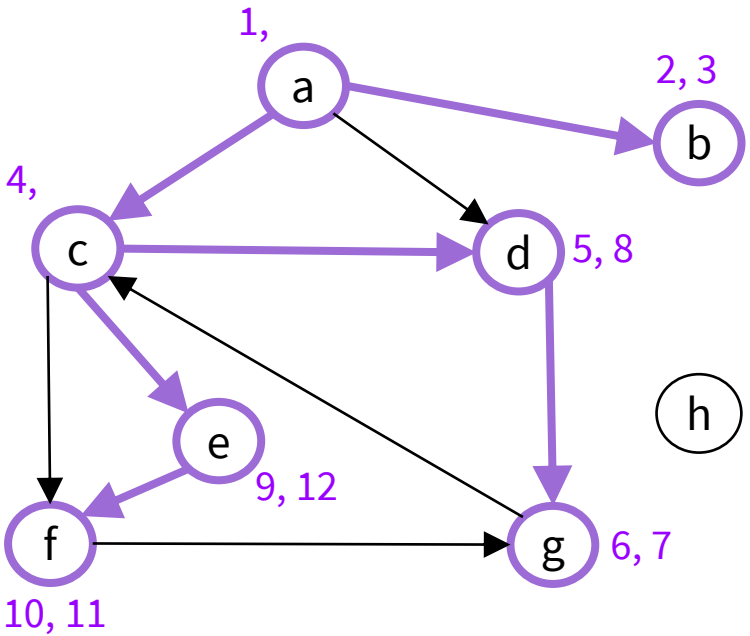

Seen:	Stack:
a	c
b	a
c	
d	
g	
e	
f	

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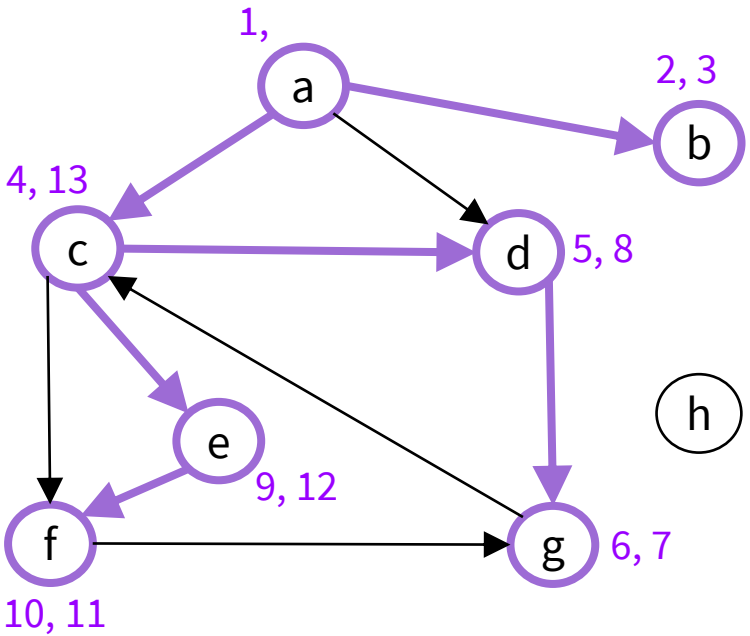

Seen:	Stack:
a	a
b	
c	
d	
g	
e	
f	

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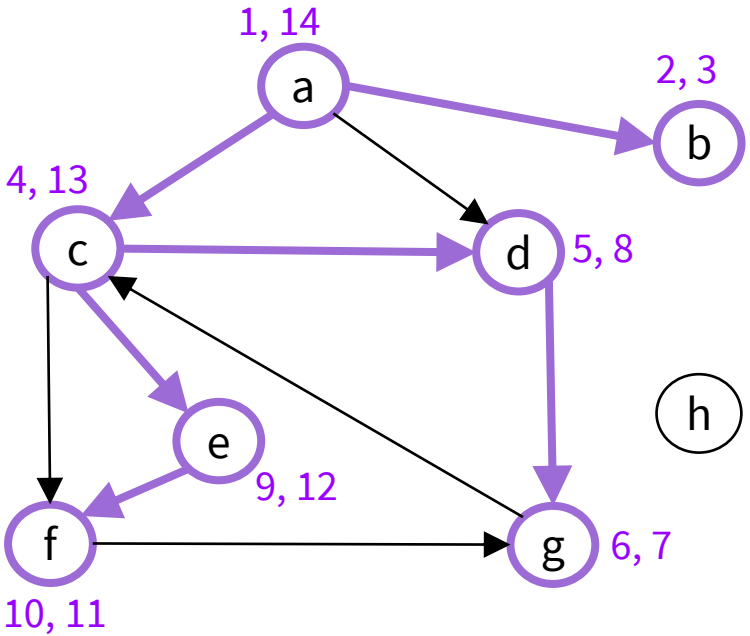

Seen: Stack:

a
b
c
d
g
e
f

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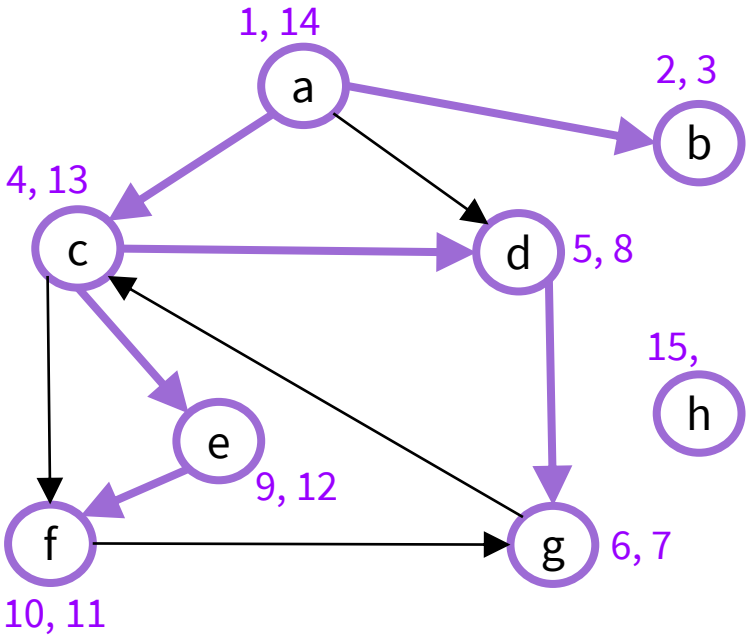

Seen:	Stack:
a	h
b	
c	
d	
g	
e	
f	
h	

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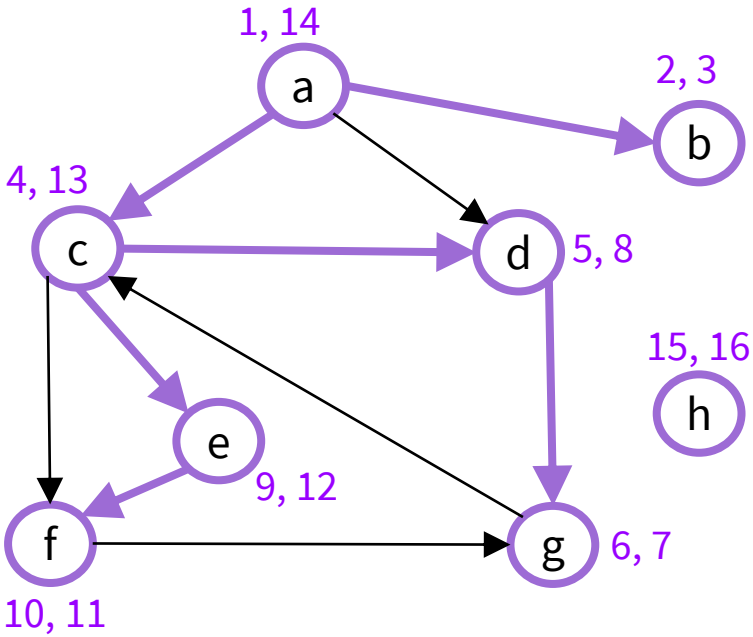

Seen: Stack:

a
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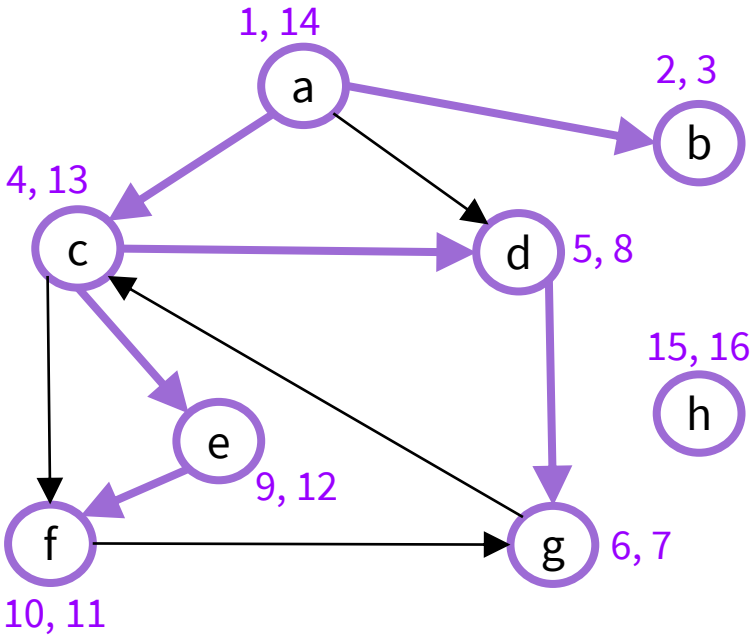

Tree Edges:
edges where you discover a new vertex

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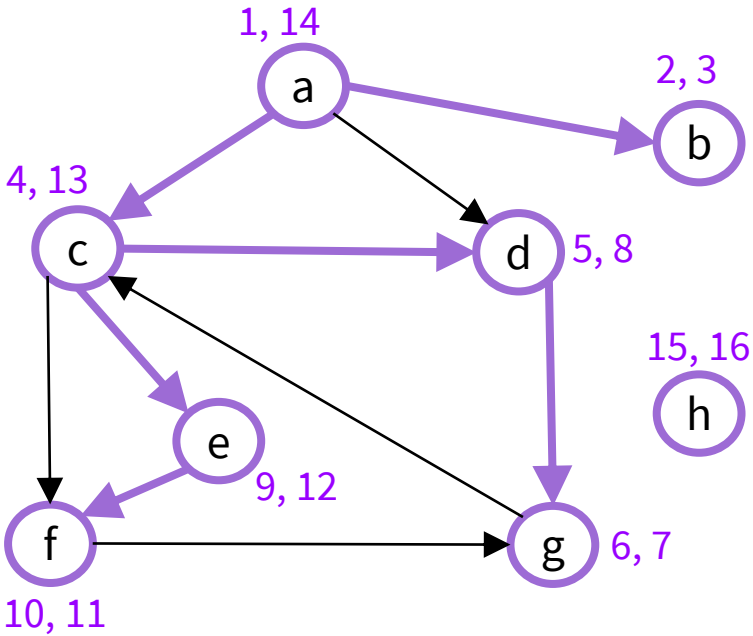

Tree Edges:

(a,b)
(a,c)
(c,d)
(d,g)
(c,e)
(f,g)

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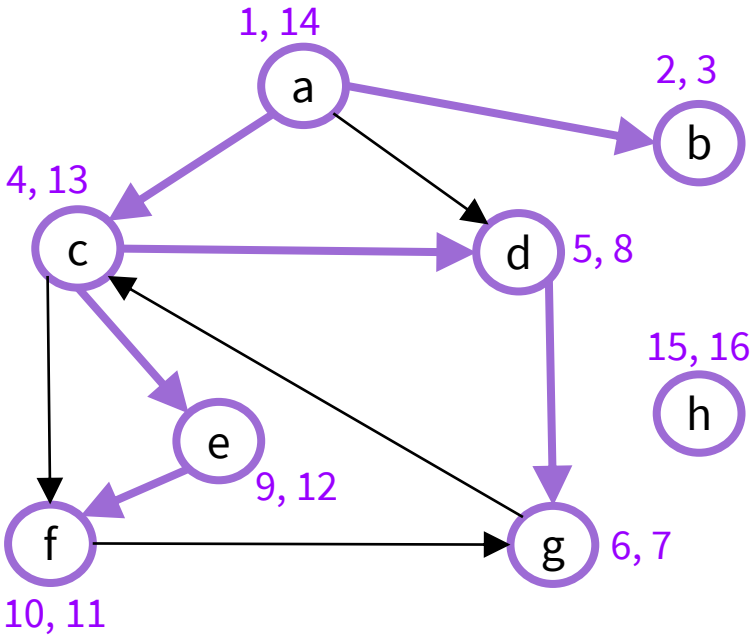

Forward Edges:
edges that go from an ancestor to a descendant

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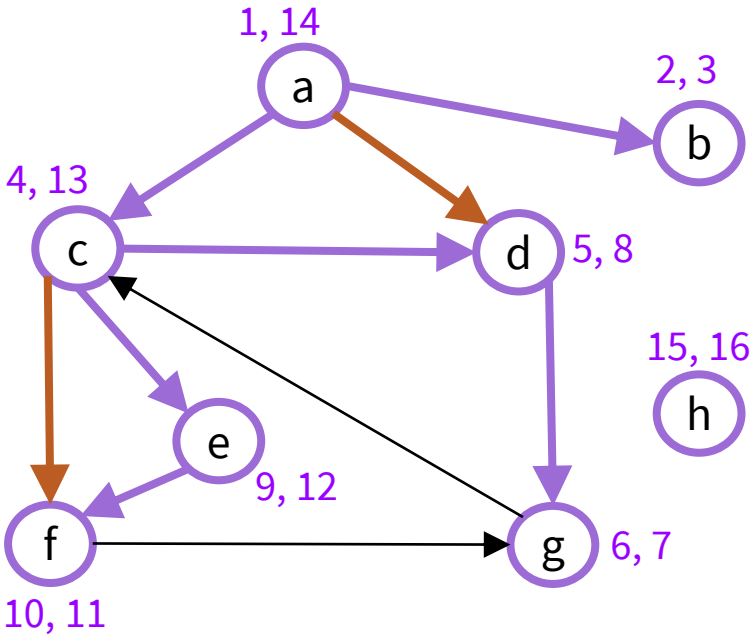

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(a, d)
(c, f)

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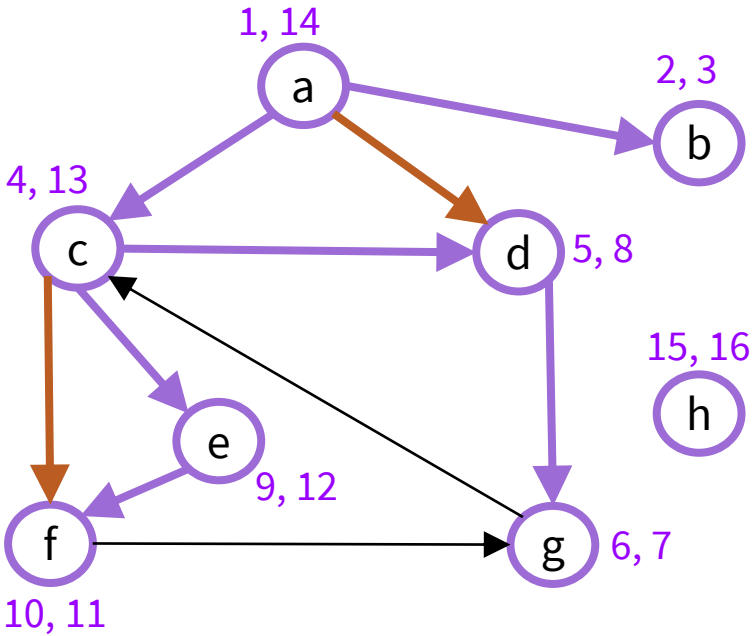

Back Edges:
edges that go from a descendant to an ancestor

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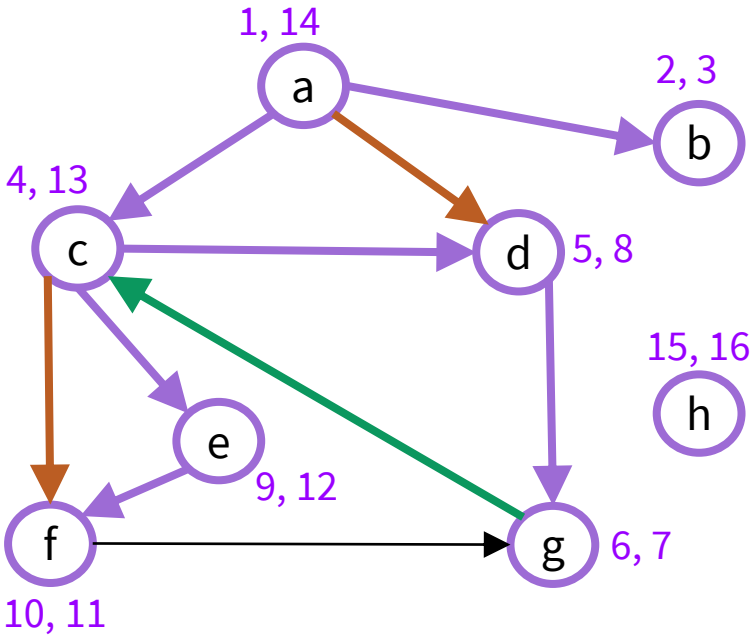
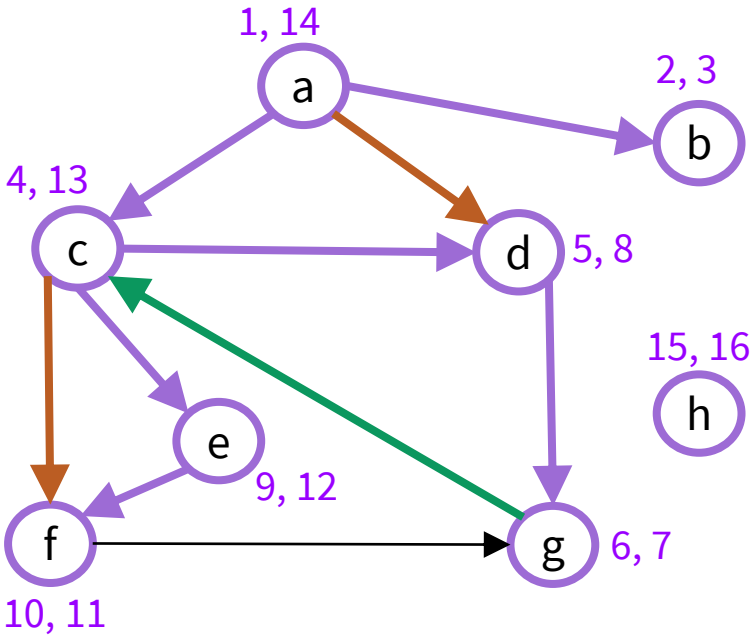


Back Edge:
(g, c)

Problem 2 – Mechanical: DFS

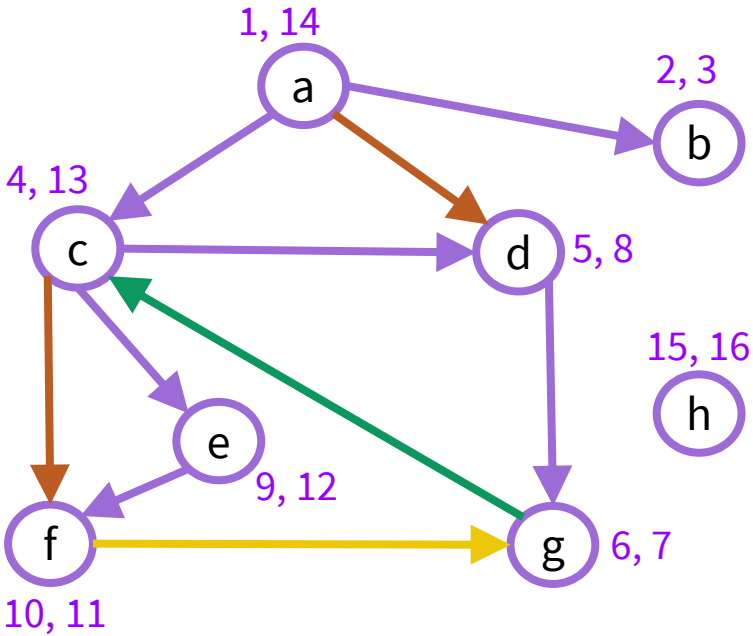
Run Depth-First-Search on the graph below to classify the edges. Mark the start and end times for each vertex.



Cross Edges:
edges going between vertices without an ancestor relationship

Problem 2 – Mechanical: DFS

Run Depth-First-Search on the graph below to classify the edges. Mark the start and end times for each vertex.



Cross Edge:
(f, g)

Graph Modeling

Modeling a Problem

- In order to write an algorithm for a word problem, first we need to translate that word problem into a form that we can interact with more easily.

- Often, that means figuring out how to encode it into data structures and identifying what type of algorithm might work for solving it

- A common form this will take is **graph modeling**, turning the problem into a graph. This will let us use graph algorithms to help us find our solution.

Graph Modeling Steps

1. Ask **"what are my fundamental objects?"** (These are usually your vertices)
2. Ask **"how are they related to each other?"** (These are usually your edges)
   - Be sure you can answer these questions:
     - Are the edges directed or undirected?
     - Are the edges weighted? If so, what are the weights? Are negative edges possible?
     - The prior two usually warrant explicit discussion in a solution. You should also be able to answer, "are there self-loops and multi-edges", though often it doesn't need explicit mention in the solution.
3. Ask **"What am I looking for, is it encoded in the graph?"** Are you looking for a path in the graph? A short(-est) one or long(-est) one or any one? Or maybe an MST or something else?
4. Ask **"How do I find the object from 3?"** If you know how, great! Choose an algorithm you know. If not, can you design an algorithm?
5. If stuck on step 4, you might need to go back to step 1! Sometimes a different graph representation leads to better results, and you'll only realize it around step 3 or 4.

Writing Algorithms Using Existing Algorithms

- Often, a problem can be solved by using an existing algorithm in one of two ways:
    - **Modifying the existing algorithm** slightly
    - **Calling the existing algorithm** (like a library function) with some additional work before and/or after the call
- Both are valid approaches, and which one you choose depends on the problem
- Whenever possible, it's a good idea to use ideas that you know work! You don't need to start from scratch to reinvent the wheel

Problem 4 – Graph Modeling

In this problem we're going to solve a classic riddle.

(a) First, you should solve the classic riddle yourself to get a feel for the problem. 2 You are on the beach with a jug that holds exactly 5 gallons, a jug that holds exactly 3 gallons, and a large bucket. Your goal is to put **exactly** 4 gallons of water into the bucket. Unfortunately, the jugs are not graduated (e.g., you can't just fill the larger jug 4/5 full). What you can do are the following operations.

- Completely fill any of your jugs.
- Pour from one of your containers into another until the first container is empty or the second is full.
- Pour out all the remaining water in a container.

How do you get 4 gallons of water into the bucket?

Work on part (a) of the problem with the people around you, and then we'll go over it together!

Problem 4 – Graph Modeling

(a) How do you get 4 gallons of water into the bucket?

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(a) How do you get 4 gallons of water into the bucket?

Fill the 5 gallon jug, pour into the 3 gallon jug until it is full (the jugs now contain 2 and 3 gallons respectively). Pour from the larger jug into the large bucket (it now has 2 gallons). Empty the 3 gallon jug, and repeat all these steps to get the desired 4 gallons.

Problem 4 – Graph Modeling

(a) How do you get 4 gallons of water into the bucket?

Fill the 5 gallon jug, pour into the 3 gallon jug until it is full (the jugs now contain 2 and 3 gallons respectively). Pour from the larger jug into the large bucket (it now has 2 gallons). Empty the 3 gallon jug, and repeat all these steps to get the desired 4 gallons.

Alternatively, Fill the 3 gallon jug, pour into the 5 gallon jug. Fill the 3 gallon jug again, pour until the 5 gallon jug is full (they now contain 5 gallons and 1 gallon respectively) . Pour the 1 gallon from the 3 gallon jug into the bucket. Refill the 3 gallon jug and pour into the bucket to bring the total to 4 gallons.

There may be other solutions.

Problem 4 – Graph Modeling

(b) Now, write an algorithm to solve any instance of this puzzle. You are given a list of 10 jugs with (positive integer) capacities $c_1, \ldots, c_{10}$, ranging from 1 to C. Your goal is to determine whether it is possible to get exactly t gallons into the bucket

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Intuition:
The "state" of the puzzle can be represented as the number of gallons in each of the jugs and the bucket.

We encode the rules of the puzzle such that each possible step is an edge.

Problem 4 – Graph Modeling

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Intuition:
The "state" of the puzzle can be represented as the number of gallons in each of the jugs and the bucket.

We encode the rules of the puzzle such that each possible step is an edge.

Work on this problem with the people around you. First see if you can model it as a graph, and then think about how you could use that graph in an algorithm. Then we'll go over it together!

Problem 4 – Graph Modeling

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What are my fundamental objects?:

How are they related to each other?:

What am I looking for, is it encoded in the graph?:

How do I find the object from 3:

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What are my fundamental objects?: How much water is in each jug and how much is in the final bucket. We need a vertex for each possible state the jugs and bucket can be in.

How are they related to each other?:

What am I looking for, is it encoded in the graph?:

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How are they related to each other?: We can fill a jug, empty a jug, or pour one jug into another or the bucket. So, we need edges that connect the different states for each of these valid operations.

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What am I looking for, is it encoded in the graph?: The series of pours that will get t gallons of water into the bucket. So, any path that gets us from 0 gallons in all the jugs and the bucket, to one of the states where the amount of water in the bucket is equal to t, if one exists, is the solution. To make things easier, we could add an extra "end" vertex that connects to every such state where there are t gallons in the bucket. Then, any path starting from the 0 vertex ending at this "end" vertex is a possible solution.

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How do I find the object from 3: Work on this part with the people around you, see if you can put it together into an algorithm, and we'll go over it together!

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Let S be the set of all 11-tuples, where for the first 10 entries, the entry is an integer between 0 and c_i , and the final entry is an integer between 0 and t. There are $(C+1)^{10} \cdot (t+1)$ such states.

We make a graph with a vertex for every element of S. And add an edge from u to v if and only if the states meet one of these conditions.

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From a given state $(j_1, j_2, \ldots, j_{10}, b)$, you can move to another state $(j'_1, j'_2, \ldots, j'_{10}, b')$ if and only if one of the following hold:

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From a given state $(j_1, j_2, \ldots, j_{10}, b)$, you can move to another state $(j'_1, j'_2, \ldots, j'_{10}, b')$ if and only if one of the following hold:

- There is only one index, k, where the tuples differ, and $j'_k = 0$. (we emptied a jug or the bucket)
- There is only one index, k, $(k \leq 10)$ where the tuples differ, and $j_k = c_k$. (we filled a jug)
- There are two indices, k, l where the tuples differ:
  - $j'_k = 0$ or $j'_l = c_l$
  - $j_k + j_l = j'_k + j'_l$

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Finally, we add a target vertex z, to the graph. Add an edge from every tuple where $b = t$ to z.

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Finally, we add a target vertex z, to the graph. Add an edge from every tuple where $b = t$ to z.

We then use [B/D]FS, starting from the all 0's tuple, and searching to see if z is reachable. If it is, we can return true (and predecessor edges will show the steps to take). If t is not reachable, then return false.

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Correctness

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Correctness

Suppose our algorithm returns true. Then [B/D]FS found a walk from all 0's to z. By construction of the graph, each edge corresponds to a valid rule we can apply in the original puzzle. Since the only edges going into t are from states where $b = t$, the walk must reach such a state, thus the puzzle can be solved by doing the steps on the edges of that walk.

Conversely, suppose the puzzle is solvable. Then there is a series of steps that can be taken to put t gallons into the bucket. Each legal step has a corresponding edge in the graph to the next state by construction, so there is a path to a valid stopping state (i.e., a tuple where $b = t$), we added an edge from all such vertices to z, so there is a path from all 0's to z. [B/D]FS will discover this path, so we will return true.

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Running Time

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Running Time

Let $n = (t+1)\mathbf{\Pi}(ci+1)$. Note that this is the number of vertices in our graph. Each vertex has a constant number of edges leaving it (as you are performing one of three operations (dump, pour, fill) among a constant number of jugs. So the graph can be has O(n) edges and can be constructed in O(n) time. Running [B/D]FS in a graph with O(n) vertices and edges takes O(n) time, so the overall running time is O(n).

That's All, Folks!

Thanks for coming to section this week!
Any questions?