CSE 421 Winter 2021
Homework 3

Due: Friday, January 29, 2021, 6:00 pm

Problem 1:
Let S be a set of intervals, where $S = \{I_1, \ldots, I_n\}$ with $I_j = [s_j, f_j]$, where $s_j < f_j$ for each j. A set of real numbers $P = \{p_1, \ldots, p_k\}$ is said to be a *cover* of a set of intervals S, if every interval I_j contains at least one element in P. (Interval $[s, f]$ contains p iff $s \le p \le f$.)

Describe an efficient algorithm that, given a set S of n intervals, finds an minimum size cover of S. You should be able to find one that runs in $O(n \log n)$ time; prove your claims.

Problem 2:
Suppose that you have a connected undirected graph $G(V, E)$ with $|V| = n$ and $|E| = m$ describing a network, where each edge e has an associated weight $w(e)$ that describes the strength of the connection between its end-points. The strength of a tree is the minimum strength of any edge in it.

Design an efficient algorithm to find a spanning tree of G of maximum possible strength. Its running time should be $O(m \log m)$ or better; prove your claims.

Problem 3: A *vertex cover* in an undirected graph $G = (V, E)$ is a subset $C \subset V$ such that every edge in E has an endpoint in C. Suppose that we associate a cost with each vertex equal to its vertex degree in G. With this measure, design an efficient algorithm to find a minimum total cost of a vertex cover in an arbitrary *tree* T.

Problem 4 (Extra Credit):
Consider the following algorithm for a weighted undirected graph $G = (V, E)$, where each edge e has a non-negative *length* $w(e)$.

```
Sort the edges of E in increasing order of weight, e_1,...,e_m
F ← ∅
for i = 1 to m
    if e_i = (u,v) such that "u and v are not connected in (V,F) or
    the shortest u to v path in (V,F) has length > 5w(e_i)" then
        add e_i to F.
end for
```

a) Prove that the graph $H = (G, F)$ constructed by the above algorithm has the property that distances between vertices in H are at most 5 times those in G.

b) Prove that $|F|$ is $o(n^2)$, no matter what G is and no matter what the lengths are.