Running Times, BFS

CSE 421 Fall 2022
Lecture 3

Proposer-Optimality

Some agents might have more than one possible match in a stable matching.

We say that h is a feasible partner for r if there is at least one stable matching where r and h are matched.

When there's more than one stable matching, there is a tremendous benefit to being the proposing side.

Proposer-Optimality

Every member of the proposing side is matched to their favorite of their feasible partners.

Proposer-Optimality

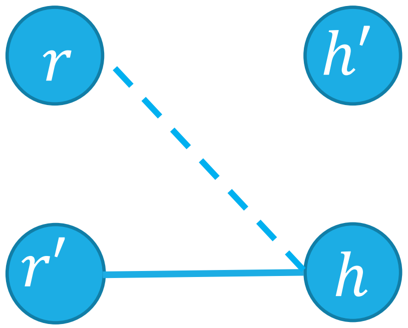
Intuition

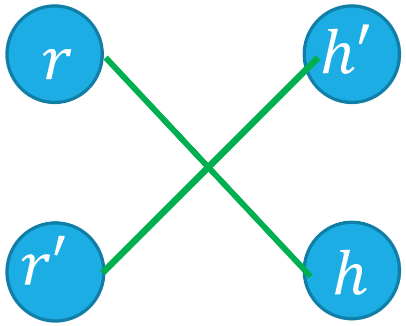
This isn't a full proof (coming in a second)

Suppose r is not matched to their favorite feasible partner, h. It has to be rejected by h during the algorithm (otherwise r matched to h or better). How did that happen? Well h had to have a better offer. But what's the source of that better offer? Call them r'.

There's some stable matching where (r, h) are matched, and r' is matched to some h'. For stability h prefers r to r' or r' prefers h' to h.

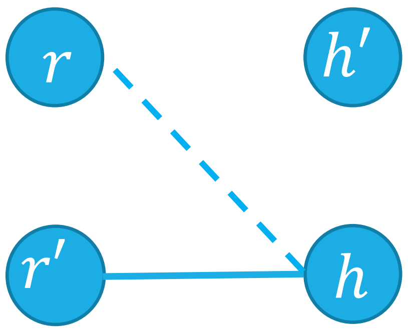
Must be the second, so r' was also already rejected by a feasible partner.





Proposer-Optimality

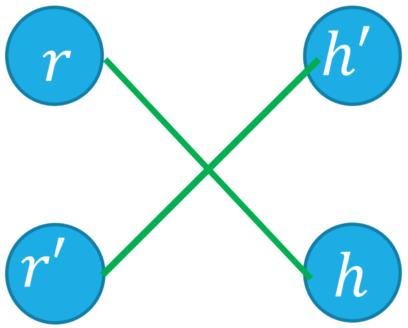
Every member of the proposing side is matched to the favorite of their feasible partners.



Intuition:

The riders start at the top of their lists. For the claim to be false, some rider r has to be the first to be rejected by their favorite feasible horse, h.

When that happens, h says it prefers some r' (and it does that while r' is still in the "favorite feasible partner" or "too good for you" sections of their list). So r' and h would block any matching



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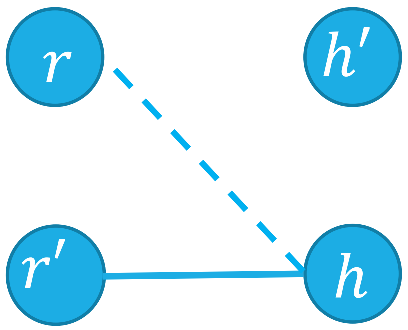


Let's prove it – again by contradiction

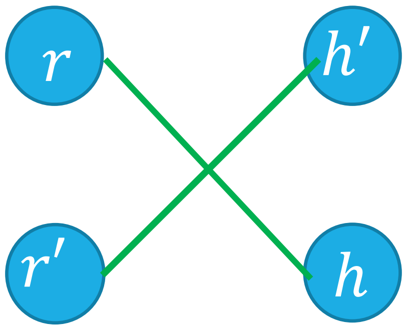
Suppose some rider is not matched to their favorite feasible partner.
Then some r must have been the **first** to be rejected by their favorite feasible partner, h. (Observation A)
And there is an r' that h (temporarily) matched to causing that rejection.

Since r and h are feasible for each other, there is some stable matching (call it M') where (r, h) are matched. The rider r' is matched to some horse h'.

What can we say about r'? They had never been rejected by a feasible partner. So they prefer h to h'.

And h prefers r' to r (by the run of the algorithm).

But then (r', h) are a blocking pair in M'!



Contradiction With Extremality

We used a trick to make the proof nicer.

Instead of saying "r' was already rejected by a feasible partner, let's go back and analyze **that** rejection." And repeat the argument over and over, we jump straight back to the first rejection.

If in your proofs, you're saying "repeat this step until…" you can probably make a cleaner proof with this trick.

Another example of this trick is in section 1.

Implications of Proposer Optimality

Proposer-Optimality

Every member of the proposing side is matched to their favorite of their feasible partners.

We didn't specify which rider proposes when more than one is free

Proposer-optimality says it doesn't matter! You always get the proposer-optimal matching.

So what happens to the other side?

Chooser-Pessimality

A similar argument (it's a quite tricky proof-by-contradiction, but very similar to proposer-optimality), will show that choosing among proposals is a much worse position to be in.

Chooser-Pessimality

Every member of the choosing (non-proposing) side is matched to their least favorite of their feasible partners.

Some More Context and Takeaways

Stable Matching has another common name: “Stable Marriage”

The metaphor used there is “men” and “women” getting married.

When choosing or analyzing an algorithm, or choosing which parts of a problem to model and which ones to ignore, think about everyone involved, not just the people you’re optimizing for; you might not be able to have it all.

Takeaways

Stable Matchings always exist, and we can find them efficiently.

The GS Algorithm gives proposers their best possible partner

At the expense of those receiving proposals getting their worst possible.

When doing a proof by contradiction, it sometimes helps to analyze the first time something happens instead of just some time where it happens.

Where Are We?

Last Week:

A useful algorithm for matching up agents in two groups.

This Week:

Not a single algorithm – a method of designing your own algorithms.

How do we search through graphs?

- You’ve already seen the basic tools – BFS and DFS
- Don’t just want to see what BFS/DFS can do, want you to be able to use them in new scenarios.

Today

What running times are good?

Review some graph terms

BFS and applications

Running Times

Recall the definition of big-O, big-Omega, big-Theta

Big-O is "at most" – it's a fancy version of "$\leq$"

Big-Omega is "at least" – it's a fancy version of "$\geq$ "

Big-Theta is "about equal to" – it's a fancy version of "$\approx$"

Big-O

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \leq c \cdot g(n)$$

Big-Omega

$f(n)$ is $\Omega(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \geq c \cdot g(n)$$

Big-Theta

$f(n)$ is $\Theta(g(n))$ if
$f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

What don’t we care about?

Ignore lower-order terms.

If there’s a $5n^2$ that’s more important than $10n$ for very large n

Ignore constant factors.

We can’t see clearly what will happen when we convert from pseudocode to Java code (and Java code to machine code)

Ignore small inputs.

Small enough and it happens in the blink of an eye anyway…

Big-O isn't perfect!

$f(n) = 1{,}000{,}000{,}000{,}000{,}000{,}000{,}000n$ is slower than $\mathrm{g}(n) = 2n^2$ for "practical" sizes of n. (but big-O, Ω, Θ says treat f as faster)

$f(n) = n$ and $g(n) = 1000000n$ aren't the same for practical purposes. But big-O, Ω, Θ treat them identically.

Polynomial vs. Exponential

We'll say an algorithm is "efficient" if it runs in **polynomial time**

Polynomial Time

We say an algorithm runs in polynomial time if on an input of size n, the algorithm runs in time $O(n^c)$ for some constant c.

Sorting algorithms (e.g. the $\Theta(n \log n)$ ones) – polynomial time.

Graph algorithms from 332 – polynomial time

Why Polynomial Time?

Most “in-practice efficient” algorithms are polynomial time, and most polynomial time algorithms **can be made** “in-practice efficient.”

Not all of them! But a good number.

It’s an easy definition to state and check.

It’s easy to work with (a polynomial time algorithm, run on the output of a polynomial time algorithm is overall a polynomial time algorithm).

e.g. you can find a minimum spanning tree, then sort the edges. The overall running time is polynomial.

It lets us focus on the big-issues.

Thinking carefully about data structures might get us from $O(n^3)$ to $O(n^2)$, or $O(2^n n)$ to $O(2^n)$, but we don’t waste time doing the second one.

Polynomial vs. Exponential

If you have an algorithm that takes exactly $f(n)$ microseconds, how large of an n can you handle in the given time?

	1 second	1 minute	1 hour	1 day	1 month	1 year	1 century
$\lg n$	2^{10^6}	$2^{6 \cdot 10^7}$	$2^{36 \cdot 10^8}$	$2^{864 \cdot 10^8}$	$2^{25920 \cdot 10^8}$	$2^{315360 \cdot 10^8}$	$2^{31556736 \cdot 10^8}$
$\sqrt{n}$	10^{12}	$36 \cdot 10^{14}$	$1296 \cdot 10^{16}$	$746496 \cdot 10^{16}$	$6718464 \cdot 10^{18}$	$994519296 \cdot 10^{18}$	$995827586973696 \cdot 10^{16}$
n	10^6	$6 \cdot 10^7$	$36 \cdot 10^8$	$864 \cdot 10^8$	$2592 \cdot 10^9$	$31536 \cdot 10^9$	$31556736 \cdot 10^8$
$n \lg n$	62746	2801417	133378058	2755147513	71870856404	797633893349	68654697441062
n^2	1000	7745	60000	293938	1609968	5615692	56175382
n^3	100	391	1532	4420	13736	31593	146677
2^n	19	25	31	36	41	44	51
$n!$	9	11	12	13	15	16	17

Polynomial vs. Exponential

For polynomial time, throwing (a lot) more time/compute power can make a significant difference. For exponential (or worse) time, the improvement is minimal.

With polynomial time, the increase in size is multiplicative..

For exponential time, it's only additive.

Running Time	Handle $n =$…	Twice as fast processor
$O(n)$	10^6	$10^6 \cdot 2$
$O(n^3)$	100	$100 \cdot 2^{1/3} \approx 126$
$O(2^n)$	19	$19 + 1 = 20$

Polynomial Time isn't perfect.

It has all the problems big-O had.

$f(n) = n^{10000}$ is polynomial-time. $g(n) = 1.0000000001^n$ is not. You'd rather run a $g(n)$ time algorithm.

Just like big-O, it's still useful, and we can handle the edge-cases as they arise.

Tools for running time analysis

Recurrences

Solved with Master Theorem, tree method, or unrolling

Facts from 332

Known running times of data structures from that course– just use those as facts.

We have a reference for you on the webpage

Style of analysis you did in 332

How many iterations do loops need, and what's the running time of each?

Occasionally, summations to answer those questions.

Be Careful with hash tables

In-practice hash tables are amazing -- $O(1)$ for every dictionary operation.

But what about in-theory? In the worst-case $O(n)$ operations are possible.

Only use a dictionary if you can be sure you'll have $O(1)$ operations.

Usually the way we accomplish that is by assuming our input comes to us numbered.

E.g. our riders and horses were numbered 0 to $n-1$.

And for graphs are vertices are numbered 0 to $n-1$.

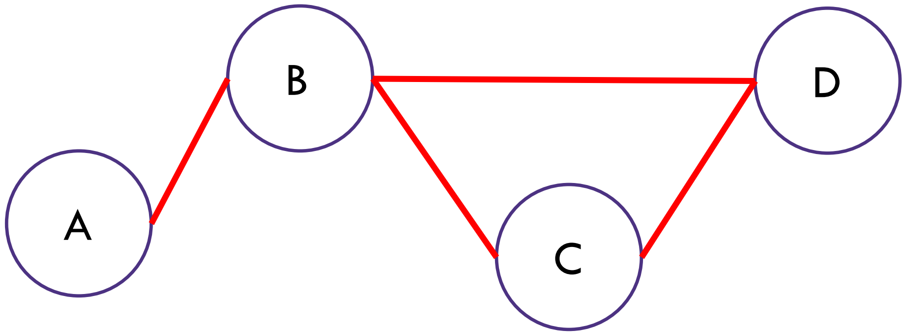
Graphs

332 review

Graphs

Represent data points and the relationships between them.

That's vague.



Formally:

A graph is a pair: G = (V,E)

V: set of **vertices** (aka **nodes**) $\{A, B, C, D\}$

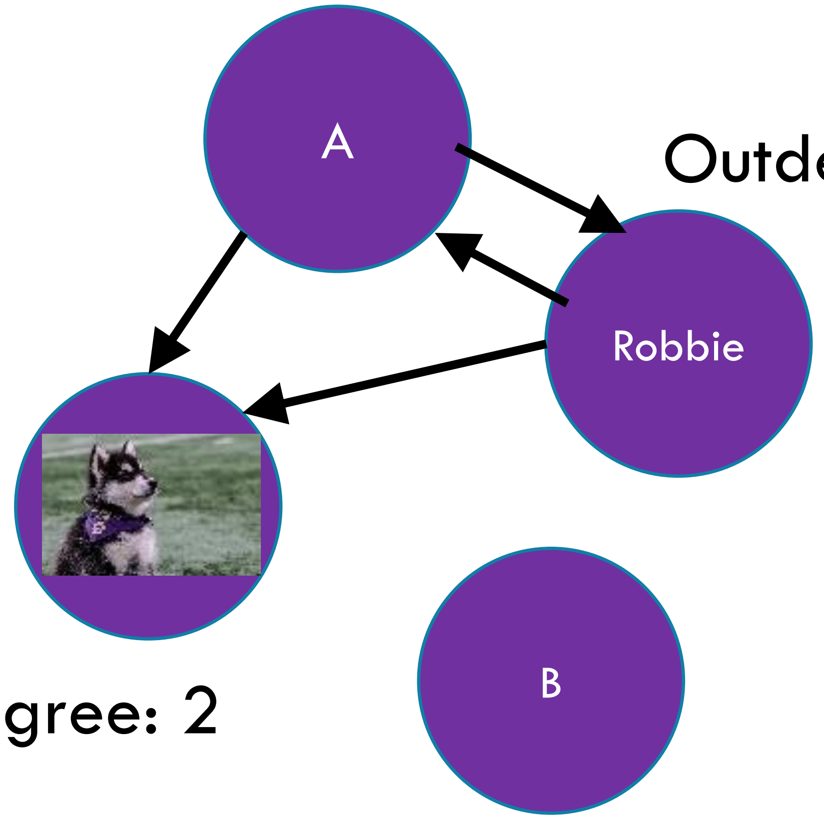
E: set of **edges** $\{(A, B), (B, C), (B, D), (C, D)\}$

Each edge is a pair of vertices.

Graph Terms

Graphs can be directed or undirected.

Following on twitter.

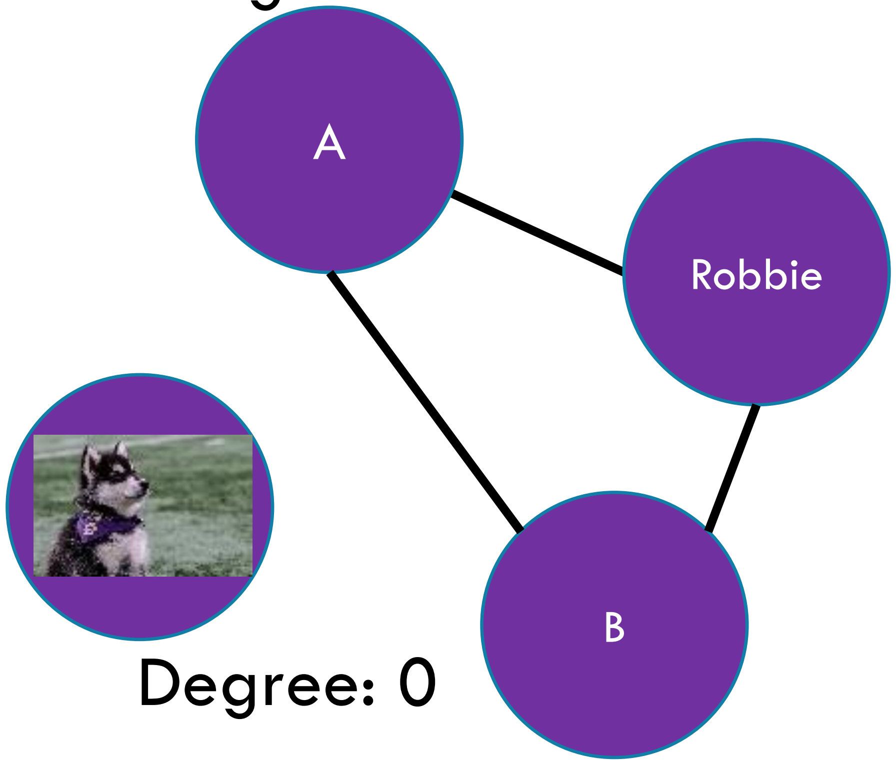


Outdegree: 2

Indegree: 2

This graph is **disconnected.**

Degree: 2



Degree: 0

Friendships on Facebook.

Making Graphs

If your problem has **data** and **relationships**, you might want to represent it as a graph

How do you choose a representation?

Usually:

Think about what your "fundamental" objects are

- Those become your vertices.

Then think about how they're related

- Those become your edges.

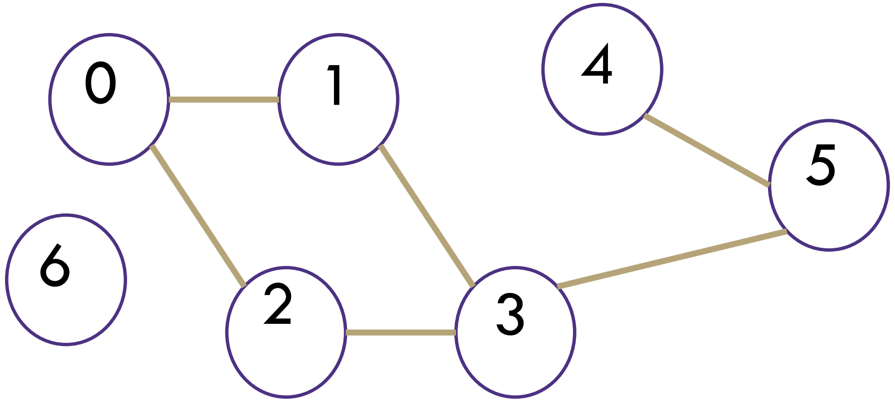
Adjacency Matrix

In an adjacency matrix a[u][v] is 1 if there is an edge (u,v), and 0 otherwise.

Worst-case Time Complexity (|V| = n, |E| = m):

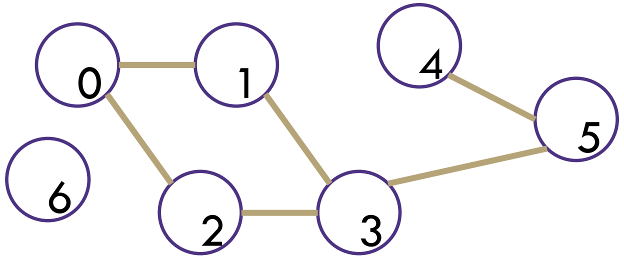
- Add Edge: $\Theta(1)$
- Remove Edge: $\Theta(1)$
- Check edge exists from (u,v): $\Theta(1)$
- Get outneighbors of u: $\Theta(n)$
- Get inneighbors of u: $\Theta(n)$

Space Complexity: $\Theta(n^2)$



	0	1	2	3	4	5	6
0	0	1	1	0	0	0	0
1	1	0	0	1	0	0	0
2	1	0	0	1	0	0	0
3	0	1	1	0	0	1	0
4	0	0	0	0	0	1	0
5	0	0	0	1	1	0	0
6	0	0	0	0	0	0	0

Adjacency List

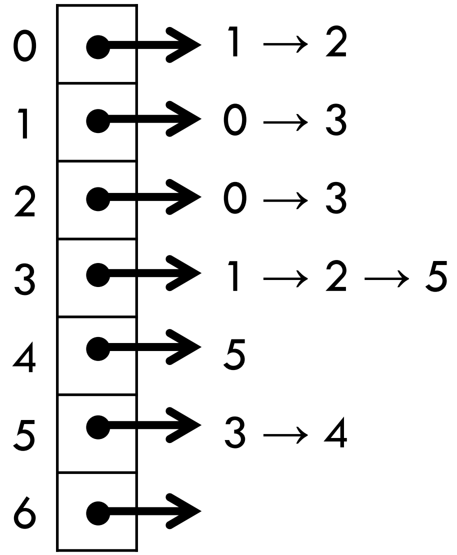


An array where the u^{th} element contains a list of neighbors of u.

Directed graphs: list of out-neighbors (a[u] has v for all (u,v) in E)

Time Complexity (|V| = n, |E| = m):

- Add Edge: $\Theta(1)$
- Remove Edge: $\Theta(1)$
- Check edge exists from (u,v): $\Theta(1)$
- Get neighbors of u (out): $\Theta(\deg(u))$
- Get neighbors of u (in): $\Theta(n)$



Assume we have hash tables AND linked lists

Space Complexity: $\Theta(n + m)$

Tradeoffs

Adjacency Matrices take more space, and have slower $\Theta()$ bounds, why would you use them?

- For **dense** graphs (where m is close to n^2), the running times will be close
- And the constant factors can be much better for matrices than for lists.
- Sometimes the matrix itself is useful ("spectral graph theory")

For this class, unless we say otherwise, we'll assume we're using Adjacency Lists and the following operations are all $\Theta(1)$

- Checking if an edge exists.
- Getting the next edge leaving u (when iterating over them all)
- "following" an edge (getting access to the other vertex)

To make this work, we usually assume the vertices are numbered.

Graph Algorithms

From 332 you already know:

How to find a topological sort

Use Dijkstra's Algorithm to find shortest paths in (positively) weighted graphs

Use Prim's and Kruskal's Algorithms to find minimum spanning trees.

Depending on which quarter you took 332, you also know:

BFS, and at least one application.

That might have been 2-coloring, unweighted shortest paths, or something else.

DFS, and at least one application.

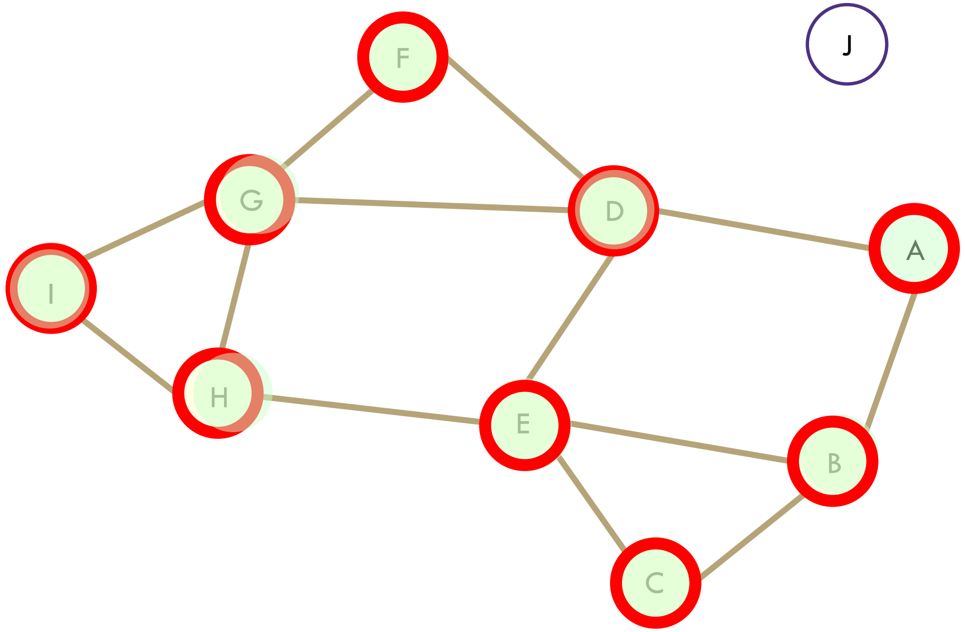
That might have been finding SCCs, a different topo sort algorithm, or something else.

Our goal is ***not*** to memorize algorithms! Our goal is to solve new problems. Even if you've seen these applications, we're coming from a new angle this week.

Traversals

Breadth First Search

```
search(graph)
    toVisit.enqueue(first vertex)
      mark first vertex as seen
    while(toVisit is not empty)
       current = toVisit.dequeue()
       for (v : current.neighbors())
           if (v is not seen)
              mark v as seen
              toVisit.enqueue(v)
```



Current node: I

Queue: B D E C F G H I

Finished: A B D E C F G H I

Breadth First Search

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search(graph)
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   while(toVisit is not empty)
     current = toVisit.dequeue()
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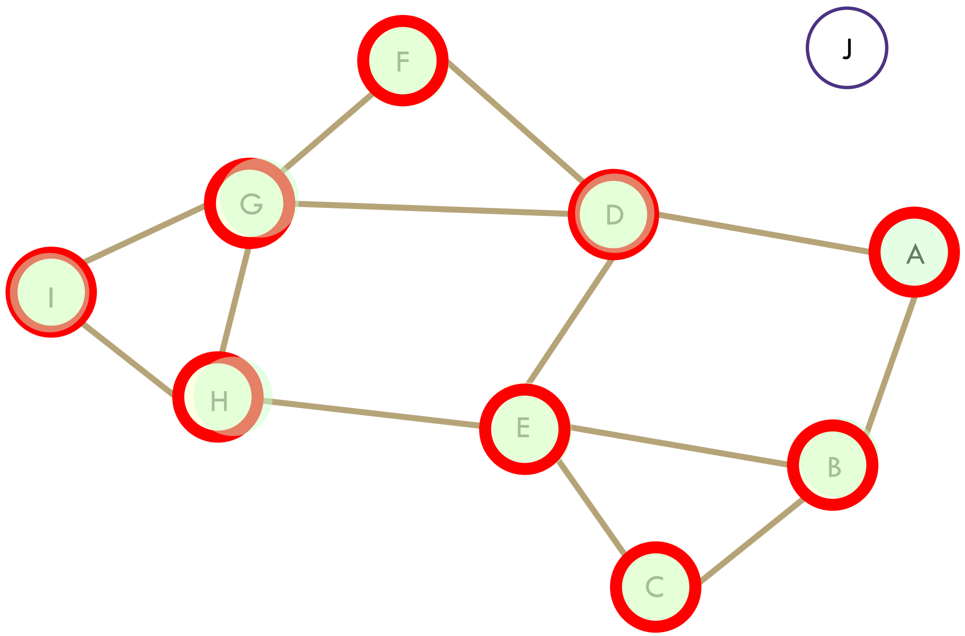
```



Hey we missed something…

We're only going to find vertices we can "reach" from our starting point.

If you need to visit everything, just start BFS again somewhere you haven't visited until you've found everything.

Running Time

```
search(graph)
    toVisit.enqueue(first vertex)
      mark first vertex as seen
    while(toVisit is not empty)
       current = toVisit.dequeue()
       for (v : current.neighbors())
          if (v is not seen)
             mark v as seen
             toVisit.enqueue(v)
```

This code might look like:
a loop that goes around m times
Inside a loop that goes around n times,
So you might say $O(mn)$.

That bound is not tight,
Don't think about the loops, think about what happens overall.
How many times is `current` changed?
How many times does an edge get used to define `current.neighbors`?

We visit each vertex at most twice, and each edge at most once: $\Theta(|V| + |E|)$

Old Breadth-First Search Application

Shortest paths in an **unweighted** graph.

Finding the connected components of an **undirected** graph.

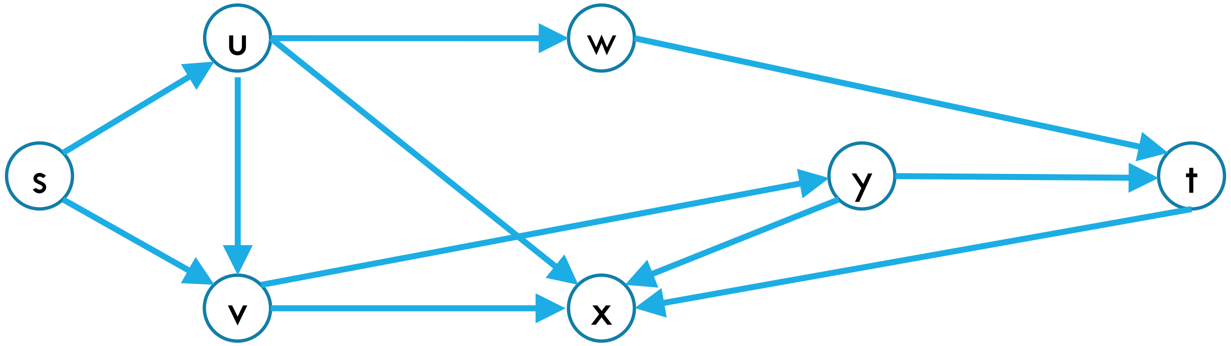
Both run in $\Theta(m+n)$ time,

where m is the number of edges (also written E or $|E|$)

And n is the number of vertices (also written V or $|V|$)

Unweighted Graphs

If the graph is unweighted, how do we find a shortest paths?



What's the shortest path from s to s?

Well....we're already there.

What's the shortest path from s to u or v?

Just go on the edge from s

From s to w,x, or y?

Can't get there directly from s, if we want a length 2 path, have to go through u or v.

A detailed application

Bipartite (also called "2-colorable")

A graph is bipartite (also called 2-colorable) if the vertex set can be divided into two sets V_1, V_2 such that the only edges go between V_1 and V_2.

Called "2-colorable" because you can "color" V_1 red and V_2 blue, and no edge connects vertices of the same color.

We'll adapt BFS to find if a graph is bipartite

And prove a graph theory result along the way.

A detailed application

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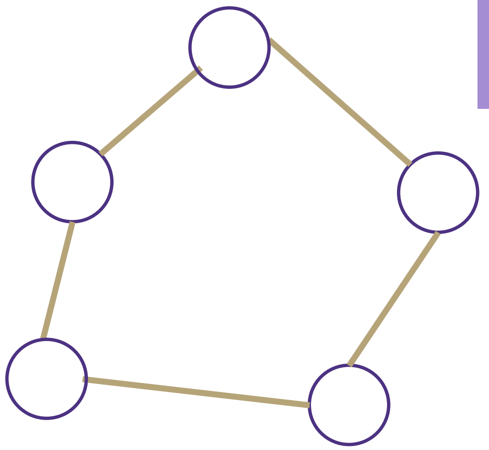


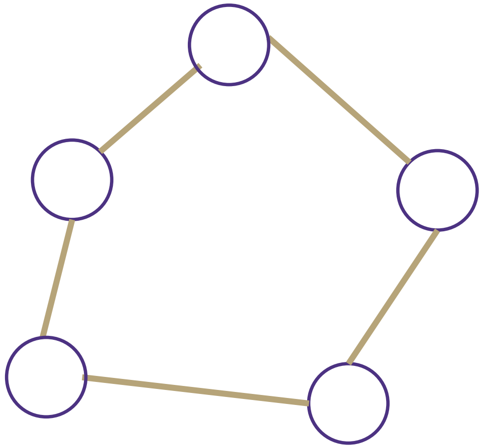
If a graph contains an odd cycle, then it is not bipartite.

Try the example on the right, then proving the general theorem in the light purple box.

Help Robbie figure out how long to make the explanation
Pollev.com/robbie

Lemma 1

If a graph contains an odd cycle, then it is not bipartite.



Start from any vertex, and give it either color.
Its neighbors **must** be the other color.
Their neighbors must be the first color
…
The last two vertices (which are adjacent) must be the same color.
Uh-oh.

BFS with Layers

Why did BFS find distances in unweighted graphs?

You started from u (“layer 0”)

Then you visited the neighbors of u (“layer 1”)

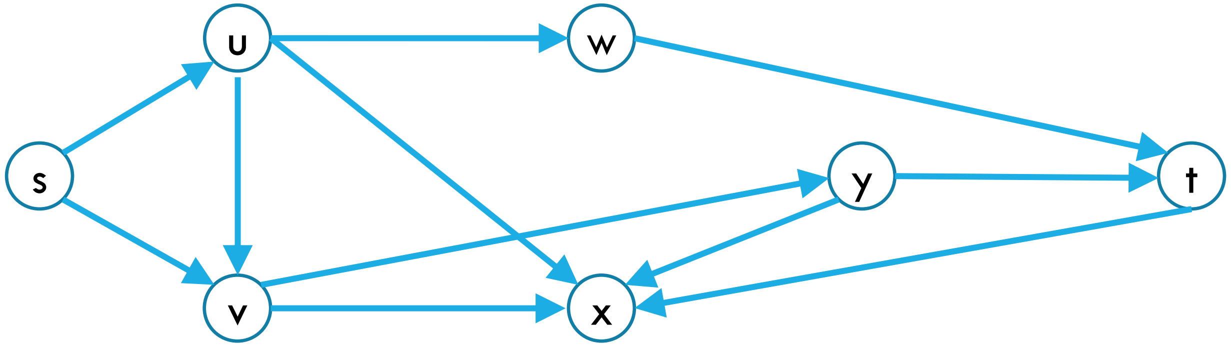
Then the neighbors of the neighbors of u, that weren’t already visited (“layer 2”)

…

The neighbors of layer $i - 1$, that weren’t already visited (“layer i”)

Unweighted Graphs

If the graph is unweighted, how do we find a shortest paths?



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From s to w,x, or y?

Can't get there directly from s, if we want a length 2 path, have to go through u or v.

BFS With Layers

```
search(graph)
   toVisit.enqueue(first vertex)
   mark first vertex as seen
   toVisit.enqueue(end-of-layer-marker)
   l=1
   while(toVisit is not empty)
      current = toVisit.dequeue()
      if(current == end-of-layer-marker)
          l++
          toVisit.enqueue(end-of-layer-marker)
      current.layer = l
      for (v : current.neighbors())
         if (v is not seen)
            mark v as seen
            toVisit.enqueue(v)
```

It's just BFS!

With some extra bells and whistles.

Layers

Can we have an edge that goes from layer i to layer $i + 2$ (or lower)?

No! If u is in layer i, then we processed its edge while building layer $i + 1$, so the neighbor is no lower than layer i.

Can you have an edge within a layer?

Yes! If u and v are neighbors and both have a neighbor in layer i, they both end up in layer $i + 1$ (from their other neighbor) before the edge between them can be processed.

Testing Bipartiteness

How can we use BFS with layers to check if a graph is 2-colorable?

Well, neighbors have to be “the other color”

Where are your neighbors?

Hopefully in the next layer or previous layer…

Color all the odd layers red and even layers blue.

Does this work?

Lemma 2

If BFS has an intra-layer edge, then the graph has an odd-length cycle.

An “intra-layer” edge is an edge “within” a layer.

Follow the “predecessors” back up, layer by layer.

Eventually we end up with the two vertices having the same predecessor in some level (when you hit layer 1, there’s only one vertex)

Since we had two vertices per layer until we found the common vertex, we have $2k + 1$ vertices – that’s an odd number!

Lemma 3

If a graph has no odd-length cycles, then it is bipartite.

Lemma 3

If a graph has no odd-length cycles, then it is bipartite.

Prove it by **contrapositive**

We want to show "if a graph is not bipartite, then it has an odd-length cycle.

Suppose G is not bipartite. Then the coloring attempt by BFS-coloring must fail.

Edges between layers can't cause failure – there must be an intra-level edge causing failure. By Lemma 2, we have an odd cycle.

The big result

Bipartite (also called "2-colorable")

A graph is bipartite if and only if it has no odd cycles.

Proof:
Lemma 1 says if a graph has an odd cycle, then it's not bipartite (or in contrapositive form, if a graph is bipartite, then it has no odd cycles)

Lemma 3 says if a graph has no odd cycles then it is bipartite.

The Big Result

The final theorem statement doesn't know about the algorithm – we used the algorithm to prove a graph theory fact!

Wrapping it up

```
BipartiteCheck(graph) //assumes graph is connected!
   toVisit.enqueue(first vertex)
   mark first vertex as seen
   toVisit.enqueue(end-of-layer-marker)
   l="odd"
   while(toVisit is not empty)
      current = toVisit.dequeue()
        if(current == end-of-layer-marker)
              l++
          toVisit.enqueue(end-of-layer-marker)
      current.layer = l
      for (v : current.neighbors())
         if (v is not seen)
               mark v as seen
            toVisit.enqueue(v)
           else //v is seen
            if(v.layer == current.layer)
                 return "not bipartite" //intra-level edge
  return "bipartite" //no intra-level edges
```

Testing Bipartiteness

Our algorithm should answer “yes” or “no”

“yes G is bipartite” or “no G isn’t bipartite”

We need to show an if-and-only-if

“Our algorithm outputs `true` if and only if G is bipartite”

There are ***two*** implications to prove!

If we output `true` then G really is bipartite.

If G is bipartite then our algorithm outputs `true`.

Proving Algorithm Correct

If the graph is bipartite, then by Lemma 1 there is no odd cycle. So by the contrapositive of lemma 2, we get no intra-level edges when we run BFS, thus the algorithm (correctly) returns the graph is bipartite.

If the algorithm returns that the graph is bipartite, then we have found a bipartition. We cannot have any intra-level edges (since we check every edge in the course of the algorithm). We proved earlier that there are no edges skipping more than one level. So if we assign odd levels to "red" and even levels to "blue" the algorithm has verified that there are no edges between vertices of the same color.